Kan Injectivity & (KZ-Monads)

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<u>Intro</u> Orthogonal Subcat Problem (OSP)

$\mathcal{C}$ cat, $\mathcal{H}$ fam. maps $\mathcal{C}$.

X orth. w.r.t. $\mathcal{H}$ if

$$\begin{tikzcd} A \arrow[r, "h \in \mathcal{H}"] \arrow[d, "f"'] & A' \arrow[dl, dashed, "\exists !"] \\ X & \end{tikzcd}$$

$\mathcal{H}^{\perp} \xhookrightarrow{i} \mathcal{C}$

<u>Rmk</u> i creates all limits.

... well ... $\mathcal{H}^{\perp} \underset{\perp}{\hookrightarrow} \mathcal{C}$? OSP

{Previous work} 1966 Lambek, 1968 Kennison, 1972 Kelly and Freyd
1994 Adámek and Rosický
2015 Adámek, Sousa, Velebil

Ort. Sub. Prob.

$\mathcal{C}$ cat, $\mathcal{H}$ maps, $\mathcal{H}^{\perp} \hookrightarrow \mathcal{C}$ $\rightsquigarrow$ $\mathcal{K}$ 2-cat, $\mathcal{H}$ 1-cells, $\mathrm{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ KAN INJ

<u>Interlude</u>: Why KZ-Monads

$\mathcal{C}$ cat $\quad \mathcal{L} \underset{i}{\overset{L}{\underset{\perp}{\hookrightarrow}}} \mathcal{C} \rightsquigarrow iL$ id. monad.

$\mathrm{Alg}(T) \underset{\perp}{\hookrightarrow} \mathcal{C} \leftsquigarrow T : \mathcal{C} \to \mathcal{C}$ id.

$T : \mathcal{C} \to \mathcal{C}$ idemp. monad $\rightsquigarrow$ $\mathcal{K}$ 2-cat T KZ-Monads (or lax id. monads)

Kan Injectivity

MEMO: $\mathcal{K}$ 2-cat then
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & A' \\ X \end{tikzcd}$$
a (left) KAN EXT. of f along h is given by

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\xi \Rightarrow"] & A' \arrow[dl, dashed, "f/h"] \\ X \end{tikzcd}$$

1-dim

s.t. $\forall g\ \forall \alpha : f \Rightarrow g \cdot h\ \exists ! \bar{\alpha} : f/h \Rightarrow g$ s.t.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Rightarrow"] & A' \arrow[dl, "g"] \\ X \end{tikzcd} = \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\xi \Rightarrow"] & A' \arrow[dl, "f/h"', bend right] \arrow[dl, "g", bend left] \arrow[dl, phantom, "\bar{\alpha} \Rightarrow"] \\ X \end{tikzcd}$$

2-dim

$\mathcal{K}$ 2-cat, $\mathcal{H}$ a fam. of 1-cells in $\mathcal{K}$.

def
- $X \in \mathcal{K}$ is (left) KAN INJ. w.r.t. $\mathcal{H}$ iff

$$\forall f \quad \begin{tikzcd} A \arrow[r, "\forall h \in \mathcal{H}"] \arrow[d, "f"'] \arrow[dr, phantom, "\xi \cong \Rightarrow"] & A' \arrow[dl, "\exists f/h"] \\ X \end{tikzcd} \quad \text{s.t. } \xi \text{ isom.}$$

- $p : X \to Y \in \mathcal{K}$ is (left) KAN INJ w.r.t. $\mathcal{H}$ iff X, Y are, and p preserves Kan-ext. along any $h \in \mathcal{H}$.

i.e.
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\xi \Rightarrow"] & A' \arrow[dl, "f/h"] \\ X \arrow[d, "p"'] \\ Y \end{tikzcd} \cong \quad pf/h$$

$$\rightsquigarrow \quad \mathrm{LInj}(\mathcal{H}) \overset{i}{\hookrightarrow} \mathcal{K}$$

{SANITY CHECK} i creates all bilimits and pseudolimits.

Ex • $\mathcal{K} = \mathrm{Cat}$:

$$\begin{tikzcd} \{0 \quad 1\} \arrow[r, hook, "J"] \arrow[d] \arrow[dr, phantom, "\cong"] & \{0 \to 2 \leftarrow 1\} \arrow[dl, dashed] \\ \mathcal{C} & \end{tikzcd}$$

A B

$\cong A \to A \sqcup B \leftarrow B \cong$

$\mathrm{LInj}(\{J\}) = ?$
cat. w/ copr.
fct. pr. copr.

• $\mathcal{K} = \mathrm{Cat}$: Rex, actually any sort of colim.

• $\mathcal{K} = \mathrm{Lex}$: Coh, Ex, Reg $\rightsquigarrow$ $\boxed{\text{Logic}}$

$$\dots \qquad \mathrm{LInj}(\mathcal{H}) \underset{\longleftarrow}{\overset{\longleftarrow}{\perp}} \mathcal{K} \ ?$$

RECEIPE [~ Marmolejo & Wood]

Thm $\mathcal{A} \hookrightarrow \mathcal{K}$ locally full (and locally replete)

let $\{ d_X : X \to DX \mid X \in \mathcal{K} \}$ be a fam. of 1-cells $d_X \in \mathcal{A}$
s.t.

(1) $\mathcal{A} \subseteq \mathrm{LInj}(\{ d_X \mid X \in \mathcal{K} \})$

(2) $\forall X \in \mathcal{K}$, $DX \in \mathcal{A}$ and $\forall f : X \to A$ w/ $A \in \mathcal{A}$, $f / d_X \in \mathcal{A}$.

(3) $\forall X \in \mathcal{K}$, d_X is dense i.e.

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, "d_X"'] \arrow[dr, phantom, "\cong"] & DX \\ DX \arrow[ur, equal] & \end{tikzcd}$$

is a Kan ext. of d_X along itself.

$$\Longrightarrow \quad \begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{K} \arrow[l, bend right, "L"'] \end{tikzcd}$$

[moreover iL KZ-monad & $\mathcal{A} \simeq \mathrm{Alg}(iL)$]

⇝ <u>AIM</u> Find such a fun. of $d_X : X \to DX$.

$$\mathcal{A} := \mathrm{LInj}(\mathcal{H})$$

Small Object Argument

$$X \in \mathcal{H}$$

$$X = X_0 \to X_1 \to \dots \to X_\alpha \to X_{\alpha+1} \to \dots \rightsquigarrow X_\kappa \in \mathrm{LInj}(\mathcal{H})$$

Let $\mathcal{K}$ be a 2-cat w/ all weighted bicolim. and $X \in \mathcal{K}$

- $X_0 := X$
- for any lim. ord. i $\quad X_i := \mathrm{bicolim}_{j<i}\, X_j$
- i even ⇝ $i+1$ & $i+2$

1-dim

2-dim

bi-push. + wide bipush.

bi-coequaliser.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[dd, bend left, "f//h"] \\ X_i \arrow[dr, dashed] & \\ & X_{i+1} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A' \arrow[d, "g"] \\ X_j \arrow[r, dashed] & X_{i+1} \end{tikzcd} \rightsquigarrow \begin{tikzcd} A' \arrow[d, "g//h"'] \arrow[r, "g"] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & X_{i+1} \arrow[dd] \\ X_{j+1} \arrow[d, dotted] & \\ X_{i+1} \arrow[r, two heads] & X_{i+2} \end{tikzcd}$$

s.t. ...

<u>Our Setting</u>: $\mathcal{H}$ set and

$\forall x \in \mathcal{H}$, $\exists \lambda_x$ inf. reg. card. s.t. x is λ_x-bipres.

[I.B.F. we "only" need all dom. and cod. of 1-cells in $\mathcal{H}$ to be κ-bipres. for a fixed κ.]

$\Downarrow$

$$X_\kappa \in \mathrm{LInj}(\mathcal{H}).$$