Equivalence between Orthocomplemented Quantales and Complete Orthomodular Lattices.

Kohei Kishida[1], Soroush Rafiee Rad[2], **Joshua Sack**[3], Shengyang Zhong[4]

[1] Dalhousie University
[2] University of Bayreuth
[3] California State University Long Beach
[4] Peking University

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Context

Hilbert spaces are popular for reasoning about quantum theory, but in many ways extraneous
(quantum states are one-dimensional subspaces, abstracting away individual vectors)

Different simpler quantum structures highlight different aspects of quantum reasoning

- Complete orthomodular lattice: ortholattice of testable properties
  gives a *static* perspective
- Orthomodular dynamic algebra: quantale of quantum actions enriched with an orthogonality operator
  gives *dynamic* perspective

A categorical equivalence between these structures clarifies how these perspectives are related.

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Complete orthomodular lattice

A complete orthomodular lattice

A structure $(L, \leq, -^{\perp})$ such that

- $(L, \leq)$ is a complete lattice (has arbitrary joins)
- $\perp$ is a lattice orthocomplement:
  - $\perp$ is a complement: $a \wedge a^{\perp} = O$ and $a \vee a^{\perp} = I$.
  - $\perp$ is involutive: $(a^{\perp})^{\perp} = a$
  - $\perp$ is order reversing: $a \leq b$ implies $b^{\perp} \leq a^{\perp}$.
- orthomodular (weakened distributivity) law holds: $q \leq p$ implies $p \wedge (p^{\perp} \vee q) = q$.

Example (Hilbert lattice)

closed subspaces of a Hilbert space.

The points of lattice are quantum testable properties.

Temporal structure of a complete orthomodular lattice

What about dynamics?

Sasaki hook and projection

Given testable properties p, q

- $f^{p}(q) \stackrel{\text{def}}{=} p^{\perp} \vee (p \wedge q)$ (hook)
  The precondition of a projection onto p resulting in q
- $f_{p}(q) \stackrel{\text{def}}{=} p \wedge (p^{\perp} \vee q)$ (projection)
  The result of projecting q onto p

Quantales: giving dynamics higher status

Definition

A quantale ("quantum locale") is a tuple $(Q, \sqsubseteq, \cdot)$, such that

- $(Q, \sqsubseteq)$ is sup-lattice (complete lattice)
- $(Q, \cdot)$ is a monoid satisfying the following distributive laws

$$a \cdot \bigsqcup S = \bigsqcup \{a \cdot b \mid b \in S\} \qquad \bigsqcup S \cdot a = \bigsqcup \{b \cdot a \mid b \in S\}$$

Perspective

Quantales relate to operator algebras: the points of a quantale can be thought of as operators on a Hilbert space.

Temporal meaning from monoidal composition

$a \cdot b$ read "a after b" (quantum observables are not commutative)

An application: dynamics acting on states

- Q - a *quantale* (a set with certain algebraic structure)
  Elements of Q: nondeterministic "actions" or "observations"
- M - *module* over Q
  Elements of M: nondeterministic "states" or "processes"
- $\star : Q \times M \to M$
  "action" of quantale Q on module M

Abramsky & Vickers. Quantales, observational logic and process semantics. MSCS 1993.

Quantum dynamic algebra

Baltag and Smets introduce a Quantum dynamic algebra: A quantale augmented with an orthogonality operator $\sim$

Baltag and Smets. Complete Axiomatizations for Quantum Actions. International Journal of Theoretical Physics, 2005.

We modify their definition to ensure categorical equivalences with complete orthomodular lattices.

Generalized dynamic algebra

A quantum dynamic algebra is a type of generalized dynamic algebra.

Definition (Generalized dynamic algebra)

A Genaralized dynamic algebra is a tuple $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, such that

- Q is a set of quantum actions (typically infinite)
- $\bigsqcup : \mathcal{P}(Q) \to Q$ (for choice),
- $\cdot : Q \times Q \to Q$ (for sequential observation or action)
- $\sim : Q \to Q$ (similar to an orthocomplement)

Generalized dynamic algebra concepts

Given a generalized dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$

$$(x \sqsubseteq y) \text{ iff } (x \sqcup y = y)$$

Potential lattice of "projectors" inside $\mathfrak{Q}$:

$$\begin{array}{llll} \mathcal{P}_{\mathfrak{Q}} & \stackrel{\text{def}}{=} & \{\sim x \mid x \in Q\} & \\ \bigvee X & \stackrel{\text{def}}{=} & \sim\sim \bigsqcup X & \text{for all } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\ \bigwedge X & \stackrel{\text{def}}{=} & \sim \bigsqcup \sim X & \text{for all } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\ A \preceq B & \Leftrightarrow & A \wedge B = A & \text{for all } A, B \in \mathcal{P}_{\mathfrak{Q}} \end{array}$$

Observed action and equivalence:

$$\begin{array}{lll} \ulcorner x \urcorner & \stackrel{\text{def}}{=} & \lambda y. \sim\sim(x \cdot y) \\ x \equiv y & \leftrightarrow & \ulcorner x \urcorner(p) = \ulcorner y \urcorner(p) \text{for all } p \in \mathcal{P}_{\mathfrak{Q}} \end{array}$$

Potential "atoms" of $\mathfrak{Q}$ built from $\mathcal{P}_{\mathfrak{Q}}$.

- $\mathcal{T}_{\mathfrak{Q}}$ is the smallest superset of $\mathcal{P}_{\mathfrak{Q}}$ closed under composition

Concrete example: a Hilbert space realization

$\mathcal{H}$ - Hilbert space
$\mathcal{P}_{\mathcal{H}}$ - the set of singleton sets of projectors P_A onto closed linear subspaces A.

Example

$\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, where

- $Q = \mathcal{P}(\mathcal{T}_{\mathcal{H}})$ where $\mathcal{T}_{\mathcal{H}}$ is the smallest superset of $\mathcal{P}_{\mathcal{H}}$ closed under composition. (An element of Q is a set)
- $\bigsqcup$ is just the union operation
(union of sets of functions, not unions of functions)
- $\cdot$ is defined by $A \cdot B = \{a \circ b \mid a \in A, b \in B\}$
(function composition of each pair of functions)
- $\sim$ is defined by $\sim A = \{P_{B^\perp}\}$ where $B = \mathrm{Im}(\bigcup_{a \in A} a)$.

Quantale inside our Hilbert space realization

The Hilbert space realization satisfies:

- $(Q, \sqsubseteq, \cdot\,)$ is a quantale:
  - $(Q, \sqsubseteq)$ is a complete lattice
  - $(Q, \cdot)$ is a monoid, where

$$a \cdot \bigsqcup S = \bigsqcup \{a \cdot b \mid b \in S\}$$

$$\bigsqcup S \cdot a = \bigsqcup \{b \cdot a \mid b \in S\}$$

- $\mathcal{P}_{\mathfrak{Q}} = \mathcal{P}_{\mathcal{H}}$
- $\mathcal{T}_{\mathfrak{Q}} = \mathcal{T}_{\mathcal{H}}$.
- $(\mathcal{P}_{\mathfrak{Q}}, \preceq, \sim)$ is a Hilbert lattice, and hence a complete orthomodular lattice.

The orthogonality operator $\sim$ is not a lattice orthocompletent for the quantale lattice, but for the induced lattice $(\mathcal{P}_{\mathfrak{Q}}, \preceq, \sim)$.

Orthomodular dynamic algebra (ODA)

A *generalized dynamic algebra* $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ is an orthomodular dynamic algebra if for all $p, q \in \mathcal{P}_{\mathfrak{Q}}$, $x, y \in \mathcal{T}_{\mathfrak{Q}}$, and $X, Y \subseteq \mathcal{T}_{\mathfrak{Q}}$:

1. $(Q, \sqsubseteq, \cdot)$ is a quantale and $\bigsqcup$ is its arbitrary join.
2. $(\mathcal{P}_{\mathfrak{Q}}, \preceq, \sim)$ is a complete orthomodular lattice
3. Q is generated from $\mathcal{P}_{\mathfrak{Q}}$ by $\cdot$ and $\bigsqcup$ (minimality)
   (ensures Q does not have too many elements.)
4. $x = y$ iff $x \equiv y$ (completeness)
   (ensures distinct behavior of distinct elements.)
5. $\bigsqcup X = \bigsqcup Y$ iff $X = Y$ (atomicity)
6. $\ulcorner p \urcorner(q) = f_p(q)$ (i.e. $\sim\sim(p \cdot q) = p \wedge (\sim p \vee q)$) (Sasaki projection)
   (connects monoidal to orthomodular lattice dynamics)
7. $\ulcorner x \urcorner(y) = \ulcorner x \urcorner(\sim\sim y)$ (composition)
   ($\ulcorner x \urcorner$ acting on Q is fully determined by its action on $\mathcal{P}_{\mathfrak{Q}}$)

Category of Complete Orthomodular Lattices

Let $\mathbb{L}$ be the category with

Object: Complete orthomodular lattices

Morphisms: Ortholattice isomorphisms:

Bijections k preserving order and orthocomplementation:

- $p \leq_1 q$ if and only if $k(p) \leq_2 k(q)$
- $k(p^{\perp_1}) = (k(p))^{\perp_2}$.

Category of Orthomodular Dynamic Algebras

Let $\mathbb{Q}$ be the category with

Objects: Orthomodular dynamic algebras

Morphisms: Functions $\theta : \mathfrak{Q} \to \mathfrak{R}$ satisfying:

- θ preserves $\cdot$, $\bigsqcup$.
- The restriction of θ to $\mathcal{P}_{\mathfrak{Q}}$ (the image of Q under $\sim$) is on ortholattice isomorphism (hence maps $\mathcal{P}_{\mathfrak{Q}}$ to $\mathcal{P}_{\mathfrak{R}}$)

Categorical equivalence

Definition (Categorical Equivalence)

An equivalence between categories $\mathbb{L}$ and $\mathbb{Q}$ is a pair of covariant functors

$$(\mathbf{F} : \mathbb{L} \to \mathbb{Q}, \mathbf{U} : \mathbb{Q} \to \mathbb{L})$$

such that

1. there is a natural isomorphism $\eta : 1_{\mathbb{Q}} \to \mathbf{F} \circ \mathbf{U}$
2. there is a natural isomorphism $\tau : 1_{\mathbb{L}} \to \mathbf{U} \circ \mathbf{F}$

Translation $\mathbf{F} : \mathbb{L} \to \mathbb{Q}$ from lattice to algebra

on objects

Let $\mathcal{L} = (L, \leq, -^{\perp})$ be a complete orthomodular lattice. Define

$$
\begin{aligned}
\mathcal{F}_T &= \text{smallest set containing } \{f_p \mid p \in L\}, \\
& \text{closed under composition} \\
\mathcal{Q} &= \mathcal{P}(\mathcal{F}_T) \\
A \cdot B &= \{f \circ g \mid f \in A, g \in B\} \\
\sim A &= f_{\bigvee\{a(I) \mid a \in A\}}, \quad (\text{where } I = \bigwedge \emptyset \text{ is the top element})
\end{aligned}
$$

Then $\mathbf{F}(\mathcal{L}) = (\mathcal{Q}, \cdot, \sim)$

on morphisms

If $k : \mathcal{L}_1 \to \mathcal{L}_2$ is a morphism (ortholattice isomorphism), then $F(k) : A \to \{k \circ a \circ k^{-1} \mid a \in A\}$ conjugates every element of input A by k.

A useful property: preservation of projectors

If $p \in L_1$, then $k \circ f_p \circ k^{-1} = f_{k(p)}$.

Proof.

For $b \in L_2$,

$$
\begin{aligned}
\psi_k(f_p)(b) &= k \circ f_p \circ k^{-1}(b) \\
&= k(p \wedge (p^\perp \vee k^{-1}(b))) \\
&= k(p) \wedge ((k(p))^\perp \vee b) \\
&= f_{k(p)}(b)
\end{aligned}
$$

□

Translation $\mathbf{U} : \mathbb{Q} \to \mathbb{L}$ from algebra to lattice

on objects

$\mathbf{U}$ maps an ODA to the orthomodular lattice it induces:
$\mathbf{U}(\mathfrak{Q}) = (\mathcal{P}_{\mathfrak{Q}}, \preceq, \sim)$.

on morphisms

$\mathbf{U}$ maps each morphism to its restriction to $\mathcal{P}_{\mathfrak{Q}}$: if $\zeta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$, then $\mathbf{U}(\zeta) = \zeta|_{\mathcal{P}_{\mathfrak{Q}}}$.

The functors $\mathbf{F} \circ \mathbf{U}$ and $\mathbf{U} \circ \mathbf{F}$

The elements of $(\mathbf{F} \circ \mathbf{U})(\mathfrak{Q})$ are

$$\{\{f_{a_1} \circ \cdots \circ f_{a_n} \mid a_1 \cdot \cdots \cdot a_n \in X, n \in \mathbb{N}\} \mid X \subseteq \mathcal{T}_{\mathfrak{Q}}\}$$

If $\zeta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$ is a $\mathbb{Q}$-morphism, then

$$\begin{aligned} &(\mathbf{F} \circ \mathbf{U})(\zeta)(\{f_{a_1} \circ \cdots \circ f_{a_n} \mid a_1 \cdot \cdots \cdot a_n \in X, n \in \mathbb{N}\}) \\ =&\{f_{\zeta(a_1)} \circ \cdots \circ f_{\zeta(a_n)} \mid a_1 \cdot \cdots \cdot a_n \in X, n \in \mathbb{N}\}. \end{aligned}$$

The elements of $(\mathbf{U} \circ \mathbf{F})(\mathfrak{L})$

$$\{\{f_p\} \mid p \in L\}$$

If $k : \mathfrak{L}_1 \to \mathfrak{L}_2$ is a $\mathbb{L}$-morphism, then

$$(\mathbf{F} \circ \mathbf{U})(k)(\{f_p\}) = \{f_{k(p)}\}$$

The natural isomorphisms

$\eta : 1_{\mathbb{Q}} \to \mathbf{F} \circ \mathbf{U}$

Let $\mathfrak{Q}$ be an ODA. Then

$$\eta_{\mathfrak{Q}} : (\bigsqcup_{i \in I} a_{i,1} \cdot \cdots \cdot a_{i,n_i})) \mapsto \{f_{a_{i,1}} \circ \cdots \circ f_{a_{i,n_i}}\}_{i \in I}.$$

$\tau : 1_{\mathbb{L}_b} \to \mathbf{U} \circ \mathbf{F}$

Let $\mathfrak{L}$ be a lattice in $\mathbb{L}$, then

$$\tau_{\mathfrak{L}} : a \mapsto \{f_a\}$$

Conclusion and future work

- Connect quantales to quantum structures: Showed what conditions can be placed on a complemented quantale (orthomodular dynamic algebra) to be categorically equivalent to a complete orthomodular lattice.
- Future work: is this the right definition of an ODA?
  - Can weaker morphisms be used?
  - Rather then sets of functions, consider relations instead
- Future work: involve unitary operations
- Future work: establish a clearer connection to operator algebras
- Future work: develop modules for ODA's to act upon
- Future work: develop a logic on ODA's and compare it to logics on lattices they are equivalent to.

THANK YOU!

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