A Framework for Universality Across Disciplines

Tomáš Gonda

Joint work with: Sebastian Stengele, Tobias Reinhart,
Gemma De las Cuevas

8 September 2022

universität
innsbruck

References

- G. De las Cuevas and T. S. Cubitt [De16],
  Simple universal models capture all classical spin physics.
  Science 351:1180-1183 (2016).

- F. W. Lawvere [La69],
  Diagonal arguments and cartesian closed categories.
  Lecture Notes in Mathematics 92:134-145 (1969).

- D. Pavlovic and M. Yahia [Pa18],
  Monoidal computer III: A coalgebraic view of computability and complexity.
  International Workshop on Coalgebraic Methods in Computer Science 167-189 (2018).

Motivation

Motivating examples

Primary:

- universal Turing machines
- universal spin models
- universal neural networks

Motivating examples

Primary:

- universal Turing machines
- universal spin models
- universal neural networks

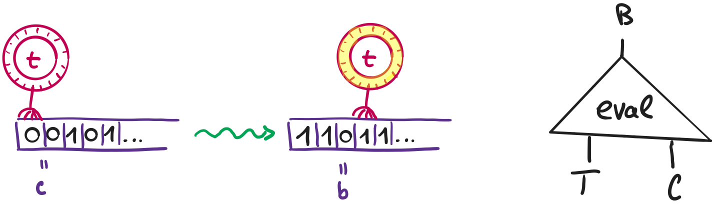
Others:

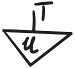
- universal grammar
- NP-completeness
- generating sets (universal gate set, ...)
- weak limits
- dense subsets
- universals in philosophy
- ...

Goals of the framework

- Capture examples of universality
- Properties of universality
  - trivial vs. non-trivial
  - necessary conditions for universality
- Relation to undecidability

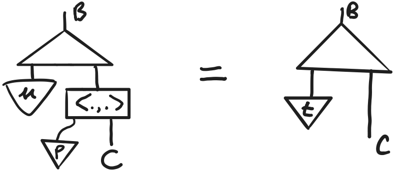
Universal Turing machine





is universal ≈ u can emulate any $t \in T$:

$\exists$ p = 0 0 1 1 1 :



Spin system

- Spin d.o.f. Σ for each vertex
- A hypergraph, edges $\sim$ local interactions

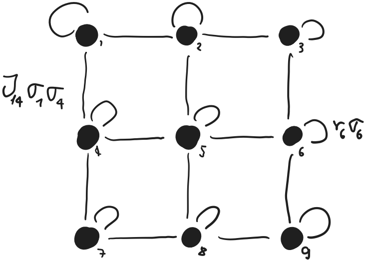
Spin system

- Spin d.o.f. Σ for each vertex
- A hypergraph, edges $\sim$ local interactions
- Hamiltonian $\Sigma^V \to \mathbb{R}$ as a sum of local coupling terms

Spin system

- ▷ Spin d.o.f. Σ for each vertex
- ▷ A hypergraph, edges $\sim$ local interactions
- ▷ Hamiltonian $\Sigma^V \to \mathbb{R}$ as a sum of local coupling terms

2D Ising spin model with fields has $\Sigma = \mathbb{Z}_2$ and interaction lattice:

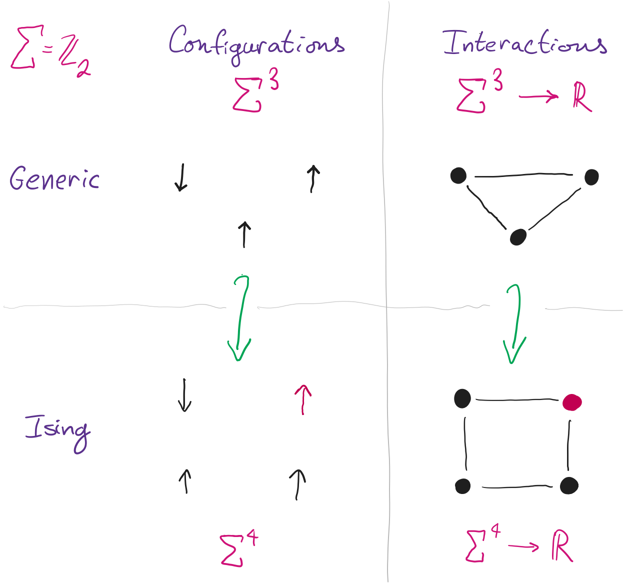


$$\sigma_i \in \Sigma$$

$$H = \sum_{i,j} J_{ij}\,\sigma_i\sigma_j + \sum_i r_i\,\sigma_i$$

Spin system simulation

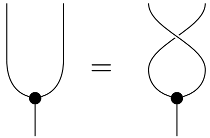
Every spin system can be simulated on a 2D Ising one [De16].

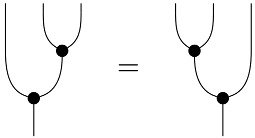


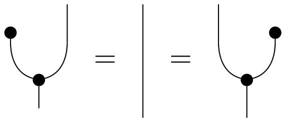
The set-up (simulators)

Ambient category

D is **gs-monoidal**, an SMC with $A \to A \otimes A$ and $A \to I$ such that

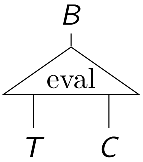






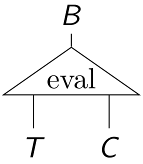
Intrinsic behavior structure

$T \in \mathsf{D}$	**things**	Turing machines	spin systems
$C \in \mathsf{D}$	**contexts**	input strings	spin configurations
$B \in \mathsf{D}$	**behaviors**	output strings	energies
$\rightmoon_B$	**relation**	$=$	$=$

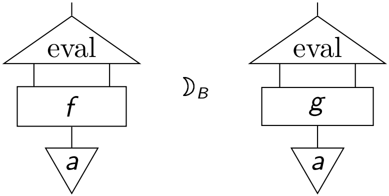


Intrinsic behavior structure

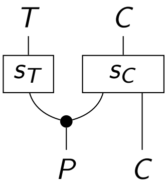
$T \in \mathrm{D}$	**things**	Turing machines	spin systems
$C \in \mathrm{D}$	**contexts**	input strings	spin configurations
$B \in \mathrm{D}$	**behaviors**	output strings	energies
$\rightmoon_B$	**relation**	$=$	$=$



$f \rightmoon g$ if, for all $a \colon I \to A$,

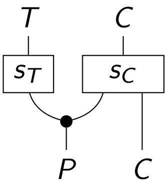


A simulator



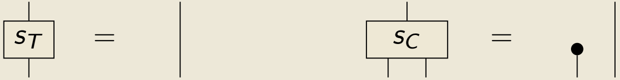
$P \in \mathsf{D}$	**programs**
$s_T \colon P \to T$	**compiler**
$s_C \colon P \otimes C \to C$	**context reduction**

A simulator

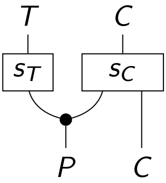


$P \in \mathrm{D}$	**programs**
$s_T : P \to T$	**compiler**
$s_C : P \otimes C \to C$	**context reduction**

Example (trivial simulator)

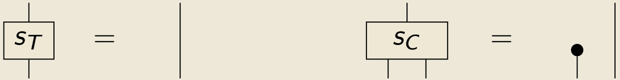


A simulator

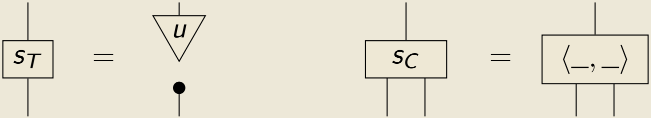


$P \in \mathrm{D}$	**programs**
$s_T : P \to T$	**compiler**
$s_C : P \otimes C \to C$	**context reduction**

Example (trivial simulator)

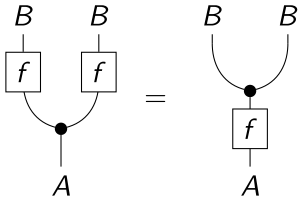


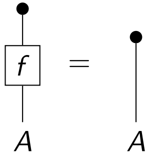
Example (singleton simulator for TM)



Deterministic subcategory

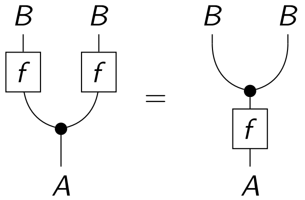
In $\mathsf{D}^{\bullet}$, every $f : A \to B$ satisfies

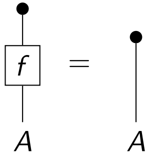




Deterministic subcategory

In $D^{\bullet}$, every $f : A \to B$ satisfies





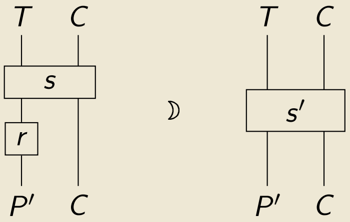
Morphisms in $D^{\bullet}$ are called **deterministic**.

Universality

Reductions and universality

Definition

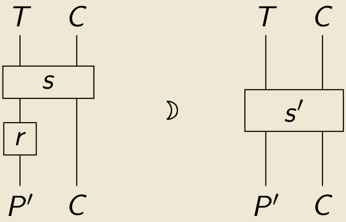
A **lax reduction** $r^*\colon s \to s'$ from simulator s to s' is a $r \in \mathrm{D}^\bullet(P', P)$ such that



Reductions and universality

Definition

A **lax reduction** $r^*\colon s \to s'$ from simulator s to s' is a $r \in \mathrm{D}^\bullet(P', P)$ such that



Definition

Simulator s is **universal** if there is a lax reduction $s \to \mathrm{id}$ to the trivial simulator.

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM
- 2D Ising spin model with fields

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM
- 2D Ising spin model with fields
- Dense subset (e.g. $T = \mathbb{R} \times \mathbb{R}_+$, $P = \mathbb{Q} \times \mathbb{R}_+$)

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM
- 2D Ising spin model with fields
- Dense subset (e.g. $T = \mathbb{R} \times \mathbb{R}_+$, $P = \mathbb{Q} \times \mathbb{R}_+$)
- A generating set (T = tuples, C = formulas)

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM
- 2D Ising spin model with fields
- Dense subset (e.g. $T = \mathbb{R} \times \mathbb{R}_+$, $P = \mathbb{Q} \times \mathbb{R}_+$)
- A generating set (T = tuples, C = formulas)
- Weak limit (T = cones, $\gimel_B$ = cone factorization, $P = C = I$)

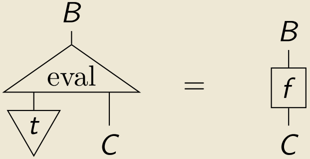
Relation to undecidability

Lawvere's Fixed Point Theorem

Definition

eval is **weakly point surjective** if for every $f\colon C \to B$

$\exists\, t \in \mathsf{D}^{\bullet}(I, T)$

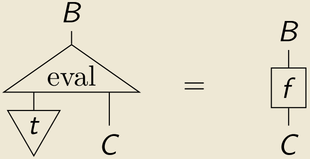


Lawvere's Fixed Point Theorem

Definition

eval is **weakly point surjective** if for every $f\colon C \to B$

$\exists\, t \in \mathsf{D}^{\bullet}(I, T)$



Theorem

Let $\rightmoon_B$ be equality, eval be w.p.s., and $s\colon C \otimes C \to T \otimes C$ be a universal simulator. Then every $g\colon B \to B$ has a fixed point.

We show that $\mathrm{eval} \circ s$ is w.p.s. and use (a stronger version of) Lawvere's Fixed Point Theorem [La69].

Undecidability from universality

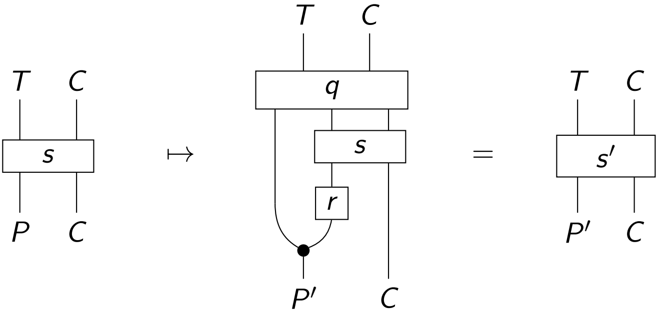
Theorem

Let $\rrbracket_B$ be equality, eval be w.p.s., and $s \colon C \otimes C \to T \otimes C$ be a universal simulator. Then every $g \colon B \to B$ has a fixed point.

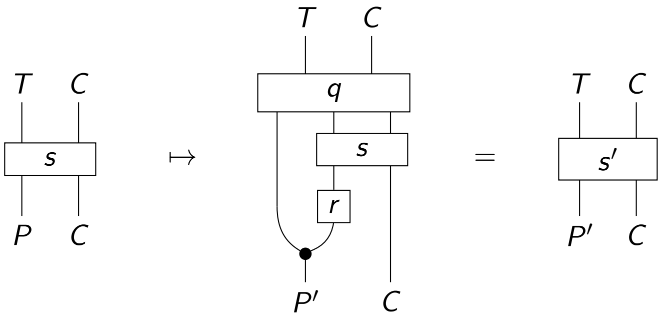
universal s + fixed-point-free $g \implies$ 'undecidability'

Hierarchy of universal simulators

Simulator morphisms

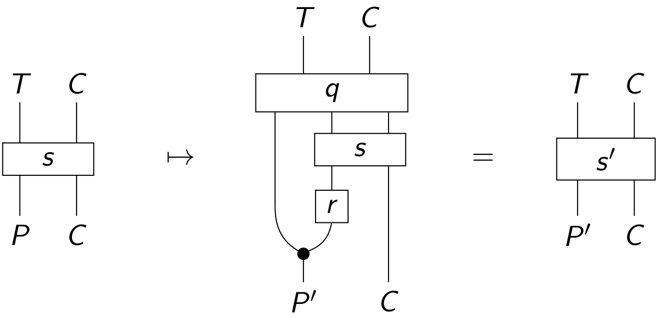


Simulator morphisms



r is deterministic $\implies$ sequential composition

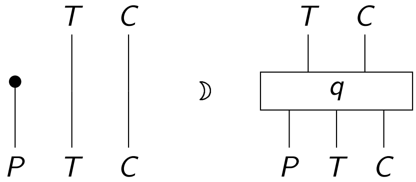
Simulator morphisms

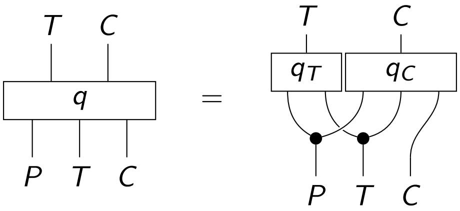


r is deterministic $\implies$ sequential composition

We also require that (s' is universal) $\implies$ (s is universal).

Processings





Strength of simulators

Definition

s' is a **stronger** simulator than s, written $s' \geq s$, if there exists a morphism $s \to s'$.

Strength of simulators

Definition

s' is a **stronger** simulator than s, written $s' \geq s$, if there exists a morphism $s \to s'$.

Theorem

The singleton simulator s_u for a universal TM is strictly stronger than the trivial simulator.

Strength of simulators

Definition

s' is a **stronger** simulator than s, written $s' \geq s$, if there exists a morphism $s \to s'$.

Theorem

The singleton simulator s_u for a universal TM is strictly stronger than the trivial simulator.

- $s_u \geq \mathrm{id}$ uses that $\exists$ right-invertible reduction.

Strength of simulators

Definition

s' is a **stronger** simulator than s, written $s' \geq s$, if there exists a morphism $s \to s'$.

Theorem

The singleton simulator s_u for a universal TM is strictly stronger than the trivial simulator.

- $s_u \geq \mathrm{id}$ uses that $\exists$ right-invertible reduction.
- $s_u \not\leq \mathrm{id}$ because $\exists\, t, t' \colon I \to T$ such that
  - t can be separated from t' and
  - $s_T(r(t)) = s_T(r(t'))$.

Conclusions

Summary

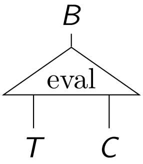
- Abstract notion of universality with many instances
- Morphisms of simulators $\rightarrow$ non-trivial universality
- Connection to Lawvere's Fixed Point Theorem

Outlook

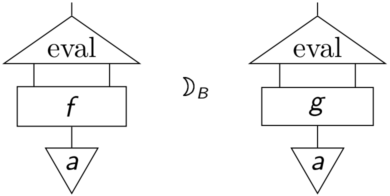
- Necessary conditions for universality - monotone observables
- Additional strengthenings of universality
- More constructive connection undecidability and trade-offs
- Comparing different instances of the framework

Intrinsic behavior structure

$T \in \mathrm{D}$	**things**	Turing machines	spin systems
$C \in \mathrm{D}$	**contexts**	input strings	spin configurations
$B \in \mathrm{D}$	**behaviors**	output strings	energies
$☽_B$	**relation**	$=$	$=$



$f ☽ g$ if, for all $a\colon I \to A$,



Extrinsic behavior structure

Definition

A **behavior structure** is a faithful, strong monoidal functor $\mathrm{Beh}\colon \mathsf{D} \to \mathsf{Set}$ and a preorder $\rightmoon_B$ on $\tilde{B} \coloneqq \mathrm{Beh}(T \times C)$.

Given $f, g\colon A \to T \times C$, f behaviorally subsumes g if

$$\begin{array}{ccc} & \mathrm{Beh}(A) & \\ {}^{\mathrm{Beh}(f)}\swarrow & & \searrow^{\mathrm{Beh}(g)} \\ \tilde{B} & \xrightarrow{\quad \rightmoon_B \quad} & \tilde{B} \end{array}$$

commutes in Rel.

This defines $\rightmoon$ on any $\mathsf{D}(A, T \times C)$.

Turing machines

Definition

A **Turing category** is a cartesian restriction category with a distinguished Turing object T and morphisms $\tau_{X,Y} \colon T \times X \to Y$ for any pair of objects X, Y such that for any $f \colon Z \times X \to Y$ there exists a unique $h \colon Z \to T$ satisfying $\tau \circ (h \times \mathrm{id}_X) = f$.

Example (Simulators of Turing machines)

Σ is a finite alphabet and $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ its Kleene star. T is given by the set of Turing machines. Further objects are $C = \Sigma^* = B$ and finite products thereof. Morphisms are partial computable maps. There is a pairing function $\langle \_, \_ \rangle \colon C \times C \to C$.
The relation $\mathbb{D}_B$ is equality among strings and eval is given by $\tau_{C,C}$.

Spin models

- $T = P =$ set of spin systems specified by size, interaction hypergraph, couplings, and fields.
- $C =$ spin configurations in Σ^*
- $B =$ energies (e.g. $\mathbb{Q}_+$)
- s_T takes a generic spin system t to a (larger) Ising system.
- s_C encodes its configurations into those of $s_T(t)$ via flag spins.

Dense subset

- $T = \mathbb{R} \times \mathbb{R}_+$, i.e. points and precisions
- $C = I$, the singleton set
- $B = \mathcal{P}(\mathbb{R})$ with $\sqsupseteq_B$ the subset inclusion
- eval maps (t, δ) to the open ball of radius δ centered at t.
- $P = \mathbb{Q} \times \mathbb{R}_+$ and s_T is the inclusion into T
- The reduction $r\colon T \to P$ maps (t, δ) to $\left(q_{(t,\delta)}, \delta/2\right)$

Generating family (of a group)

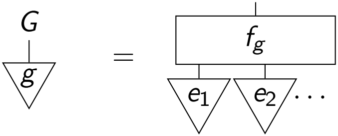
- $T = P = G^*$ are families of group elements
- C consists of formulas $G^k \to G$, e.g.

$$(g, h) \mapsto hg^{-1}h^2$$

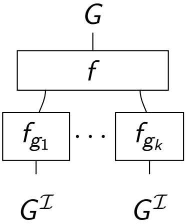
- $B = G$ with $\sqsupset_B$ the equality
- eval evaluates formulas on families with enough elements.

Generating family (of a group)

- s_T discards P and returns the generating family $(e_i)_{i\in\mathcal{I}}$
- For each g, we have a formula $f_g\colon G^{\mathcal{I}} \to G$ with



- s_C acts by mapping the pair of a family (g_j) and a formula $f\colon G^k \to G$ to

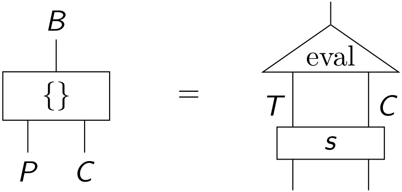


Weak limits

- $T = B =$ the set of cones over a given diagram $F\colon \mathsf{J} \to \mathsf{C}$
- $C = I$, the singleton set
- $\mathrm{eval} = \mathsf{id}_T$
- $\psi \rightmoon_B \phi$ if ϕ factors through ψ.
- $P = I$ and s_T is the weak limit of F.
- Can be generalized to scenarios when $\lim F$ doesn't exist by using other P.

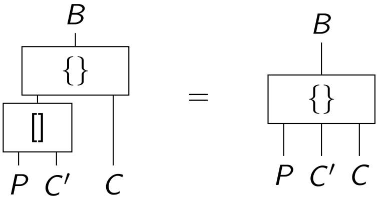
Monoidal computer

Specify a family of universal evaluators [Pa18]



for fixed P, every $C \in$ D, and a corresponding w.p.s. eval and a universal simulator s.

Plus there are (deterministic) partial evaluators relating them:

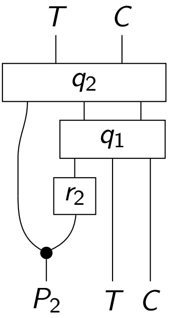


Morphism composition

Given two morphisms $(r_1^*, q_{1*})\colon s \to s_1$ and $(r_2^*, q_{2*})\colon s_1 \to s_2$ of simulators, we define the **sequential composition**

$$(r_2^*, q_{2*}) \circ (r_1^*, q_{1*})\colon s \to s_2$$

to be the morphism whose processing is given by the map



with reduction given by $(r_1 \circ r_2)^*$.

Necessary conditions for universality