Cyclic Causal Networks via Partial Markov Categories

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Collaboration

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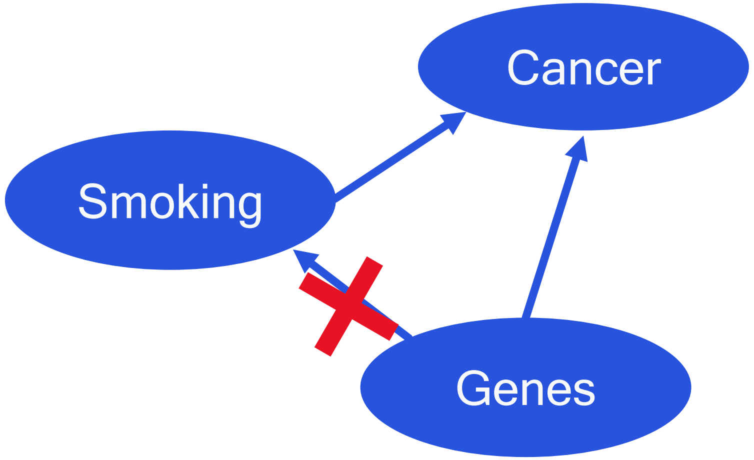
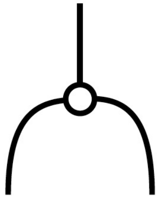
Causal Models

Cyclicity and interventions



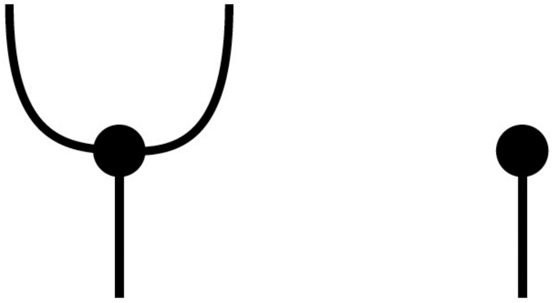
- Bayesian Cauasl Network [Pearl, 2009]
  - Acyclic graph
  - Each vertex has a measurable space
  - Each vertex has a stochastic map given the parents
- Example: $P(G), P(S|G), P(C|S,G)$
- Categorical formulation [Fong, 2013]
- Cyclic models [Bongers et al, 2021]
- Intervention: remove causal mechanism
  - Correspond to experiment
  - As an endofunctor [Jacobs et al, 2019]
- Goal: bicategorical formulation of cyclic models with interventions as 2-cells
- Partial Markov category with comparator



Markov Category

Symmetric Monoidal Category $(\mathcal{C},\otimes,I)$ is Markov [Fritz 2020] if

- Each object X has a commutative comonoid structure
  - copy: $X \to X \otimes X$
  - delete: $X \to I$
  - Satysfying associativity, unitality and commutativity axioms
  - Uniform: compatible with $\otimes$
- $\mathcal{C}$ is semicartesian: I terminal



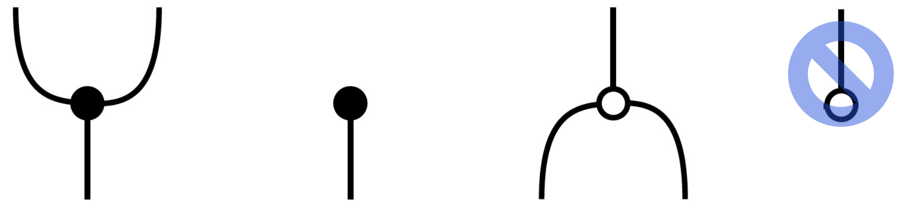
Example:

- Finitary distribution monad $D: Set \to Set$,
  - Maps set X to set DX of finitely supported probability measures on X
- Kleisly category $Kl(D)$ is Markov

Partial Markov Category

Symmetric Monoidal Category $(\mathcal{C}, \otimes, I)$ is partial Markov if

- Each object X has a commutative comonoid structure
  - copy: $X \to X \otimes X$,
  - delete: $X \to I$
  - Satysfying associativity, commutativity and unitality
- Each object X has a commutative semimonoid structure
  - compare: $X \otimes X \to X$
  - Satisfying associativity and commutativity
- Special uniform Frobenius axioms, except monoidal untality
  - Not compact closed
- ~~$\mathcal{C}$ is semicartesian: I terminal~~



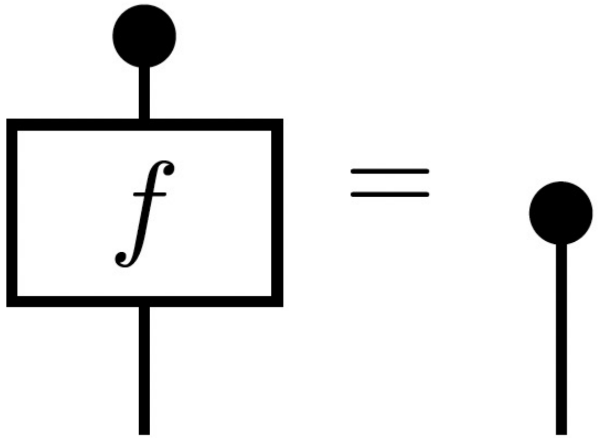
Example of Partial Markov Categories

- Partial functions
  - Monad $(\cdot +1) : Set \to Set$
  - Kleisli category $Kl(\cdot +1)$ is partial Markov
  - Comparator: $(x_1, x_2\,) \mapsto \begin{cases} x_1 & \text{if } x_1 = x_2 \\ \bot & \text{otherwise} \end{cases}$

- Conditional subdistributions
  - Finitary subdistribution monad $D_{\leq 1}\!: Set \to Set$
    - Maps set X to set $D_{\leq 1}X$ of finitely supported measures on X with norm in $[0, 1]$
    - $D_{\leq 1}X = D(X + 1)$
  - Kleisli category $Kl(D_{\leq 1})$ is partial Markov
  - Comparator $P(X = x|X_1 = x_1, X_2 = x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 = x \\ 0 & \text{otherwise} \end{cases}$

Causal Morphisms

- Morphism $f: A \to B$ is causal if [Kissinger & Uilen, 2017]

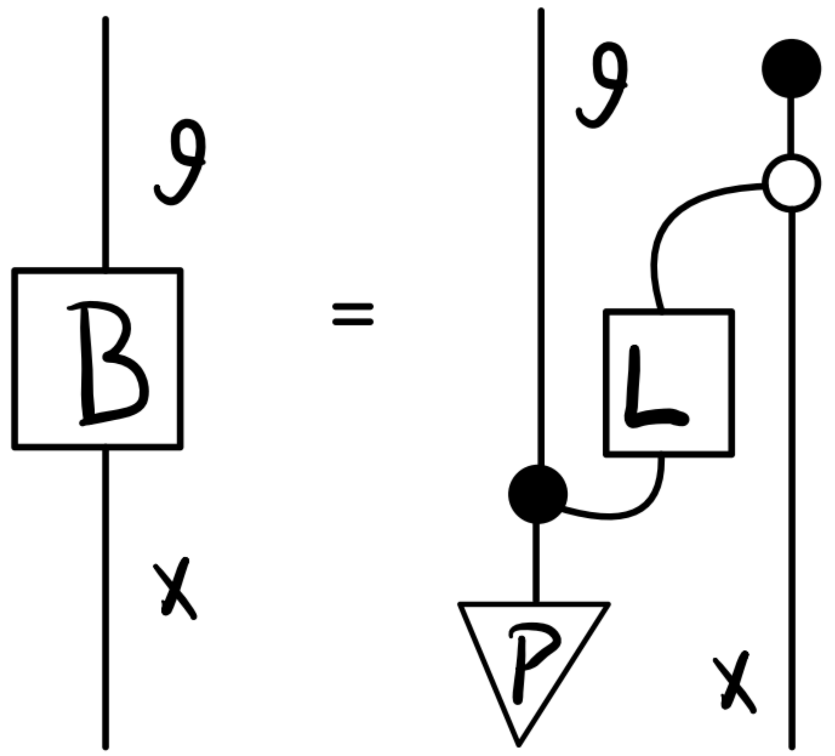


- The wide subcategory of causal morphisms is a Markov category.

Bayesian update

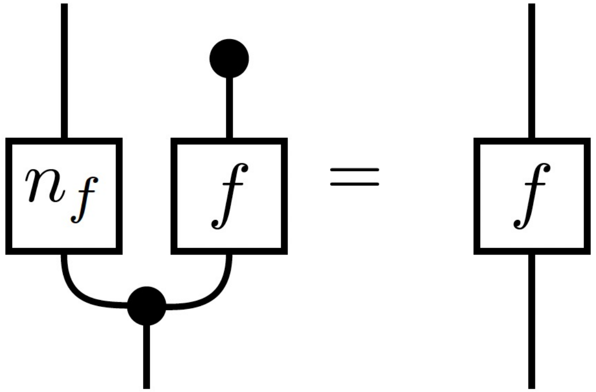
Belief of unfair coin

- In $Kl(D_{\leq 1})$
- Coin values $X = \{0, 1\}$, probability $\Theta = [0,1]$
- Prior $P\colon \mathrm{I} \to \Theta$
  - Finitely supported distribution over probabilities
- Likelihood $L\colon \Theta \to X$, $L(\theta|x) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \end{cases}$
- Updated belief $B\colon X \to \Theta$
  - $B(\theta|x) = \sum_{x'} P(\theta)L(x'|\theta)[x' = x] = P(\theta)L(x|\theta)$
- Bayesian update $\mathbb{P}(\theta|x) = \frac{P(\theta)L(x|\theta)}{\sum_{\theta'} P(\theta)L(x|\theta')}$
  - Equal up to normalization



Normalization

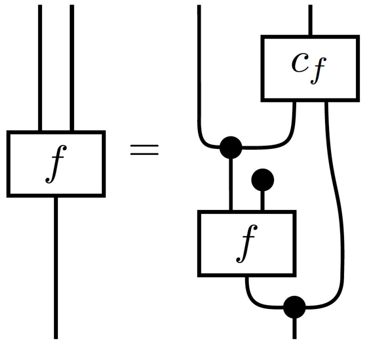
- For morphism $f: A \to B$, normalization is a morphism $n_f: A \to B$ such that



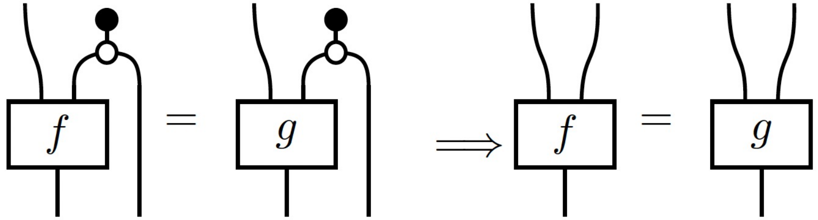
- In $Kl(D_{\leq 1})$, $n_f(b|a) \sum_{b'} f(b'|a) = f(b|a)$
- If $\sum_{b'} f(b'|a) \neq 0$, $n_f(b|a) = \frac{f(b|a)}{\sum_{b'} f(b'|a)}$

Conditionals

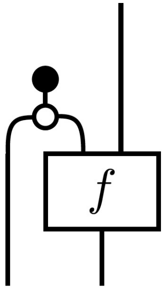
- A morphism $f: X \to Y \otimes Z$ has a conditional $c_f: X \otimes Y \to Z$ if



- A partial Markov category is uniform if for any $f, g: X \otimes Y \to Z$

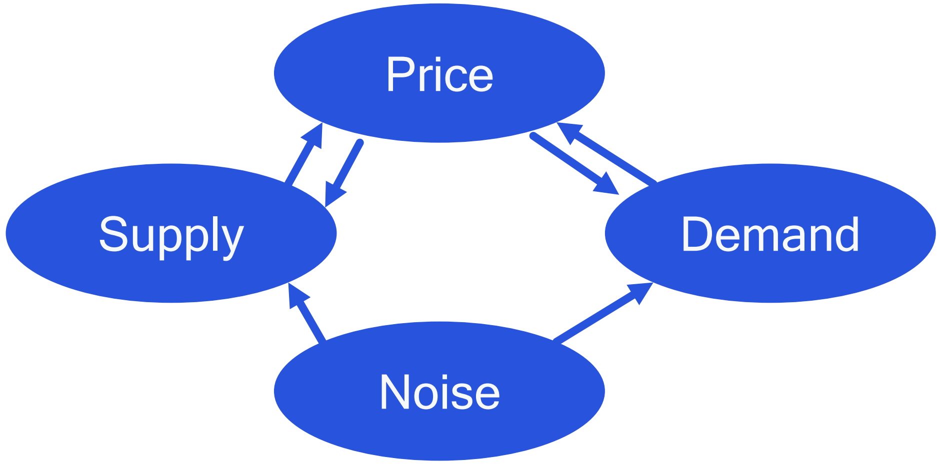


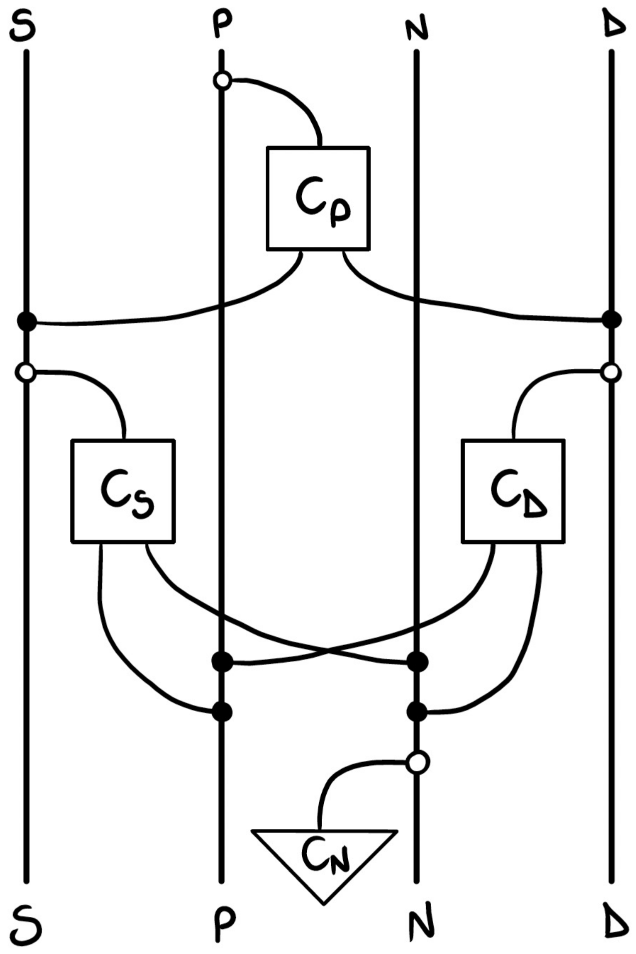
- For a uniform partial Markov category, a conditional c_f of f is a normalization of
  - Proof via Frobenius axioms



Cyclic causal models

Encoding in a partial Markov Category



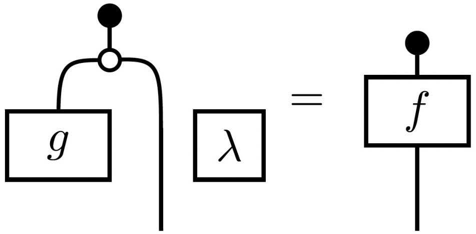


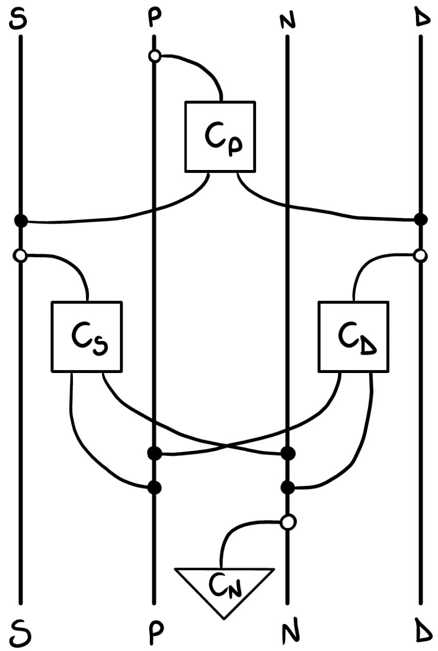
- Endomorphism $S \otimes P \otimes N \otimes D \to S \otimes P \otimes N \otimes D$
- For each varable, we get a morphism given the parents & comparator
- Endomorphism composed out of commuting constraint endomorphisms

Solutions

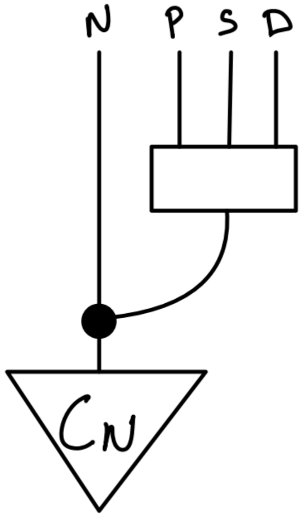
Sampling from a causal model

- Given causal network $f : \mathbb{X} \to \mathbb{X}$
- A solution is a causal morphism $g : I \to \mathbb{X}$ if there exists a scalar $\lambda : I \to I$ such that



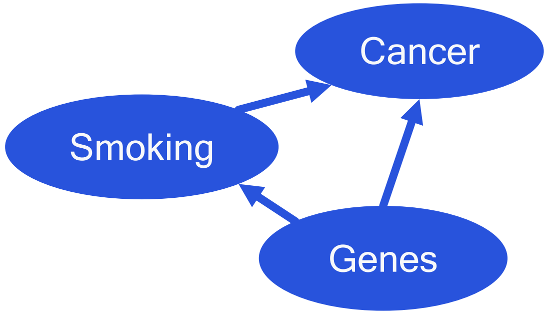


- If causal model has deterministic leaf variables, it is a structural causal model
  - Solution concides with fixpoints in prior work [Bongers et al, 2021]

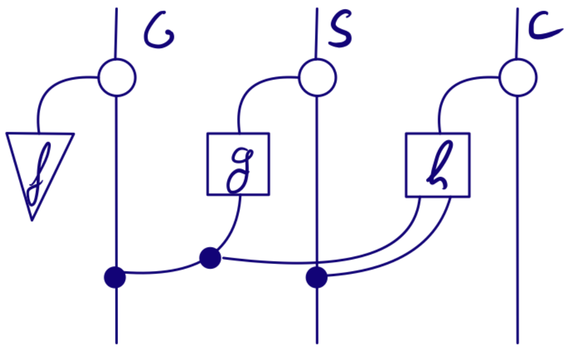


Acyclic case

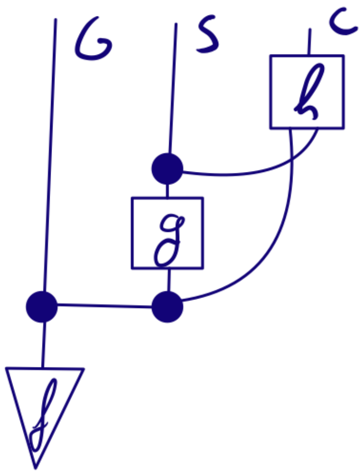
- Acyclic model



- Encodes as



- Has solution

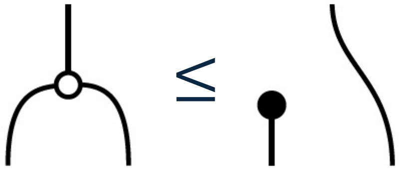


- In general, coincides with [Fong, 2013]

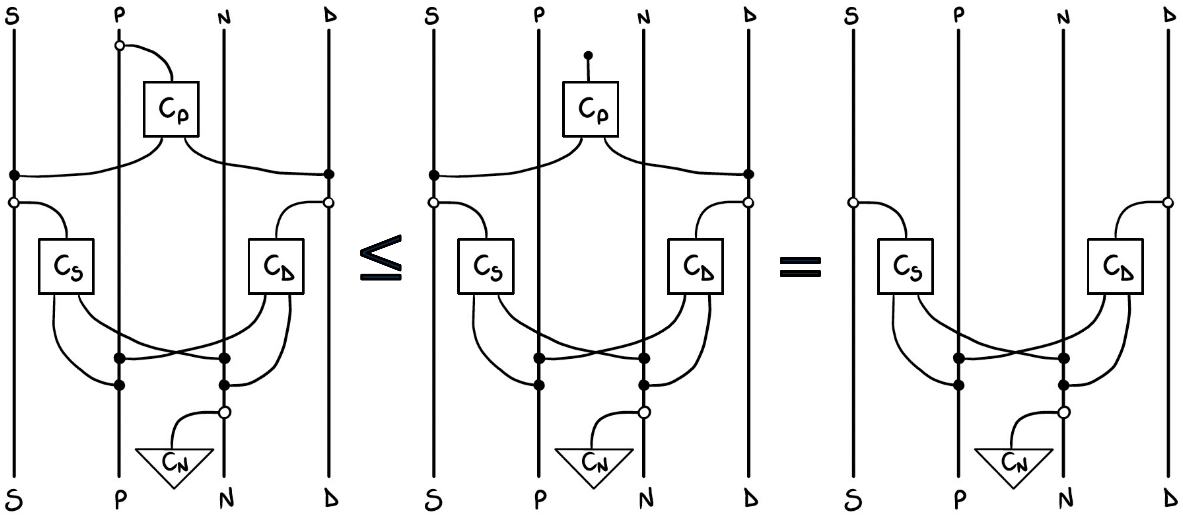
Interventions

Via bicategory

- Make partial Markov category a posetal symmetric monoidal bicategory
- 2-cells generated by



- Example



- Consistent with posetal bicategorical structure on $Kl(D_{\leq 1})$: $f, f': A \to B$, $f \leq f'$ if $\forall a, b, f(b|a) \leq f'(b|a)$

Future directions

- Continuous spaces
- Implement with probabilistic programming
  - Directly sample from causal model with comparators
- Perform causal inferences
  - Estimate effects of interventions from partially observed data [Jacobs et al, 2019]

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