All $(\infty, 1)$-toposes have strict univalent universes

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The theorem

Theorem

Every Grothendieck ∞-topos can be presented by a model category that interprets homotopy type theory with strict univalent universes.

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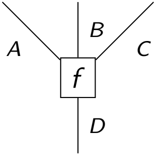
Goals for today:

1. A general idea of these words mean.
2. Why you might care / what it's good for.
3. A bit about the proof.

Outline

1. Type theories for categories
2. Type theories for higher categories
3. $(\infty, 1)$-toposes
4. Sketch of proof
5. Applications

Syntaxes for categories

<table>
<tr><td>Traditional
(arrow-theoretic)</td><td>$f : A \times B \times C \to D$</td></tr>
<tr><td>Graphical calculus
(string diagrams)</td><td>
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<tr><td>Type-theoretic</td><td>$x : A, y : B, z : C \vdash f(x, y, z) : D$</td></tr>
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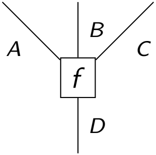
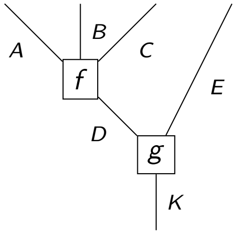
Syntaxes for categories

Traditional (arrow-theoretic)	$f : A \times B \times C \to D$
Graphical calculus (string diagrams)	
Type-theoretic	$((x : A), (y : B), (z : C)) \vdash (f(x, y, z) : D)$

Syntaxes for composition

$$A \times B \times C \times E \xrightarrow{f \times 1_E} D \times E \xrightarrow{g} K$$



$$x : A, y : B, z : C, w : E \vdash g(f(x, y, z), w) : K$$

General principle of alternative syntax

Idea

Any construction of free categories (of a given sort) yields an alternative syntax for reasoning in arbitrary categories (of that sort).

(We reason in the free category, then map it into an arbitrary one.)

Example

String diagrams (of any sort) with a given set of labels, modulo deformation-equivalence (of the appropriate sort), form the free category (of the appropriate sort) generated by the labels.

Example

Terms (in any type theory) built from a given set of base symbols, modulo definitional equality (of the appropriate sort), form the free category (of the appropriate sort) generated by the base symbols.

From type theories to categories

Let X be any “doctrine” (CCCs, LCCCs, toposes, etc.).

$$\begin{tikzcd}
\text{X type theory} \arrow[loop above, "\text{reasoning}"] \arrow[r, "\text{constructs}"] & \text{free X category} \arrow[r, "\text{maps into}"] & \text{arbitrary X category} \\
& \text{(Lawvere theory, prop, etc.)} \arrow[u, dashed] &
\end{tikzcd}$$

A zoo of type theories

Type theory	Category theory
Simply typed λ-calculus	Cartesian closed category
Intuitionistic linear logic	Symmetric monoidal category
Intuitionistic affine logic	Semicartesian monoidal category
Classical linear logic	$*$-autonomous category
Intuitionistic first-order logic	Heyting category
Intuitionistic higher-order logic	Elementary topos
Extensional MLTT	Locally cartesian closed category
Intensional MLTT / HoTT	LCC $(\infty, 1)$-category
HoTT with univalence	$(\infty, 1)$-topos

Dependent type theory

Sometimes additional work is required on the categorical side.

Example

In dependent type theory (DTT), types can depend on variables too:

$$((x : A), (y : B(x)), (z : C(x, y))) \vdash (f(x, y, z) : D(x, y, z))$$

Think of B as a family of types $B(x)$ indexed by "elements" $x : A$.
Categorically, a morphism $B \to A$ with $B(x)$ the "fibers".

But the direct semantics is a category with families (CwF), with

1. A category $\mathbf{C}$ (contexts) with terminal object (empty context)
2. A functor $\mathcal{T} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ (types) — a *separate datum*
3. Context extension $\Gamma \in \mathbf{C}$, $A \in \mathcal{T}(\Gamma) \mapsto \Gamma.A \in \mathbf{C}$

Expect $\mathcal{T}(\Gamma) \approx \mathbf{C}/\Gamma$; but need a coherence theorem to strictify this.
(Also, usually require $\mathbf{C}$ to be LCCC, for Π-types.)

From DTT to LCCCs

$$\begin{tikzcd}
 & & \text{arbitrary LCCC} \arrow[d, "\text{strict slices}"] \\
\text{dependent type theory} \arrow[loop above, "\text{reasoning}"] \arrow[r, "\text{constructs}"] & \text{free CwF} \arrow[r, "\text{maps into}"] & \text{arbitrary CwF}
\end{tikzcd}$$

Outline

Type theories for higher categories, directly

Example

A type 2-theory has

1. Types $A, B, \ldots$
2. Terms $((x : A), (y : B)) \vdash (f(x, y) : C)$
3. "2-Terms" $((x : A), (y : B)) \vdash (\alpha(x, y) : f(x, y) \Rightarrow g(x, y))$

$\rightsquigarrow$ objects, morphisms, and 2-morphisms in a 2-category.

This works, but gets less practical for ∞-categories!

At least for $(\infty, 1)$-categories (all morphisms of dim > 1 invertible), there is another way...

Right homotopies

A standard trick for working with $(\infty, 1)$-categories uses special 1-categories called Quillen model categories.

Idea

A homotopy between $f, g : X \to Y$ is a lift to the path space:

$$\begin{tikzcd} & Y^{[0,1]} \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

sending $x \in X$ to the path $H_x : [0,1] \to Y$, where $H_x(0) = \mathrm{ev}_0(H_x) = f(x)$ and $H_x(1) = \mathrm{ev}_1(H_x) = g(x)$.

Similarly, higher homotopies are detected by higher path spaces. So it suffices to characterize the path spaces categorically.

Weak factorization systems

The path space $Y^{[0,1]}$ is a *factorization of the diagonal*

$$\begin{tikzcd} & Y^{[0,1]} \arrow[dr, two heads, "p"] & \\ Y \arrow[ur, tail, "r"] \arrow[rr, "\Delta"'] & & Y \times Y \end{tikzcd}$$

such that p is a fibration and r is an acyclic cofibration.

It doesn't matter exactly what those words mean, so much as the abstract structure that they form.

Definition (Quillen)

A model category is a complete and cocomplete category equipped with three classes of maps $\mathcal{F}$ (fibrations), $\mathcal{C}$ (cofibrations), and $\mathcal{W}$ (weak equivalences) satisfying some axioms.

$\mathcal{C} \cap \mathcal{W} =$ acyclic cofibrations, $\mathcal{F} \cap \mathcal{W} =$ acyclic fibrations.

Path objects

Definition

A path object in a model category is a factorization of the diagonal

$$\begin{tikzcd} & PY \arrow[dr, two heads, "p"] & \\ Y \arrow[ur, tail, "r"] \arrow[rr, "\Delta"'] & & Y \times Y \end{tikzcd}$$

such that p is a fibration and r is an acyclic cofibration.
A homotopy is a lift to a path object.

Theorem (Quillen, Dwyer–Kan, Joyal, Rezk, Dugger, Lurie, . . .)

Every model category presents an $(\infty, 1)$*-category, and every locally presentable* $(\infty, 1)$*-category is presented by some model category.*

Type theories for higher categories, indirectly

Given a model category $\mathbf{C}$, define a category with families where $\mathcal{T}(\Gamma)$ is a strictification of the subcategory of fibrations in $\mathbf{C}/\Gamma$.

Magical Observation (Awodey–Warren)

A path object in $\mathbf{C}$ corresponds exactly* to an identity type from Martin-Löf's intensional type theory.

* As long as $\mathbf{C}$ is sufficiently well-behaved.

$PY \twoheadrightarrow Y \times Y$	$x : Y, y : Y \vdash \mathsf{Id}(x, y)$
$r : Y \rightarrowtail PY$	$x : Y \vdash \mathsf{refl}_x : \mathsf{Id}(x, x)$
r is an acyclic cofibration	Id-elimination (indiscernability of identicals)

Type theory inspired by this is called homotopy type theory (HoTT).

Model categories for almost all of type theory

Theorem (Awodey–Warren, van den Berg–Garner, Cisinski, Gepner–Kock, Lumsdaine–Shulman, etc.)

Any locally presentable, locally cartesian closed $(\infty, 1)$-category can be presented by a model category that interprets homotopy type theory with Σ, Π, Id, HITs, etc.

$$
\begin{tikzcd}
 & & \text{arbitrary l.p. } (\infty,1)\text{-LCCC} \arrow[d, "\text{presented by}"] \\
 & & \text{well-behaved model category} \arrow[d, "\text{strict slices of fibrations}"] \\
\text{homotopy type theory} \arrow[loop above, "\text{reasoning}"] \arrow[r, "\text{constructs}"] & \text{free CwF} + \cdots \arrow[r, "\text{maps into}"] & \text{arbitrary CwF} + \cdots
\end{tikzcd}
$$

Outline

Why toposes?

Definition

A (Grothendieck 1-)topos consists of the objects obtained by gluing together those in some category of specified basic ones.

Objects of topos	Basic objects
(Generalized) manifolds	open subsets $U \subseteq \mathbb{R}^n$
Sequential spaces	convergent sequences $\{0, 1, 2, \ldots, \infty\}$
Time-varying sets	elements that exist starting at a time t
Graphs	vertices and edges
Decorated graphs	"atomic" decorations
G-sets	orbits G/H
Quantum systems	consistent classical observations
Nominal sets	co-(finite sets)

Type theory for toposes

A topos is distinguished among LCC 1-categories by having a subobject classifier: a monomorphism $\top : 1 \to \Omega$ of which every monomorphism is a pullback, uniquely.

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] \\
B \arrow[r, dashed, "\exists!"'] & \Omega
\end{tikzcd}$$

In the internal type theory, Ω is a type whose elements are the propositions — making it into “higher-order logic”.

Why $(\infty, 1)$-toposes?

Definition (Toen-Vezossi, Rezk, Lurie, ...)

A (Grothendieck) $(\infty, 1)$-topos consists of objects obtained by ∞-gluing together those in some $(\infty, 1)$-category of basic ones.

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1. Need to keep track of isomorphisms (gauge transformations, internal categories, pseudofunctors, homotopies, ...)
2. Sometimes the basic objects live in a higher category.
   - 2-actions of a 2-group are glued together from 2-orbits.
   - (Generalized) orbifolds are glued together from orbit groupoids.
   - Parametrized spectra are glued together from co-(finite spaces).

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   - Parametrized spectra are glued together from co-(finite spaces).
3. 1-categorical gluing is badly behaved for non-monos.

$$\begin{tikzcd} 1+1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & 1 \end{tikzcd} \quad \text{vs.} \quad \begin{tikzcd} 1+1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^1 \end{tikzcd}$$

∞-gluing remembers "gluing shape", enabling cohomology.

Type theory for $(\infty, 1)$-toposes

An $(\infty, 1)$-topos is distinguished among LCC $(\infty, 1)$-categories by having an object classifier: a small morphism $\widetilde{U} \to U$ of which every small morphism is a pullback, uniquely up to homotopy.

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U} \arrow[d] \\ B \arrow[r, dashed, "\exists!"'] & U \end{tikzcd}$$

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Actually we have one object classifier for every suitable notion of "small" (parametrized by certain regular cardinals). This is a *size* restriction, not a *dimension* restriction.

Univalent universes

In the internal dependent type theory, an object classifier U is a universe: a type whose elements are (some) other types.

Since homotopies of classifying maps correspond to equivalences of objects, U must satisfy Voevodsky's univalence axiom: for types $A : U$ and $B : U$, the canonical map

$$\mathsf{Id}(A, B) \to \mathsf{Equiv}(A, B)$$

is an equivalence.

The coherence problem

- An object classifier in an $(\infty, 1)$-topos classifies things up to homotopy pullback.
- But type theory is interpreted in a model category using strict 1-categorical pullback.

Question

Can we present an $(\infty, 1)$-topos by a model category containing strict univalent universes: small fibrations $\widetilde{U} \twoheadrightarrow U$ of which every small fibration is a *strict* pullback, uniquely up to homotopy?

- Voevodsky, 2009ish: Yes for the "fundamental" $(\infty, 1)$-topos $\infty\mathbf{Gpd}$, using the model category of simplicial sets.
- Partial additional results since then (e.g. inverse diagrams).
- General case was open until now.

From univalent universes to $(\infty, 1)$-toposes

$$
\begin{tikzcd}
 & & \text{arbitrary} \\ & & (\infty,1)\text{-topos} \arrow[d, dashed, "\text{presented by}"] \\
 & & \text{model category with universes} \arrow[d, "\text{strict slices of fibrations}"] \\
\text{homotopy type theory w/ univalence} \arrow[loop above, "\text{reasoning}"] \arrow[r, "\text{constructs}"] & \text{free } \mathrm{CwF}+\cdots \arrow[r, "\text{maps into}"] & \text{arbitrary } \mathrm{CwF}+\cdots
\end{tikzcd}
$$

The theorem, again

Theorem

Every Grothendieck ∞-topos can be presented by a model category that interprets homotopy type theory with strict univalent universes.

Caveats:

- The bookkeeping in the free-CwF hasn't all been written down.
- The universes aren't known to be closed under HITs yet.

Outline

The model

Any Grothendieck $(\infty, 1)$-topos can be presented as a left exact left Bousfield localization $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ of the injective model structure on simplicial presheaves over some small simplicial category $\mathscr{C}$.

- objects: simplicially enriched functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{sSet}$.
- morphisms: strict enriched natural transformations.
- cofibrations: pointwise monomorphisms.
- weak equivalences: generated by pointwise weak homotopy equivalences and S.

This is well-behaved (a "right proper Cisinski model category"), so it interprets all of homotopy type theory except for universes.

What are the injective fibrations?

The injective fibrations are, by definition, the maps having the right lifting property with respect to all pointwise acyclic cofibrations. But this is unhelpful for constructing a universe in general.

Lemma

A pointwise fibration $f : X \twoheadrightarrow Y$ in $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ has a relative pseudomorphism classifier $\mathcal{R}f \to Y$ and a natural bijection between

1. *(Strict) natural transformations $A \to \mathcal{R}f$.*
2. *Homotopy coherent transformations $A \rightsquigarrow X$ such that the composite $A \rightsquigarrow X \to Y$ is strict.*

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Lemma

$f : X \to Y$ in $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ is an injective fibration if and only if

1. *it is a pointwise fibration, and*
2. *the canonical map $X \to \mathcal{R}f$ has a retraction over Y.*

Presheaf universes

Define a semi-algebraic injective fibration to be a pointwise fibration equipped with a retraction of $X \to \mathcal{R}f$.

Lemma

In $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$, a universe can be "defined" by

$$U(c) = \Big\{ \textit{small semi-algebraic injective fibrations over } \mathscr{C}(-, c) \Big\}.$$

- Choose an inaccessible cardinal to define "small"
- Need to choose iso representatives, etc., to strictify
- Semi-algebraicity ensures fibrations can be glued together to make a universal one over U.

Sheaf universes

Given a left exact localization $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$:

1. Using a technical result of Anel–Biedermann–Finster–Joyal (2019, forthcoming), we can ensure that left exactness of S-localization is pullback-stable.
2. Then for any $f : X \twoheadrightarrow Y$ we can construct *in the internal type theory of* $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ a fibration $\mathsf{isLocal}_S(f) \twoheadrightarrow Y$.
3. Define a semi-algebraic local fibration to be a semi-algebraic injective fibration equipped with a section of $\mathsf{isLocal}_S(f)$.
4. Now use the same approach.

Outline

Application #1: internal languages

Type-theoretic reasoning can prove things about arbitrary Grothendieck $(\infty, 1)$-toposes.

Example

- Hou-Finster-Licata-Lumsdaine proved the Blakers-Massey theorem in type theory.
- Rezk and Anel-Biedermann-Finster-Joyal translated this by hand to the first $(\infty, 1)$-topos-theoretic proof, and generalized it to modalities and Goodwillie calculus.
- Now, the translation is automatic.

You don't have to read *Higher Topos Theory* to use $(\infty, 1)$-categories.

Application #2: synthetic homotopy theory

Even in classical homotopy theory, type-theoretic proofs are new!

Example

- The circle S^1 is "inductively generated" by a point $b : S^1$ and a loop $\ell : \mathsf{Id}(b, b)$.
- Thus we can reason about it "by induction", with a "base case" for b and a "varying case" for ℓ.
- For instance, we prove $\Omega S^1 :\equiv \mathsf{Id}(b, b) \simeq \mathbb{Z}$ by simple inductive and recursive arguments, and similarly for higher homotopy groups of spheres, etc.

You don't have to learn model category theory
to use abstract homotopy theory.

Application #3: internalization for free

Homotopy type theory is powerful enough
to serve as a foundation for all of mathematics.

Example

- The "0-truncated" types behave just like (structural) *sets*.
- Can build set-level math out of them (constructively).
- In an $(\infty, 1)$-topos, internalizes in the corresponding 1-topos.

All of your (constructive) theorems
are automatically true in all $(\infty, 1)$-toposes.

Application #4: the principle of equivalence

We can make definitions that enforce any desired invariance.

Example

- When categories are defined in set theory, we could distinguish isomorphic objects; we just discipline ourselves not to.
- In HoTT, we require $\mathsf{Id}(x, y) \simeq \mathsf{Iso}(x, y)$, making isomorphic objects *formally indistinguishable*.
- In an $(\infty, 1)$-topos, such categories are automatically *stacks*.
- Similarly, equivalent categories are formally indistinguishable, and so on.

The categorical principle of equivalence
belongs to the foundations of mathematics.

Application #5: computation and formalization

Type theory is also a programming language.

Example

- Theorems in type theory can be formally verified by a computer.
- Constructive proofs can be executed as programs.*
- Conversely, type theory can verify correctness of programs.

Mathematics and computation
are two sides of the same coin.

* Still open to make this true compatibly with its $(\infty, 1)$-topos semantics.

An advertisement

International HoTT Conference 2019
Carnegie Mellon University, Pittsburgh
August 12–17

HoTT Summer School 2019
Carnegie Mellon University, Pittsburgh
August 7–10

- Cubical methods (Anders Mortberg)
- Formalization in Agda (Guillaume Brunerie)
- Formalization in Coq (Kristina Sojakova)
- Higher topos theory (Mathieu Anel)
- Semantics of type theory (Jonas Frey)
- Synthetic homotopy theory (Egbert Rijke)

Funding still available for US students

`https://hott.github.io/HoTT-2019/`