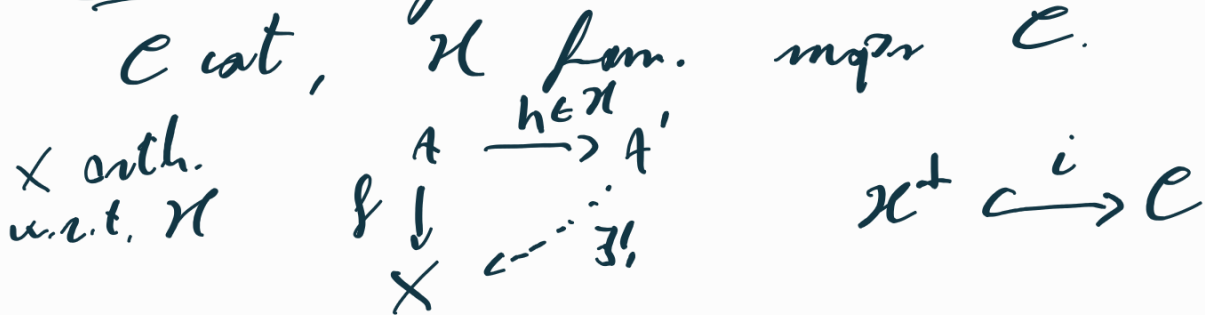


# Kan Injectivity & (KZ-Monads)

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Intro Orthogonal Subcat Problem (OSP)



Remark  $i$  creates all limits.  
... well ...  $\mathcal{X}^\perp \xrightarrow{i} \mathcal{C}$ ? OSP

{PREVIOUS WORK} 1966 Lambek, 1968 Kennison, 1972 Kelly and Freyd  
1994 Adamek and Ronicky

2015 Adamek, Sousa, Velebil

Orth. Sub. Prob.  
 $\mathcal{C}$  cat,  $\mathcal{K}$  maps,  $\mathcal{X}^\perp \hookrightarrow \mathcal{C} \rightsquigarrow \mathcal{K}$  2-cat,  $\mathcal{K}$ ,  $\text{LInj}(\mathcal{K})$  csk  
KAN INJ  
1-cats

Introduce: Why KZ-Monads

$\mathcal{C}$  cat

$\mathcal{L} \xrightarrow{i} \mathcal{C} \rightarrow \text{id. monad.}$

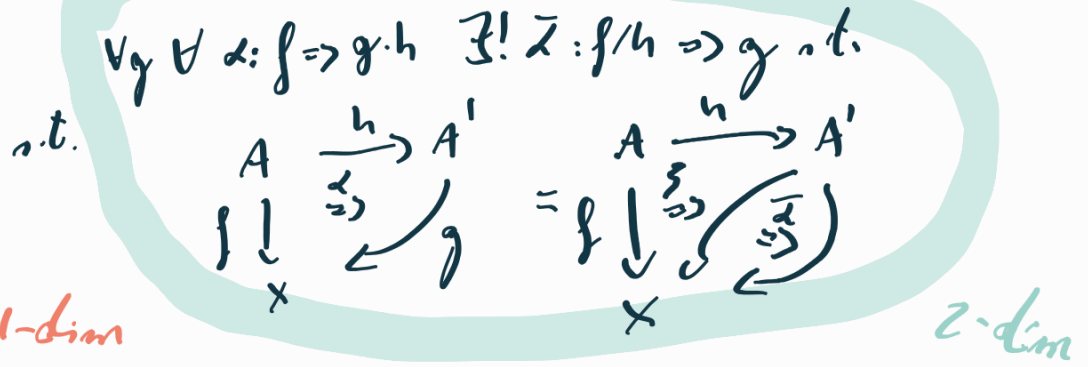
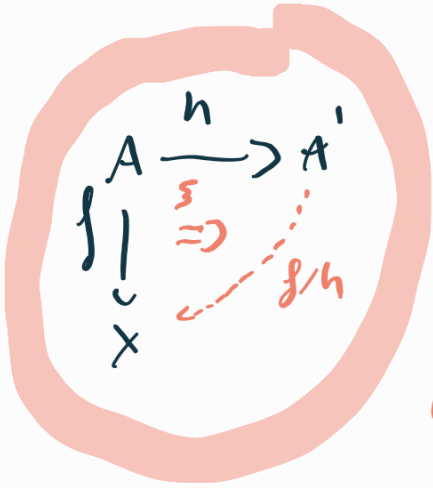
$\text{Alg}(\mathcal{L}) \xrightarrow{i} \mathcal{C} \text{ or } \tau: \mathcal{C} \rightarrow \mathcal{C} \text{ id.}$

$\tau: \mathcal{C} \rightarrow \mathcal{C}$  idemp. monad  $\rightsquigarrow \mathcal{K}$  2-cat  $\tau$  KZ-Monads  
(or lax id. monads)

# KAN Injectivity

MEMO:  $\mathcal{K}$  2-cat then

$$\begin{array}{c}
 A \xrightarrow{h} A' \\
 \downarrow f \\
 X
 \end{array}
 \quad \text{(left)} \quad
 \text{a KAN EXT. of } f \text{ along } h \text{ is given by}$$



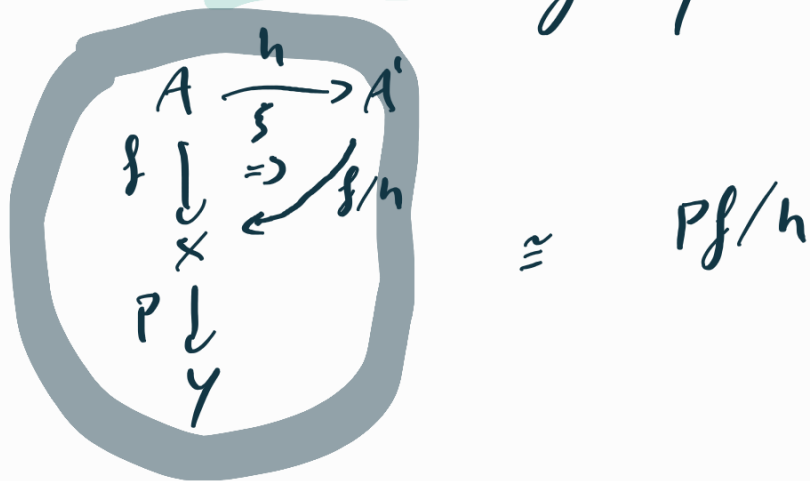
$\mathcal{K}$  2-cat,  $\mathcal{H}$  a fam. of 1-cells in  $\mathcal{K}$ .

def.  $\cdot X \in \mathcal{K}$  is (left) KAN INV. w.r.t.  $\mathcal{H}$  iff

$$\forall \begin{array}{c} A \xrightarrow{h \in \mathcal{H}} A' \\ \downarrow f \\ X \end{array} \quad \exists! \xi \Rightarrow \exists f/h \quad \text{s.t. } \xi \text{ isom.}$$

$\cdot p: X \rightarrow Y \in \mathcal{K}$  is (left) KAN INV. w.r.t.  $\mathcal{H}$  iff  $X, Y$  are, and  $p$  preserves non-ext. along any  $h \in \mathcal{H}$ .

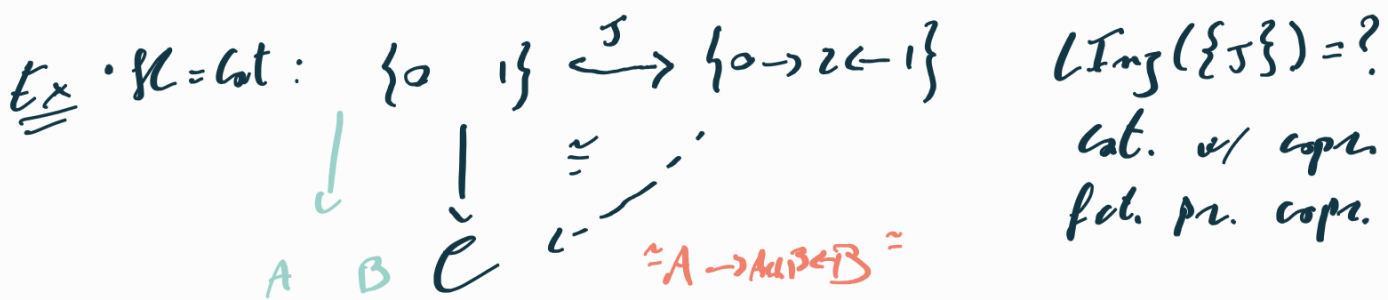
i.e.



$$\Rightarrow \text{LIm}_y(x) \xrightarrow{i} \mathcal{K}$$

{ SANITY  
CHECK }

$i$  creates all bilimits and pseudolimits.



- $\mathcal{K} = \text{Cat} : \text{Rel}$ , actually any sort of colim.
- $\mathcal{K} = \text{Lex} : \text{Coh}, \text{Ex}, \text{Reg} \rightsquigarrow \boxed{\text{Logic}}$

...  $\text{LInj}(\mathcal{K}) \xrightarrow{\perp} \mathcal{K} ?$

## RECIPE [à Marmoleyo & Wood]

Thm  $A \hookrightarrow \mathcal{K}$  locally full (and locally replete)

r.t. let  $\{d_x : X \rightarrow DX \mid x \in \mathcal{K}\}$  be a fam. of 1-cells  $d_x \in A$

(1)  $A \subseteq \text{LInj}(\{d_x \mid x \in \mathcal{K}\})$

(2)  $\forall x \in \mathcal{K}, DX \in A$  and  $\forall f : X \rightarrow A$  w/  $A \in A, f/d_x \in A$ .

(3)  $\forall x \in \mathcal{K}, d_x$  is dense i.e.

$$\begin{array}{ccc}
 X & \xrightarrow{d_x} & DX \\
 d_x \downarrow & \cong & \swarrow \\
 DX & & 
 \end{array}$$

is a Kan ext. of  $d_x$  along itself.

$\Rightarrow A \xrightarrow[\perp]{L} \mathcal{K}$  [ moreover it is  $\mathcal{K}$ -monad ]  
 &  $A \cong \text{Alg}(il)$  ]

→ AIM Find mod. a fam. of  $d_x: X \rightarrow DX$ .

$$A := \text{Inj}(\mathcal{K})$$

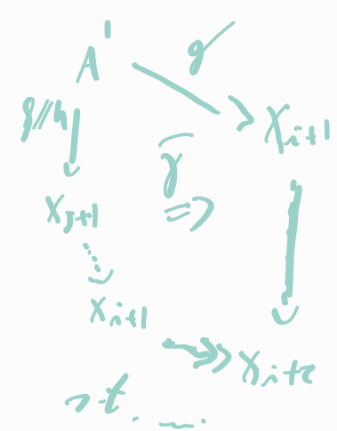
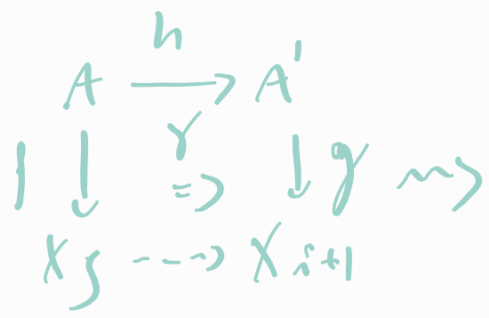
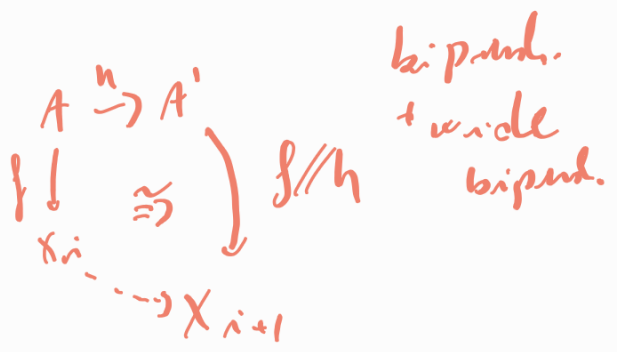
Small Object Argument

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow X_{n+1} \rightarrow \dots$$

$$\cong X_n \in \text{Inj}(\mathcal{K})$$

Let  $\mathcal{K}$  be a 2-cat w/ all weighted bicat. and  $X \in \mathcal{K}$

- $X_0 := X$
- for any bin. cond.  $i$   $X_i :=$  bi column-  $X_j$   $_{j < i}$
- $i$  even  $\rightarrow$   $i+1$   $_{1\text{-dim}}$   $\&$   $i+2$   $_{2\text{-dim}}$  bi equivalent.



Our Setting:  $\mathcal{K}$  set and  $\forall X \in \mathcal{K}, \exists \lambda_X$  inf. eq. cond. at  $X$  is  $\lambda_X$ -bipres.

[ IBF. we "only" need all dom. and cod. of 1-cells in  $\mathcal{K}$  to be  $\kappa$ -bipres. for a fixed  $\kappa$ . ]

$$X_{\mathcal{K}} \in \text{Inj}(\mathcal{K})$$