Enriched Lawvere Theories for Operational Semantics

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Introduction

How do we integrate syntax and semantics?

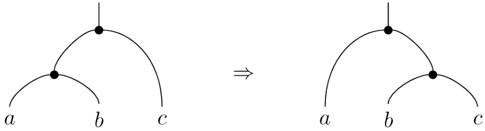
object	type
morphism	term
$*$ 2-morphism	rewrite $*$

Operational semantics

algebraic theories : *denotational* semantics

$$(ab)c = a(bc)$$

enriched theories : *operational* semantics



Lawvere theories

Th(Mon)

type	M		monoid
operations	$m\colon$	$M^2 \to M$	multiplication
	$e\colon$	$1 \to M$	identity

equations

$$\begin{tikzcd}
M^3 \arrow[r] \arrow[d] & M^2 \arrow[d] \\
M^2 \arrow[r] & M
\end{tikzcd}$$

$$\begin{tikzcd}
& M^2 \arrow[dr] & \\
1 \times M \arrow[ur] \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
& M^2 \arrow[dl] & \\
M & & M \times 1 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Enriched theories

$$\mathrm{Th}(\mathrm{PsMon})$$

type	M		pseudomonoid
operations	$\otimes \colon$	$\mathrm{M}^2 \to \mathrm{M}$	multiplication
	$\mathrm{I} \colon$	$1 \to \mathrm{M}$	identity

rewrites

$$\begin{tikzcd}
\mathrm{M}^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \alpha"] & \mathrm{M}^2 \arrow[d] \\
\mathrm{M}^2 \arrow[r] & \mathrm{M}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{M}^2 \arrow[dr] \arrow[d, phantom, "\Downarrow \lambda"] & \\
1 \times \mathrm{M} \arrow[ur] \arrow[rr] & {} & \mathrm{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{M}^2 \arrow[dl] \arrow[d, phantom, "\rho \Downarrow"] & \\
\mathrm{M} & {} & \mathrm{M} \times 1 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

equations pentagon, triangle identities

Enriched categories

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Let V be monoidal. A V-*enriched category* has hom-objects in V; composition and identity are morphisms in V, as are the components of a V-functor and a V-natural transformation:

V-**category**	$\mathrm{C}(a,b)$	$\in \mathrm{V}$
V-**functor**	$F_{ab}\colon \mathrm{C}(a,b) \to \mathrm{D}(F(a),F(b))$	$\in \mathrm{V}$
V-**transformation**	$\varphi_a\colon 1_{\mathrm{V}} \to \mathrm{D}(F(a),G(a))$	$\in \mathrm{V}.$

These form the 2-category VCat.

Our enriching category

Let V be a cartesian closed category:

$$\mathsf{V}(a \times b, c) \cong \mathsf{V}(a, [b, c]).$$

Then $\underline{\mathsf{V}} \in \mathsf{VCat}$.

Let $\mathsf{V} \in \mathsf{CCC}_{fc(1)}$, meaning assume and choose:

$$n_{\mathsf{V}} := \sum_{n} 1_{\mathsf{V}}.$$

Let $\quad \mathsf{N}_{\mathsf{V}} := \{n_{\mathsf{V}} | n \in \mathsf{N}\} \subset_{full} \mathsf{V}$

and $\quad \mathsf{A}_{\mathsf{V}} := \underline{\mathsf{N}}_{\mathsf{V}}^{\mathrm{op}}$ – our "arities".

Enriched products

The V-**product** of $(a_i) \in \mathsf{C}$ is an object $\prod_i a_i \in \mathsf{C}$ equipped with a V-natural isomorphism

$$\mathsf{C}(-, \textstyle\prod_i a_i) \cong \prod_i \mathsf{C}(-, a_i).$$

A V-functor $F \colon \mathsf{C} \to \mathsf{D}$ **preserves** V-products if the "projections" induce a V-natural isomorphism:

$$\mathsf{D}(-, F(\textstyle\prod_i a_i)) \cong \prod_i \mathsf{D}(-, F(a_i)).$$

Let VCat_{fp} be the 2-category of V-categories with finite V-products and V-functors preserving them.

Enriched Lawvere theories

Definition

A V-**theory** is a V-category $\mathrm{T} \in \mathrm{VCat}_{fp}$ whose objects are finite V-products of a distinguished object.

A morphism of V-theories is a V-functor $F\colon \mathrm{T} \to \mathrm{T}' \in \mathrm{VCat}_{fp}$. These and V-natural transformations form the 2-category of V-theories, VLaw.

Enriched models

Definition

A **context** is a V-category $\mathsf{C} \in \mathsf{VCat}_{fp}$.
A **model** of T is a V-functor

$$\mu \colon \mathsf{T} \to \mathsf{C} \in \mathsf{VCat}_{fp}.$$

The category of models is $\mathsf{Mod}(\mathsf{T}, \mathsf{C}) := \mathsf{VCat}_{fp}(\mathsf{T}, \mathsf{C})$.

Example: monoidal categories

Let $V = \mathrm{Cat}$.

$$\mathrm{Th}(\mathrm{PsMon})$$

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& \mathrm{M}^2 \arrow[dl] \arrow[d, phantom, "\rho \Downarrow"] & \\
\mathrm{M} & {} & \mathrm{M} \times 1 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

equations pentagon, triangle identities

Example: cartesian object

Let $V = \mathrm{Cat}$.

Th(Cart)

type		X	cartesian object
operations	$m\colon$	$X^2 \to X$	product
	$e\colon$	$1 \to X$	terminal element
rewrites	$\triangle\colon$	$\mathrm{id}_X \Longrightarrow m \circ \Delta_X$	unit of $m \vdash \Delta_X$
	$\pi\colon$	$\Delta_X \circ m \Longrightarrow \mathrm{id}_{X^2}$	counit of $m \vdash \Delta_X$
	$\top\colon$	$\mathrm{id}_X \Longrightarrow e \circ !_X$	unit of $e \vdash !_X$
	$\epsilon\colon$	$!_X \circ e \Longrightarrow \mathrm{id}_1$	counit of $e \vdash !_X$
equations			triangle identities

Change of base

Let $F\colon \mathsf{V} \to \mathsf{W}$ preserve finite products, and $\mathsf{C} \in \mathsf{VCat}$.

Then F induces a **change of base**:

$$F_*(\mathsf{C})(a,b) := F(\mathsf{C}(a,b)).$$

This gives a 2-functor

$$F_*\colon \mathsf{VCat} \to \mathsf{WCat}.$$

Enrichment provides semantics, so change of base should *preserve* theories to be a *change of semantics*.

Preservation of theories

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Theorem

Let $F\colon \mathsf{V} \to \mathsf{W} \in \mathsf{CCC}_{fc(1)}$.
Then F *is a* **change of semantics:**

F_* *preserves theories. For every* V*-theory* $\tau_{\mathsf{V}}\colon \mathsf{A_V} \to \mathsf{T}$,

$$\tau_{\mathsf{W}} := \mathsf{A_W} \xrightarrow{\sim} F_*(\mathsf{A_V}) \xrightarrow{F_*(\tau_{\mathsf{V}})} F_*(\mathsf{T}) \quad \textit{is a } \mathsf{W}\textit{-theory.}$$

F_* *preserves models. For every model* $\mu\colon \mathsf{T} \to \mathsf{C}$,

$$F_*(\mu)\colon F_*(\mathsf{T}) \to F_*(\mathsf{C}) \quad \textit{is a model of } (F_*(\mathsf{T}), \tau_{\mathsf{W}}).$$

Change of semantics

There is a "spectrum" of semantics:

$$
\begin{tikzcd}
\mathsf{Gph} \arrow[r, bend left, "\mathrm{FC}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "\mathrm{UG}"] \arrow[r, bend left, "\mathrm{FP}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos} \arrow[l, bend left, "\mathrm{UC}"] \arrow[r, bend left, "\mathrm{FS}"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "\mathrm{UP}"]
\end{tikzcd}
$$

FC_*	maps small-step to big-step operational semantics.
FP_*	maps big-step to full-step operational semantics.
FS_*	maps full-step to denotational semantics.

The theory of SKI

Th(SKI)

type	t		
terms	$S:$	$1 \to t$	
	$K:$	$1 \to t$	
	$I:$	$1 \to t$	
	$(-\ -):$	$t^2 \to t$	
rewrites	$\sigma:$	$(((S\ a)\ b)\ c)$	$\Rightarrow ((a\ c)\ (b\ c))$
	$\kappa:$	$((K\ a)\ b)$	$\Rightarrow a$
	$\iota:$	$(I\ a)$	$\Rightarrow a$

A model of Th(SKI)

A Gph-product preserving Gph-functor $\mu \colon \mathrm{Th(SKI)} \to \mathrm{Gph}$ yields a graph $\mu(t)$ of SKI-terms:

$$1 \cong \mu(1) \xrightarrow{\mu(S)} \mu(t) \xleftarrow{\mu((- \ -))} \mu(t^2) \cong \mu(t)^2.$$

The rewrites are transferred by the enrichment of μ:

$$\mu_{1,t} \colon \mathrm{Th(SKI)}(1, t) \to \mathrm{Gph}(1, \mu(t)).$$

The free model of SKI

The syntax and semantics of the SKI combinator calculus are given by the free model

$$\mu_{\mathsf{SKI}}^{\mathsf{Gph}} := \mathsf{Th}(\mathsf{SKI})(1,-)\colon \mathsf{Th}(\mathsf{SKI}) \to \mathsf{Gph}.$$

The graph $\mu_{\mathsf{SKI}}^{\mathsf{Gph}}(t)$ is the *transition system* which represents the **small-step operational semantics** of the SKI-calculus:

$$(\mu(a) \to \mu(b) \in \mu_{SKI}^{\mathsf{Gph}}(t)) \iff (a \Rightarrow b \in \mathsf{Th}(\mathsf{SKI})(1,t)).$$

Change of semantics

$\mathrm{FC}\colon$ Gph $\to$ Cat preserves products, hence gives a change of semantics from *small-step* to *big-step* operational semantics:

$$
\begin{tikzcd}
& (((S\ K)\ (I\ K))\ S) \arrow[dl, "\iota"'] \arrow[dr, "\sigma"] \arrow[ddl, dotted, bend right, "\iota\sigma"] \arrow[ddl, dotted, "\sigma\iota"] \arrow[ddr, dotted, bend right, "\kappa\sigma\iota"] \arrow[ddr, dotted, "\kappa\sigma"] \arrow[ddr, dotted, bend left, "\kappa\iota\sigma"] & \\
(((S\ K)\ K)\ S) \arrow[d, "\sigma"'] & & ((K\ S)\ ((I\ K)\ S)) \arrow[d, "\kappa"] \arrow[dll, "\iota"'] \\
((K\ S)\ (K\ S)) \arrow[rr, "\kappa"'] & & S.
\end{tikzcd}
$$

$\mathrm{FP}\colon$ Cat $\to$ Pos gives *full-step* (Hasse diagram), and
$\mathrm{FS}\colon$ Pos $\to$ Set gives *denotational* semantics, collapsing the connected component to a point.

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Conclusion

Enriched theories give a way to unify the structure and behavior of formal languages.

Enriching in category-like structures reifies operational semantics by incorporating rewrites between terms.

Cartesian functors between enriching categories induce change-of-semantics functors between categories of models.

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