A Category of Plane Graphs

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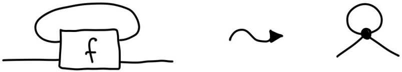
Why Graphs?

- string diagrams as syntax for monoidal categories
  – we can draw pictures!
- want a combinatorial representation to reason about them
- use (some form of) graphs and their morphisms

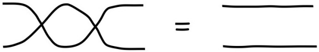
Remark

here: vertices represent generators, edges represent wires



Why Plane Graphs?

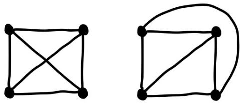
- graphs are (V, E), it's all sets
- drawings contain information about the surface as well
- in SMC the surface doesn't matter:



- other monoidal theories may require non-trivial topology
- example: string diagrams for quantum processes

What are Surface Embeddings?

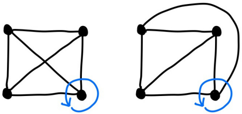
- Plane graph embeddings
  = drawing in the plane (or on the sphere)



- A graph is planar if it has a plane embedding
- (similarly for higher genus surfaces)
- we work at the level of *embedding* of a graph

Representing Graph Embeddings

Rotation Systems fix order of edges around vertices



Theorem

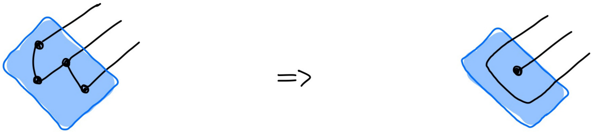
Rotation systems uniquely determine a graph embedding.

Our Plan

construct a category of graphs, then add rotation information

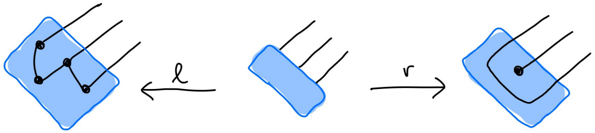
An Example of DPO Rewriting

given: rewrite rule



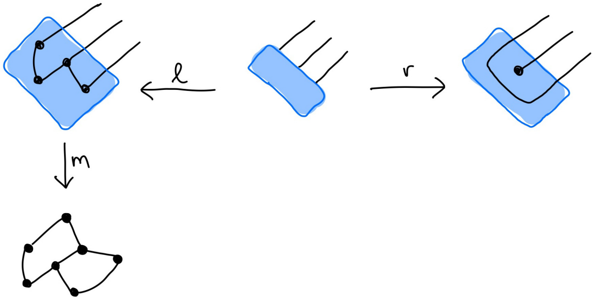
An Example of DPO Rewriting

given: rewrite rule as span with common boundary in the middle



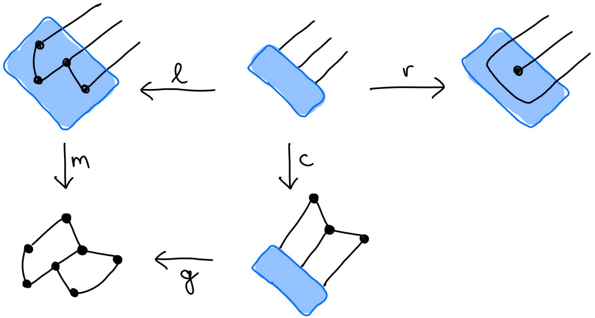
An Example of DPO Rewriting

given: matching of the LHS within a graph



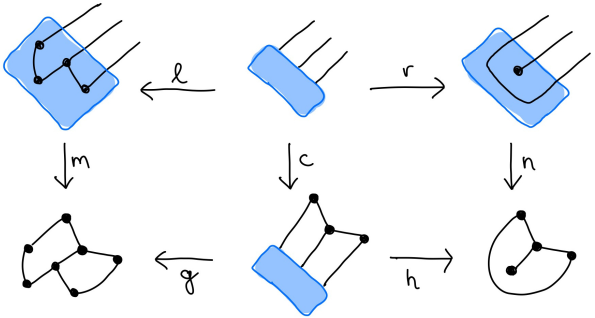
An Example of DPO Rewriting

construct: context graph by pushout complement



An Example of DPO Rewriting

construct: final graph by pushout



DPO Rewriting

$$
\begin{array}{ccccc}
L & \longleftarrow & B & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
G & \longleftarrow & G \setminus L & \longrightarrow & G[R/L]
\end{array}
$$

- rewrite rules are $L \Rightarrow R$, with common boundary B
- double-pushout diagram, all maps are embeddings
- need: pushouts, pushout complements, notion of embedding

DPO Rewriting

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DPO Rewriting

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G \setminus C & \longleftarrow & B & \longrightarrow & R \\
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- rewrite rules are $L \Rightarrow R$, with common boundary B
- double-pushout diagram, all maps are embeddings
- need: pushouts, pushout complements, notion of embedding
- $C = G \setminus L$: context with a hole
- $L = G \setminus C$: LHS with a "hole"

Category of Graphs

We start from the standard category of graphs:

- graphs are $E \overset{s}{\underset{t}{\rightrightarrows}} V$
- morphisms are pairs of edge map f_E and vertex map f_V s.t.

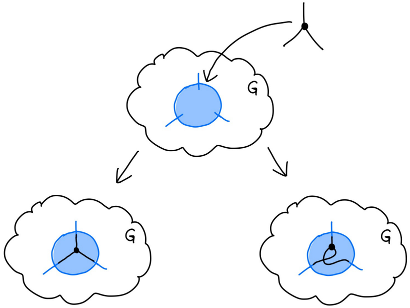
$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ V & \xrightarrow[f_V]{} & V' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ V & \xrightarrow[f_V]{} & V' \end{array}$$

Disclaimer

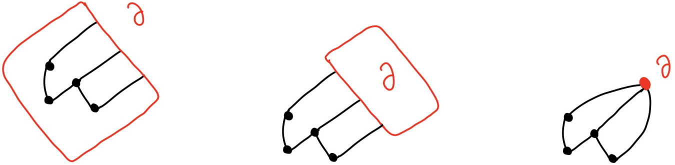
(Almost) all graphs are drawn undirected in this presentation.

Open Graphs

- have to encode inputs and outputs of the diagrams
- different approaches: open graphs, representative vertices, cospans
- morphisms for open graphs don't preserve the surface:



Boundary Vertices



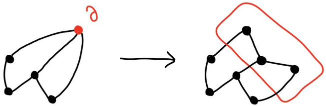
- identify the "outside" of a graph
- attach input and output edges to this region item represent the outside with a *boundary vertex*

This provides:

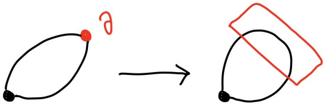
- total graphs
- strategy to deal with the outside, and any holes in a graph

Requirements for Graph Morphisms

- vertex map needs to be partial

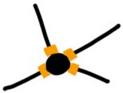


- cannot be injective on edges

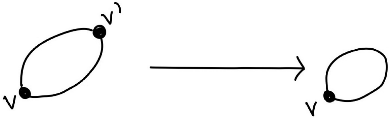


How to define graph embeddings?

Flags



- connection points between vertices and their incident edges, pairs (v, e)
- flag map (f_E, f_V) *partial* map induced by graph map
- characterise morphisms/embeddings on the flag map



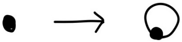
- example: flag injectivity

Flag Surjectivity

Starting with the condition for standard graph morphisms $(V, E) \to (V', E')$:

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ V & \xrightarrow[f_V]{} & V' \end{array}$$

What about vertices with no edges attached?

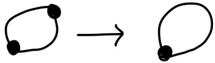


Flag Surjectivity

Condition on *vertices*, by considering the preimage:

$$\begin{array}{ccc} V & \xrightarrow{f_V} & V' \\ {\scriptstyle s^{-1}}\downarrow & & \downarrow{\scriptstyle s'^{-1}} \\ P(E) & \xrightarrow{P(f_E)} & P(E') \end{array}$$

What about vertices where f_V is undefined?



Flag Surjectivity

Flag surjectivity = lax commutation of the square:

$$\begin{array}{ccc} V & \xrightarrow{f_V} & V' \\ {\scriptstyle s^{-1}}\big\downarrow & \geq & \big\downarrow{\scriptstyle s'^{-1}} \\ P(E) & \xrightarrow[P(f_E)]{} & P(E') \end{array}$$

Graphs with Circles

Objects are total graphs, as defined above

Morphisms are (f_E, f_V) where

- f_E is total
- the flag map is surjective
  (no increase of flags at a vertex)

\+ other conditions

Graph *embeddings* are

- flag injective (no decrease of flags at a vertex)

\+ other conditions

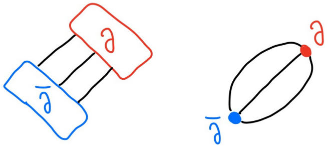
It's a category!

Rewriting for Graphs with Circles

Define rewriting for a specific case

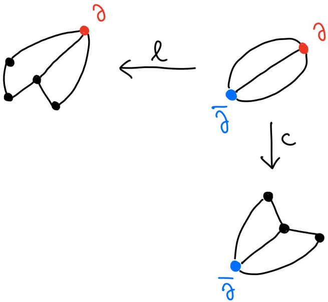
Boundary Graph

boundary vertex and dual boundary vertex, connected by edges:



Partitioning Spans

partition a graph into two (connected) parts: context and subgraph



Partitioning Spans

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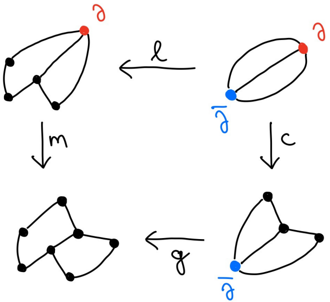
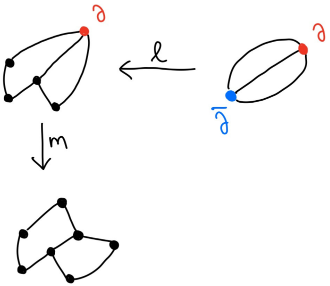


Theorem

Pushouts of partitioning spans exist, and all morphisms in the pushout square are embeddings.

Boundary Embeddings

for constructing pushout complements which give rise to partitioning spans



Boundary Embeddings

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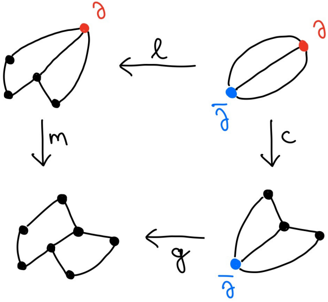
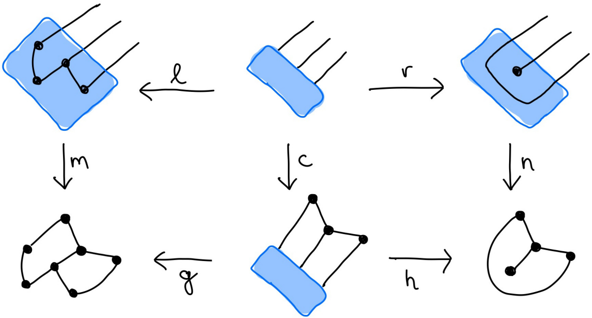


Theorem

Pushout complements of boundary embeddings exist and are unique (up to degeneracies).

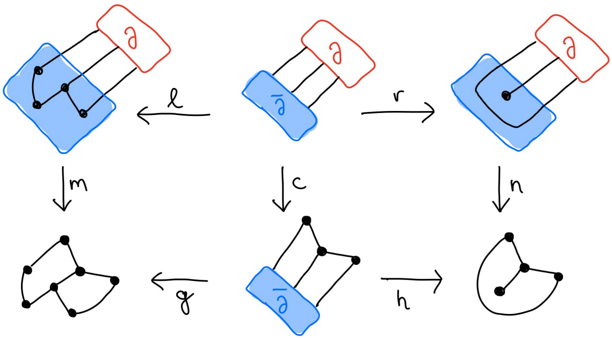
The Same Example of DPO Rewriting

Remember this example?



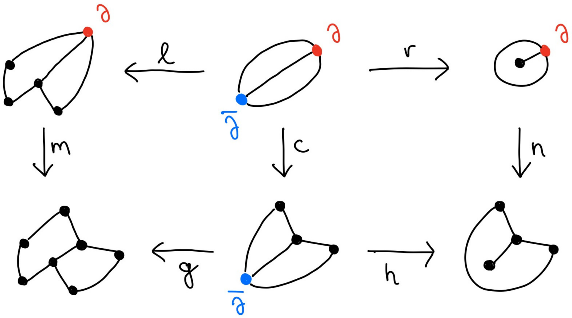
The Same Example of DPO Rewriting

Let's add some boundary regions . . .



The Same Example of DPO Rewriting

. . . and use their representative vertices



Category of Rotation Systems

obj: graphs + cyclic ordering of flags for all vertices

arr: same as graphs + order preservation condition

Example

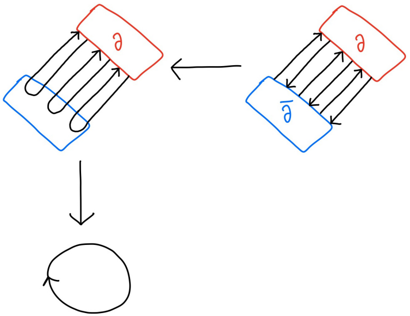
$$
\begin{array}{ccc}
V & \xrightarrow{f_V} & V' \\
{\scriptstyle t^{-1}}\downarrow & \geq & \downarrow{\scriptstyle t'^{-1}} \\
P(E) & \xrightarrow[P(f_E)]{} & P(E')
\end{array}
\qquad\qquad
\begin{array}{ccc}
V & \xrightarrow{f_V} & V' \\
{\scriptstyle t^{-1}}\downarrow & \geq & \downarrow{\scriptstyle t'^{-1}} \\
CList(E) & \xrightarrow[CList(f_E)]{} & CList(E')
\end{array}
$$

Theorem

Pushouts and pushout complements are the same as in the underlying category of graphs.

Let's talk about Loops!

problem: construct a pushout complement of a loop



Let's talk about Loops!

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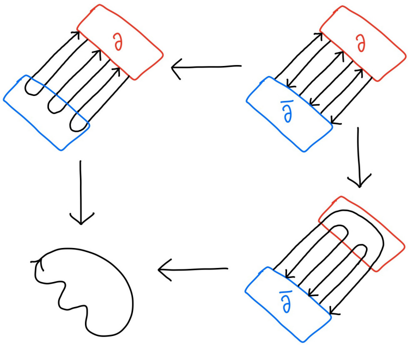

has a plane solution

Let's talk about Loops!

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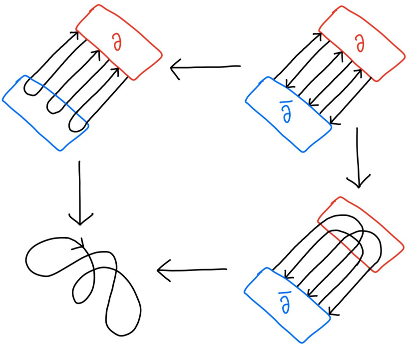


has a plane solution
and a non-plane solution

Summary

- fix inputs and outputs to control topology – boundary vertices!
- restrict your rewrite rules to meaningful cases
- category of graphs with circles extendable to rotation systems

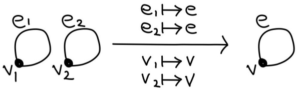
Future Thoughts

- How about surface-embedded loops?
- How about multiple boundary vertices?

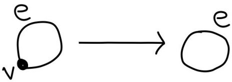
THANK YOU FOR YOUR ATTENTION!

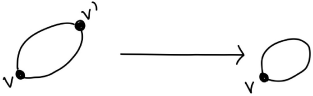
Appendix: Examples

Valid morphisms:



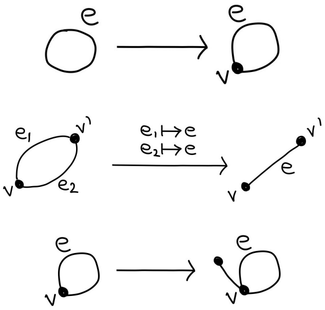
Embeddings:





Appendix: Non-Examples

These aren't morphisms in the category:



Appendix: Definition Graphs with Circles

A morphism $f : G \to G'$ between two graphs with circles consists of two (partial) functions $f_V : V \rightharpoonup V'$ as above, and $f_A : A \to A'$, satisfying the conditions listed below. Note that any such f_A factors as four maps,

$$\begin{array}{cc} f_E : E \to E' & f_{EO} : E \to O' \\ f_{OE} : O \to E' & f_O : O \to O' \end{array}$$

The following conditions must be satisfied:

- $f_A : A \to A'$ is total;
- the component $f_{OE} : O \to E'$ is the empty function;
- the pair (f_V, f_E) forms a flag surjection between the underlying graphs.

If, additionally, the following three conditions are satisfied, we call the morphism an *embedding*:

- $f_V : V \rightharpoonup V'$ is injective;
- the component f_O is injective;
- the pair (f_V, f_E) forms a flag bijection between the underlying graphs.

Appendix: Two regions on a sphere