Cyclic Causal Networks via Partial Markov Categories

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Collaboration

• Joint work with Siddharth Bhat, Elena Di Lavore, Miguel Lopez, Mario Román, Nicoletta Sabadini, Ruben Van Belle

• Special thanks to the organizers of the Applied Category Theory 2022 Adjoint School at Glasgow
Causal Models

Cyclicity and interventions

- Bayesian Causal Network [Pearl, 2009]
  - Acyclic graph
  - Each vertex has a measurable space
  - Each vertex has a stochastic map given the parents
- Example: $P(G), P(S|G), P(C|S, G)$
- Categorical formulation [Fong, 2013]
- Cyclic models [Bongers et al, 2021]
- Intervention: remove causal mechanism
  - Correspond to experiment
  - As an endofunctor [Jacobs et al, 2019]
- Goal: bicategorical formulation of cyclic models with interventions as 2-cells
- Partial Markov category with comparator
Markov Category

Symmetric Monoidal Category \((\mathcal{C}, \otimes, I)\) is Markov [Fritz 2020] if

- Each object \(X\) has a commutative comonoid structure
  - copy: \(X \to X \otimes X\)
  - delete: \(X \to I\)
- Satisfying associativity, unitality and commutativity axioms
- Uniform: compatible with \(\otimes\)
- \(\mathcal{C}\) is semicartesian: \(I\) terminal

Example:

- Finitary distribution monad \(D: \text{Set} \to \text{Set}\),
  - Maps set \(X\) to set \(DX\) of finitely supported probability measures on \(X\)
- Kleisly category \(Kl(D)\) is Markov
Partial Markov Category

Symmetric Monoidal Category \((\mathcal{C}, \otimes, I)\) is partial Markov if

- Each object \(X\) has a commutative \textit{comonoid} structure
  - copy: \(X \rightarrow X \otimes X\),
  - delete: \(X \rightarrow I\)
  - Satisfying associativity, commutativity and unitality
- Each object \(X\) has a commutative \textit{semimonoid} structure
  - compare: \(X \otimes X \rightarrow X\)
  - Satisfying associativity and commutativity
- Special uniform Frobenius axioms, except monoidal unitality
  - Not compact closed
- \(\mathcal{C}\) is semicartesian: \(I\) terminal
Example of Partial Markov Categories

- Partial functions
  - Monad $\cdot +1 : Set \rightarrow Set$
  - Kleisli category $Kl(\cdot +1)$ is partial Markov
  - Comparator: $(x_1, x_2) \mapsto \begin{cases} x_1 & \text{if } x_1 = x_2 \\ \bot & \text{otherwise} \end{cases}$

- Conditional subdistributions
  - Finitary subdistribution monad $D_{\leq 1} : Set \rightarrow Set$
    - Maps set $X$ to set $D_{\leq 1}X$ of finitely supported measures on $X$ with norm in $[0, 1]$
    - $D_{\leq 1}X = D(X + 1)$
  - Kleisli category $Kl(D_{\leq 1})$ is partial Markov
  - Comparator $P(X = x | X_1 = x_1, X_2 = x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 = x \\ 0 & \text{otherwise} \end{cases}$
Causal Morphisms

• Morphism $f: A \to B$ is causal if [Kissinger & Uilen, 2017]

• The wide subcategory of causal morphisms is a Markov category.
Bayesian update

Belief of unfair coin

- In $KL(D_{\leq 1})$
- Coin values $X = \{0, 1\}$, probability $\Theta = [0,1]$
- Prior $P: I \rightarrow \Theta$
  - Finitely supported distribution over probabilities
- Likelihood $L: \Theta \rightarrow X$, $L(\theta|x) = \begin{cases} \theta & \text{if } x = 1 \\ 1 - \theta & \text{if } x = 0 \end{cases}$
- Updated belief $B: X \rightarrow \Theta$
  - $B(\theta|x) = \sum_{x'} P(\theta)L(x'|\theta)[x' = x] = P(\theta)L(x|\theta)$
- Bayesian update $P(\theta|x) = \frac{P(\theta)L(x|\theta)}{\sum_{\theta'} P(\theta)L(x|\theta')}$
  - Equal up to normalization
Normalization

• For morphism \( f: A \rightarrow B \), normalization is a morphism \( n_f: A \rightarrow B \) such that

\[
\sum_{b'} f (b'|a) \sum_{b} f (b'|a) = f (b|a)
\]

\( \text{In } Kl(D_{\leq 1}), \quad n_f (b|a) \sum_{b'} f (b'|a) = f (b|a) \)

• If \( \sum_{b'} f (b'|a) \neq 0 \), \( n_f (b|a) = \frac{f (b|a)}{\sum_{b'} f (b'|a)} \)
Conditionals

- A morphism $f: X \rightarrow Y \otimes Z$ has a conditional $c_f: X \otimes Y \rightarrow Z$ if

- A partial Markov category is **uniform** if for any $f, g: X \otimes Y \rightarrow Z$

- For a uniform partial Markov category, a conditional $c_f$ of $f$ is a normalization of
  - Proof via Frobenius axioms
Cyclic causal models

Encoding in a partial Markov Category

- Endomorphism $S \otimes P \otimes N \otimes D \rightarrow S \otimes P \otimes N \otimes D$
- For each variable, we get a morphism given the parents & comparator
- Endomorphism composed out of commuting constraint endomorphisms
Solutions

Sampling from a causal model

• Given causal network $f : \mathcal{X} \to \mathcal{X}$

• A solution is a causal morphism $g : I \to \mathcal{X}$ if there exists a scalar $\lambda : I \to I$ such that

$$g \circ \lambda = f$$

• If causal model has deterministic leaf variables, it is a structural causal model
  • Solution concides with fixpoints in prior work [Bongers et al, 2021]
Acyclic case

- Acyclic model
- Cyclic Causal Networks via Partial Markov Categories
  - Encodes as
  - Has solution
  - In general, coincides with [Fong, 2013]
Interventions
Via bicategory

• Make partial Markov category a posetal symmetric monoidal bicategory
• 2-cells generated by

\[ f \leq f' \text{ if } \forall a, b, f(b | a) \leq f'(b | a) \]

• Example

• Consistent with posetal bicategorical structure on \( Kl(D_{\leq 1}) : f, f' : A \to B \), \( f \leq f' \) if \( \forall a, b, f(b | a) \leq f'(b | a) \)
Future directions

• Continuous spaces

• Implement with probabilistic programming
  • Directly sample from causal model with comparators

• Perform causal inferences
  • Estimate effects of interventions from partially observed data [Jacobs et al, 2019]
References


Thank you