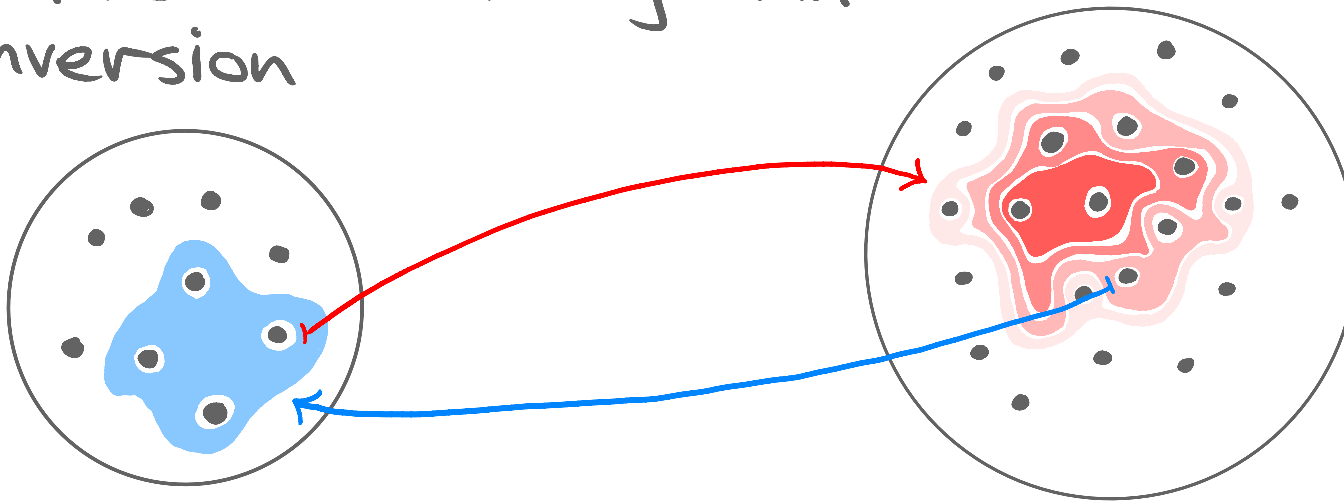
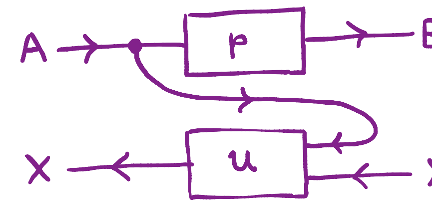
Dependent Bayesian Lenses: Categories of bidirectional with canonical Bayesian inversion



Dylan Braithwaite & Jules Hedges

LENSES

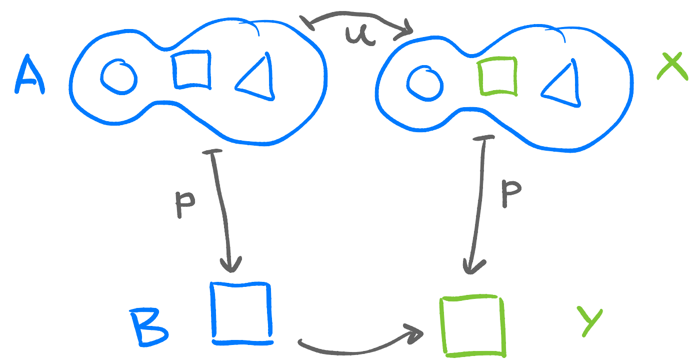
- Lenses are a model of bidirectional transformation
- Defined over an arbitrary cartesian category $\mathcal{C}$



- <u>Lens</u>($\mathcal{C}$) has
  - objects are pairs $\begin{pmatrix} A \\ X \end{pmatrix}$ of objects from $\mathcal{C}$
  - morphisms $\begin{pmatrix} A \\ X \end{pmatrix} \rightleftarrows \begin{pmatrix} B \\ Y \end{pmatrix}$ are given by two morphisms in $\mathcal{C}$

$$p : A \longrightarrow B$$

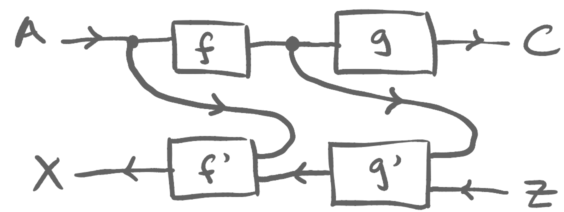
$$u : A \times Y \longrightarrow X$$



Composing Lenses

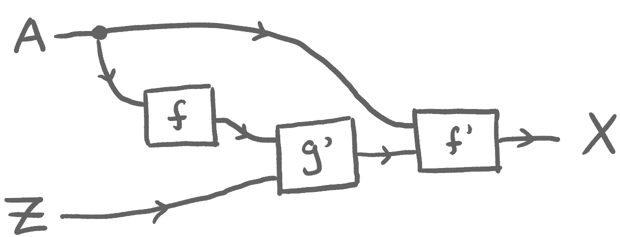
Consider a composable pair of lenses $\begin{pmatrix} A \\ X \end{pmatrix} \underset{f'}{\overset{f}{\rightleftarrows}} \begin{pmatrix} B \\ Y \end{pmatrix} \underset{g'}{\overset{g}{\rightleftarrows}} \begin{pmatrix} C \\ Z \end{pmatrix}$

we want to use their composite like this:



Obviously in the forward direction we do $A \to \boxed{f} \to \boxed{g} \to C$

In the backward direction we do:



In SET: $(gf)'(a,z) = f'(a, g'(f(a), z))$

Compare to differentiation

$$(gf)'(a) = f'(a)\, g'(f(a))$$

③

COMPOSING BAYESIAN UPDATES

Consider conditional probability distributions

$$\mathbb{P}(Y|X), \ \mathbb{P}(Z|Y)$$

$f : X \longrightarrow Y \qquad g : Y \longrightarrow Z$

MARKOV CATEGORY

Monoidal category with
- supply of comonoids
- natural delete morphisms
- satisfying compatibility eqns

These compose by marginalisation

$$\mathbb{P}(Z|X) = \int_Y \mathbb{P}(Z|Y)\mathbb{P}(Y|X)$$

$gf : X \longrightarrow Z$

$$\widetilde{gf}(z|x) = \sum_y \tilde{g}(z|y)\tilde{f}(y|x)$$

Eg FIN STOCH

$f : X \longrightarrow Y$

$\tilde{f} : Y \times X \longrightarrow [0,1]$

w/ finite support

Bayes' theorem gives us a way to invert conditional probabilities

$$\mathbb{P}(X|Z) = \frac{\mathbb{P}(Z|X)\mathbb{P}(X)}{\int_X \mathbb{P}(Z|X)\mathbb{P}(X)}$$

$$\widetilde{(gf)}^+_\pi(x|z) = \frac{(\widetilde{gf})(z|x)\cdot\tilde{\pi}(x)}{\sum_{x'}(\widetilde{gf})(z|x')\tilde{\pi}(x')}$$

given a prior probability distribution on X

$(gf)^+_\pi : Z \longrightarrow X$ for $\pi : I \longrightarrow X$

But by expanding the composite distribution we can see inversion composes in a familiar way

$$\widetilde{(gf)}^+_\pi(x|z) = \frac{\sum_y \tilde{g}(z|y)\tilde{f}(y|x)\tilde{\pi}(x)}{\sum_{x'}\sum_y \tilde{g}(z|y)\tilde{f}(y|x')\tilde{\pi}(x')} = \sum_y \tilde{f}^+_\pi(y|x)\,\tilde{g}^+_{\pi f}(z|y)$$

$$\Longrightarrow (gf)^+_\pi = \tilde{f}^+_\pi \circ \tilde{g}^+_{\pi f}$$

Compare with lenses:

$$(gf)'(a,-) = f'(a,-)\circ g'(f(a),-)$$

4

GROTHENDIECK LENSES

Indexed category $\mathcal{X}_{(-)} : \mathcal{C}^{op} \longrightarrow \underline{Cat}$

Consider the Grothendieck construction $\int_{c:\mathcal{C}} (\mathcal{X}_c)^{op}$:

- Objects are pairs $\binom{A}{x}$ where $x \in \mathcal{X}_A$
- Morphisms $\binom{A}{x} \rightleftarrows \binom{B}{y}$ consist of
  - $f : A \longrightarrow B$ in $\mathcal{C}$
  - $f' : f^* y \longrightarrow x$ in $\mathcal{X}_A$

Generalized Lens Categories via Functors $\mathcal{C}^{op} \longrightarrow \underline{Cat}$

D. Spivak

arxiv 1908.02202

These compose like lenses too!

$$\binom{A}{x} \underset{f'}{\overset{f}{\rightleftarrows}} \binom{B}{y} \underset{g'}{\overset{g}{\rightleftarrows}} \binom{C}{z} \; : \; (fg)' : f^* g^* z \longrightarrow x$$

is given by the composition $f^* g^* z \xrightarrow{f^*(g')} f^* y \xrightarrow{f'} x$

Hence, we call this $\underline{Lens}(\mathcal{X}_{(-)})$.

For $X \in \mathcal{C}$, $X \times (-)$ is a comonad

$X \longmapsto \underline{coKl}(X \times (-))$ defines an indexed category

And we have that

$$\underline{Lens}(\underline{coKl}((-)\times(=))) \cong \underline{Lens}(\mathcal{C})$$

5

Bayesian Lenses

$\mathcal{C}$ a Markov category.

Compositional Active Inference 1: Bayesian Lenses. Statistical Games
T. Smithe
arxiv 2109.04461

$\underline{\mathrm{State}}_{\mathcal{C}} : \mathcal{C}^{op} \longrightarrow \underline{\mathrm{Cat}}$

$$\underline{\mathrm{State}}_{\mathcal{C}}(A) = \begin{cases} \text{objects}: & \mathrm{ob}(\mathcal{C}) \\ X \longrightarrow Y: & \text{functions } \mathcal{C}(I,A) \longrightarrow \mathcal{C}(X,Y) \end{cases}$$

$$\underline{\mathrm{BLens}}(\mathcal{C}) = \underline{\mathrm{Lens}}(\underline{\mathrm{State}}_{\mathcal{C}}) = \begin{cases} \text{objects}: \text{ pairs } \begin{pmatrix} A \\ X \end{pmatrix} \\ \begin{pmatrix} A \\ X \end{pmatrix} \rightleftarrows \begin{pmatrix} B \\ Y \end{pmatrix}: & \text{morphism } A \longrightarrow B \\ & \text{\& function } \mathcal{C}(I,A) \longrightarrow \mathcal{C}(Y,X) \end{cases}$$

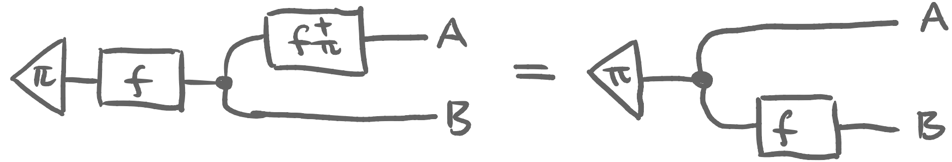
For example $\begin{pmatrix} A \\ A \end{pmatrix} \underset{f^\dagger}{\overset{f}{\rightleftarrows}} \begin{pmatrix} B \\ B \end{pmatrix}$ where $f^\dagger : \pi \longmapsto f^\dagger_\pi$

A coherent choice of Bayesian inverses makes this functorial

Bayesian Inverses

"Given $f: A \to B$ and $\pi: I \to A$, when does $f^\dagger_\pi$ exist?"

Abstract definition:



<u>FinStoch</u> always has inverses: $\tilde{f}^\dagger_\pi(a|b) = \dfrac{\tilde{f}(b|a)\,\tilde{\pi}(a)}{\sum_{a'} \tilde{f}(b|a')\,\tilde{\pi}(a')}$

But what if this is 0?

Abstract Bayesian inverses are not in general unique!

Eg the copy morphism $\Delta_X : X \to X \otimes X$

In <u>FinStoch</u> $(\Delta_X)^\dagger_\pi : X \otimes X \to X$ is defined by Bayes' rule only on the diagonal $\{(x,x) \mid x \in \mathrm{supp}(\pi)\} = \mathrm{supp}(\Delta_X \circ \pi)$

Δ_X has a LOT of Bayesian inverses!

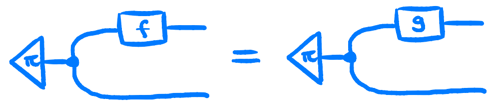
Support Objects

Can we instead define Bayesian inverses between supports?

Eg $(\Delta_x)^{+}_{\pi} : \mathrm{supp}(\Delta_x \circ \pi) \to \mathrm{supp}(\pi)$

For a state $\pi : I \to X$, a support of π is any object X_π which represents the functor

$$\mathcal{C}(X,-)/\underset{\pi}{\cong} : \mathcal{C} \to \mathrm{Set}$$



A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics
T. Fritz
arxiv 1908.07021
Defn. 13.20

+ upcoming work by Fritz, Gonda, Houghton-Larsen, Perrone, Stein

Equivalently: X_π is a support object iff there exists a section-retract pair

$$X_\pi \xrightarrow[i]{} X \xrightarrow[r]{} X_\pi$$

such that

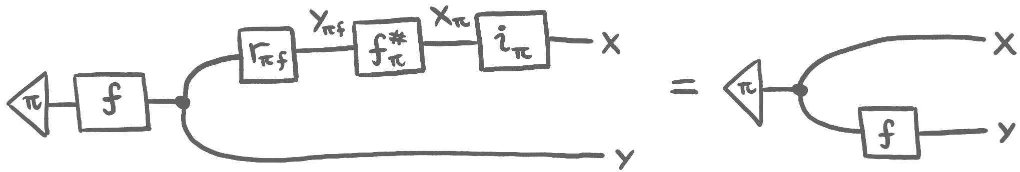
$$f \underset{\pi}{\cong} g \iff f \circ i = g \circ i$$

In FinStoch $X_\pi = \{ x \in X \mid \tilde{\pi}(x) > 0 \}$

Bayesian Inverses... Again

Given $f : X \longrightarrow Y$, $\pi : I \longrightarrow X$ and support objects X_π and $Y_{\pi \circ f}$

say $f^\#_\pi : Y_{\pi f} \longrightarrow X_\pi$ is a Bayesian inverse with support if



- $f^\#_\pi$ exists iff an ordinary Bayesian inverse exists
- if $f^\#_\pi$ exists, it is unique*

*: for fixed choice of $Y_{\pi f}$ and X_π

but support objects are unique upto isomorphism

What does this say about copying?

$(X \otimes X)_{\Delta_X \circ \pi} \cong X_\pi$

In fact $(\Delta_X)^\#_\pi : (X \otimes X)_{\Delta_X \circ \pi} \longrightarrow X_\pi$ is an iso. with inverse $\begin{pmatrix} X \longrightarrow X \\ X \longrightarrow \bullet \end{pmatrix}^\#_{\Delta_X \circ \pi}$

⑨ Dependent Bayesian Lenses

So does Bayesian inversion give a section $\mathcal{C} \longrightarrow \underline{\mathrm{BLens}}(\mathcal{C})$?

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, mapsto] & B \arrow[d, mapsto] \\
\binom{A}{?} \arrow[r, shift left, "f"] & \binom{B}{?} \arrow[l, shift left, "f^{\#}_{(-)}"]
\end{tikzcd}
$$

We need families of objects in the fibres!

Want objects of the form $\binom{A}{X}$ where X is a family of objects indexed by states on A i.e. $X : \mathcal{C}(I,A) \longrightarrow \underline{\mathrm{Set}}$

Dependent $\underline{\mathrm{Stat}}$:

$$
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "\mathcal{C}(I,-)^{op}"] & \underline{\mathrm{Set}}^{op} \arrow[r, "\underline{\mathrm{Fam}}_{\mathcal{C}}"] & \underline{\mathrm{Cat}} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(I,X) \arrow[r, mapsto] \arrow[d, "\mathcal{C}(I,f)"'] & \mathcal{C}^{\mathcal{C}(I,X)} \\
Y \arrow[r, mapsto] & \mathcal{C}(I,Y) \arrow[r, mapsto] & \mathcal{C}^{\mathcal{C}(I,Y)} \arrow[u, "\mathcal{C}^{\mathcal{C}(I,f)}"']
\end{tikzcd}
$$

$\underline{\mathrm{Lens}}(\underline{\mathrm{Fam}}_{\mathcal{C}}(\mathcal{C}(I,-)^{op}))$:

$$
\begin{tikzcd}
\binom{A}{X} \arrow[r, shift left, "f"] & \binom{B}{Y} \arrow[l, shift left, "f'"]
\end{tikzcd}
$$

$f : A \longrightarrow B$ in $\mathcal{C}$

$f' : (\pi : \mathcal{C}(I,A)) \longrightarrow \mathcal{C}(Y(f\pi), X(\pi))$

in $\underline{\mathrm{Set}}$

Functorial Bayesian Inversion

If $\mathcal{C}$ has all Bayesian inverses, then any assignment of support objects to states of $\mathcal{C}$ defines a section $\mathcal{C} \xrightarrow{T} \underline{\mathrm{DBLens}}(\mathcal{C})$

$$A \xrightarrow{f} B$$

$$\Big\downarrow T$$

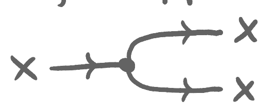
$$\begin{pmatrix} A \\ A_{(-)} \end{pmatrix} \underset{f^{\#}_{(-)}}{\overset{f}{\rightleftarrows}} \begin{pmatrix} B \\ B_{(-)} \end{pmatrix} \qquad \text{where } f^{\#}_{\pi} : B_{f\pi} \longrightarrow A_{\pi} \text{ for each } \pi : I \longrightarrow A$$

$\underline{\mathrm{DBLens}}(\mathcal{C})$ inherits monoidal structure from $\mathcal{C}$ by the monoidal Grothendieck construction...

But T is not strong monoidal because otherwise we would have

$$(X \otimes X)_{\Delta_X ; \pi} \cong X_{\pi} \otimes X_{\pi} \qquad \text{i.e. } X_{\pi} \cong X_{\pi} \otimes X_{\pi}$$

With a nice choice of supports, comonoids are sent to



$$X_{(-)} \longleftarrow X_{(-)}$$

11

So What?

- Compositional active inference

Compositional Active Inference 1: Bayesian Lenses. Statistical Games
T. Smithe
arxiv 2109.04461

- Stochastic dynamical systems

$$\begin{pmatrix} A \\ A_{(-)} \end{pmatrix} \underset{r\circ f\circ i}{\overset{f}{\rightrightarrows}} \begin{pmatrix} B \\ B_{(-)} \end{pmatrix}.$$

Double Categories of Open Dynamical Systems
D.J. Myers
arxiv 2005.05956

- Open cybernetic systems doctrine

Diagetic Representation of Feedback in Open Games
M. Capucci
arxiv 2206.12338

- PL Semantics?

Structural Foundations for Probabilistic Programming Languages
D. Stein, PhD Thesis