Transformation Structures for 2-Group Actions

(Joint work with Roger Picken)

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This talk is based on two articles by J.M. and Roger Picken:

- "Transformation double categories associated to 2-group actions" (Th. App. Cat. or arXiv:1401.0149)
- "2-Group Actions and Moduli Spaces of Higher Gauge Theory" (arXiv:1904.10865)

- Transformation Groupoids for Group Actions
- 2-Groups and Crossed Modules
- Actions of 2-Groups
- Transformation Double Groupoid
- Application to Higher Gauge Theory

Transformation Groupoids

Idea: *Categorification of symmetry*. To describe symmetry of categories, generalize group actions.
Global symmetry involves group actions:

Definition

A group action ϕ on a set X is a functor $F : G \to \mathbf{Sets}$ where the unique object of G is sent to X. Equivalently, it is a function $\hat{F} : G \times X \to X$ which commutes with the multiplication (composition) of G:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{(1_G, \hat{F})} & G \times X \\ {\scriptstyle (m, 1_X)}\big\downarrow & & \big\downarrow{\scriptstyle \hat{F}} \\ G \times X & \xrightarrow[\hat{F}]{} & X \end{array}$$

Any group action gives a groupoid:

Definition

The **transformation groupoid** of an action of a group G on a set X is the groupoid $X /\!\!/ G$ with:

- **Objects**: X (that is, all $x \in X$
- **Morphisms**: $G \times X$ (that is, pairs $(g, x) : x \to gx$)
- **Composition**: $(g', gx) \circ (g, x) = (g'g, x)$

Note: The relies on the fact that G is a group object in **Sets**. We use this to categorify the "transformation groupoid" construction.

2-Groups and Crossed Modules

Idea: *Categorification of symmetry*. To describe symmetry of categories, generalize group actions.

Definition

A **2-group** $\mathcal{G}$ is a 2-category with one object, and all morphisms and 2-morphisms invertible.

A **categorical group** is a group object in **Gpd**: a category $\mathcal{G}$ with $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an inverse map satisfying the usual group axioms. In particular, a monoidal category $(G, \otimes)$ where every object and morphism is invertible with respect to $\otimes$.

I will use "2-group" for both except where the distinction is critical.

2-groups are classified by crossed modules:

Definition

A crossed module consists of $(G, H, \triangleright, \partial)$, where:

- G, $H \in \mathbf{Grp}$
- $G \triangleright H$ an action of G on H by automorphisms
- $\partial : H \to G$ a homomorphism

satisfying

- $\partial(g \triangleright h) = g\partial(h)g^{-1}$
- $\partial(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}$

A homomorphism of crossed modules is a pair of homomorphisms $G \to G'$ and $H \to H'$ which is compatible with $\triangleright$.

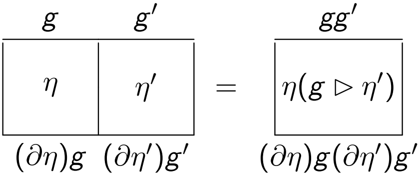
Theorem

The category of crossed modules and homomorphisms is equivalent to the category of 2-groups under with a correspondence determined by $\mathbf{G}(G, H, \triangleright, \partial)$ *with:*

- **Objects**: *elements of G*
- **Morphisms**: *elements of the semidirect product* $G \times H$, *with* $(g, h) : g \to (\partial h)g$ *and*

$$(\partial(\eta)g, \zeta) \circ (g, \eta) = (g, \zeta\eta).$$

- **Monoidal Structure**:



Examples of 2-Groups

Examples of 2-Groups

- Any group G has a corresponding shifted 2-group given by $(1, G, id, 1)$ and the adjoint 2-group $(G, G, Ad, 1)$
- If G is a group and M a left G-module, then the crossed module $(G, M, \triangleright, 1)$ defines a 2-group
- If $i : N \to K$ is the inclusion of a normal subgroup, the crossed module $(K, N, \cdot, i)$ defines a 2-group
- For any category $\mathbf{C}$, the 2-group $Aut(\mathbf{C})$ has:
  - Objects: invertible functors $F : \mathbf{C} \to \mathbf{C}$
  - Morphisms: invertible natural transformations $n : F \to F'$
- For any monoidal category $(\mathbf{C}, \otimes)$, the Picard groupoid of invertible objects and morphisms is a 2-group

Definition

A (strict) action of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$ is a functor $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ (strictly) satisfying the action diagram in $\mathbf{Cat}$:

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id_{\mathbf{C}}} & \mathcal{G} \times \mathbf{C} \\ {\scriptstyle Id_{\mathcal{G}} \times \hat{\Phi}} \downarrow & & \downarrow {\scriptstyle \hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C} \end{array}$$

Actions of 2-groups make sense in any 2-category, but only take this special form (by *currying*) in the $\mathbf{Cat}$:

Lemma

For any $\mathbf{C} \in \mathbf{Cat}$*, a strict monoidal functor* $\Phi : \mathcal{G} \to End(\mathbf{C})$ *is equivalent to a strict action* $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$*.*

Three Group Actions in a 2-Group Action

If $\mathcal{G}$ is a 2-group classified by the crossed module $(G, H, \rhd, \partial)$, and $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ is a strict action, by abuse of notation we denote by $\blacktriangleright$ three interconnected group actions. Two actions of G on objects and morphisms of $\mathbf{C}$:

- Given $\gamma \in Ob(\mathcal{G}) = G$ and $x \in Ob(\mathbf{C})$, let

$$\gamma \blacktriangleright x = \Phi_\gamma(x) = \hat{\Phi}(\gamma, x)$$

- Given $\gamma \in Ob(\mathcal{G}) = G$ and $f \in Mor(\mathbf{C})$, let

$$\gamma \blacktriangleright f = \Phi_\gamma(f) = \hat{\Phi}((\gamma, 1_H), f)$$

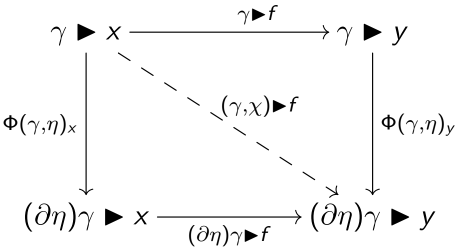
(Which are compatible with the structure maps of $\mathbf{C}$: source, target, composition, etc.)

One action of $G \ltimes H$ on morphisms of $\mathbf{C}$:

- Given $(\gamma, \chi) \in Mor(\mathcal{G}) = G \ltimes H$ and $(f : x \to y) \in Mor(\mathbf{C})$, let

$$(\gamma, \chi) \blacktriangleright f \quad = \quad \hat{\Phi}((\gamma, \chi), f)$$

be the diagonal whose existence is guaranteed by the naturality of the square associated to $f : x \to y$ in $\mathbf{C}$:



(Note $\Phi_{(\gamma,\chi)}$ typically assigns a nonidentity morphism to x, so there is no action of $G \ltimes H$ on objects of $\mathbf{C}$)

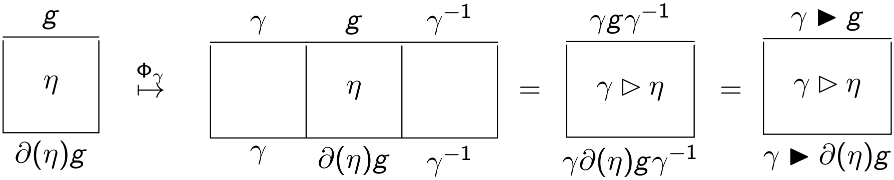
Example: Adjoint Action of $\mathcal{G}$

Definition (Part 1)

If $\mathcal{G} \sim (G, H, \triangleright, \partial)$, then the *adjoint action* of $\mathcal{G}$ (seen as a 2-group) on $\mathcal{G}$ (seen as a categorical group) is given by the 2-functor:

- $\gamma \in Ob(\mathcal{G})$ gives an endofunctor $\Phi_\gamma : \mathcal{G} \to \mathcal{G}$ which itself is defined by:
  - $\Phi_\gamma(g) = \gamma g \gamma^{-1}$
  - $\Phi_\gamma(g, \eta) = (\gamma g \gamma^{-1}, \gamma \triangleright \eta)$

Draw this as:



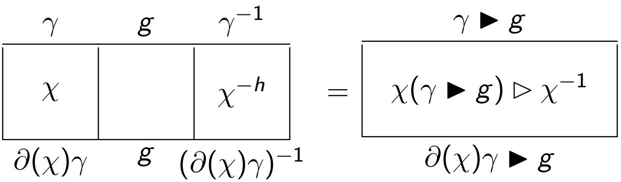
Definition (Part 2)

- $(\gamma, \chi) \in Mor(\mathcal{G})$ gives a natural transformation $\Phi_{(\gamma,\chi)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\chi)\gamma}$, defined by:

$$\Phi_{(\gamma,\chi)}(g) = (\gamma g \gamma^{-1}, \chi(\gamma g \gamma^{-1}) \triangleright \chi^{-1}))$$

Draw this as:

$\Phi_{(\gamma,\chi)}(g) =$



Transformation Double Categories

Definition

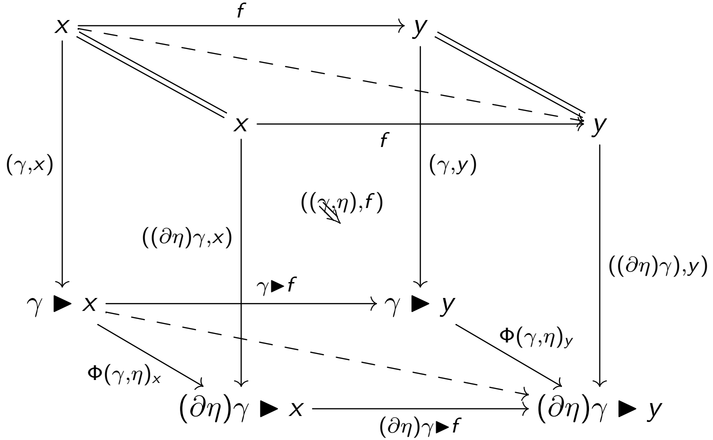
Given a 2-group $\mathcal{G}$, a category $\mathbf{C}$, and an action of $\mathcal{G}$ on $\mathbf{C}$, the transformation 2-groupoid $\mathbf{C}/\!\!/\mathcal{G}$ is the groupoid in **Cat** with:

- $Ob(\mathbf{C}/\!\!/\mathcal{G}) = \mathbf{C}$.
- $Mor(\mathbf{C}/\!\!/\mathcal{G}) = \mathcal{G} \times \mathbf{C}$, with
  - **Source** functor $s = \pi_{\mathbf{C}} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - **Target** functor $t = \hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - **Composition**: (given by the action condition)

This is really a *double category* - the morphisms of $Mor(\mathbf{C}/\!\!/\mathcal{G})$ are squares:

$$
\begin{array}{ccc}
x & \xrightarrow{\quad f \quad} & y \\
{\scriptstyle (\gamma,x)}\big\downarrow & {\scriptstyle ((\gamma,\eta),f)} \Downarrow & \big\downarrow{\scriptstyle ((\partial\eta)\gamma,y)} \\
\gamma \blacktriangleright x & \xrightarrow[(\gamma,\eta)\blacktriangleright f]{} & (\partial\eta)\gamma \blacktriangleright y
\end{array}
$$

The squares describe the action of $G \ltimes H$ on $Mor(\mathbf{C})$. They come from diagonals of the cube:



Structure of Transformation Groupoid

Theorem

If $\mathcal{G}$ is given by a crossed module $(G, H, \partial, \triangleright)$, acts on a category **C***, the double category has:*

- *Horizontal Category:* **C**
- *Vertical Category: $Ob(\mathbf{C}) /\!/ G$, the transformation groupoid for the action of G on the objects of* **C**
- *Horizontal Category of Squares:* $\mathbf{C} \times \mathcal{G}$
- *Vertical Category of Squares: $Mor(\mathbf{C}) /\!/ (G \ltimes H)$, the transformation groupoid for the action of $G \ltimes H$ onthe morphisms of* **C**

So $\mathbf{C} /\!/ \mathcal{G}$ contains transformation groupoids for our three group actions ▶ as:

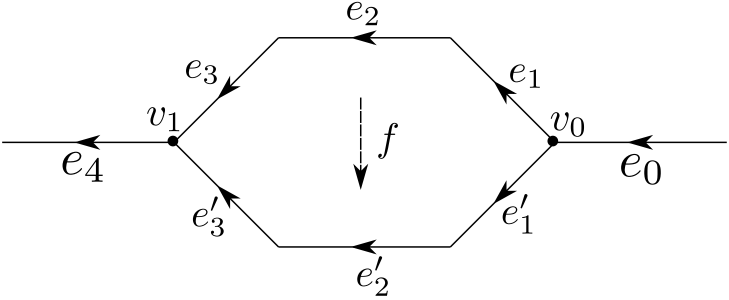
$$Ob(\mathbf{C} /\!/ \mathcal{G}) \subset \mathbf{C}^{(1)} /\!/ \mathcal{G}^{(0)} \subset Ver(\mathbf{C} /\!/ \mathcal{G})$$

related by identity-inclusion maps.

Our Motivating Example

Our original motivation in developing this machinery was to describe the **symmetry** of **moduli spaces** of **connections on gerbes**. Think of these as determining **holonomies** valued in a 2-group $\mathcal{G} \sim (G, H, \triangleright, \partial)$ to paths and homotopies of paths in a manifold.

We simplified things by using discrete paths and homotopies made of chosen edges and faces:



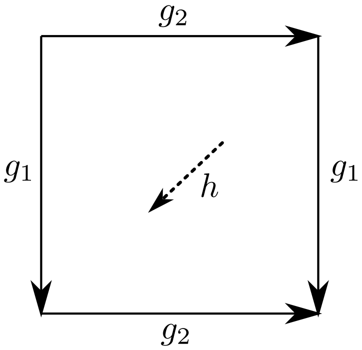
Definition

The **category of connections**, **Conn** has:

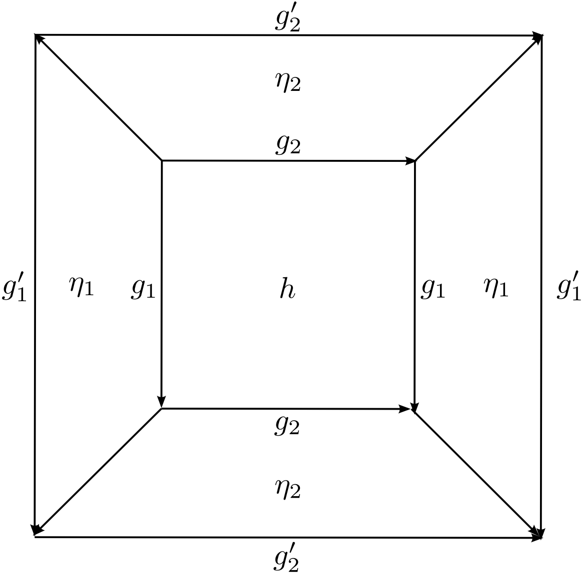
- **Objects**: pairs (g, h), where $g : E \to G$, $h : F \to H$ (with some compatibility condition)
- **Morphisms**: $((g, h), \eta)$ where $\eta : E \to H$, seen as $((g, h), \eta) : (g, h) \to (g', h')$ where: for each edge e. (And a condition for each face.)

This does form a category (MP2).

Intuitively, given 2-group $\mathcal{G}$, the object of **Conn** are connections - which assign G-holonomies to edges and H-holonomies to faces:



A Connection on $(T^2, \mathcal{D})$

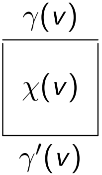


Horizontal Morphism in **Conn**$(T^2, \mathcal{D})$

Definition

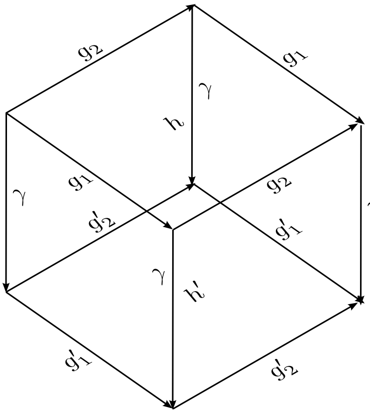
The **2-group of gauge symmetries** is **Gauge** $= \mathcal{G}^V$, with:

- **Objects** are the set of maps $\gamma : V \to G$
- **Morphisms** are the set of pairs (γ, χ) where γ is an object and $\chi : V \to H$, such that at each $v \in V$:

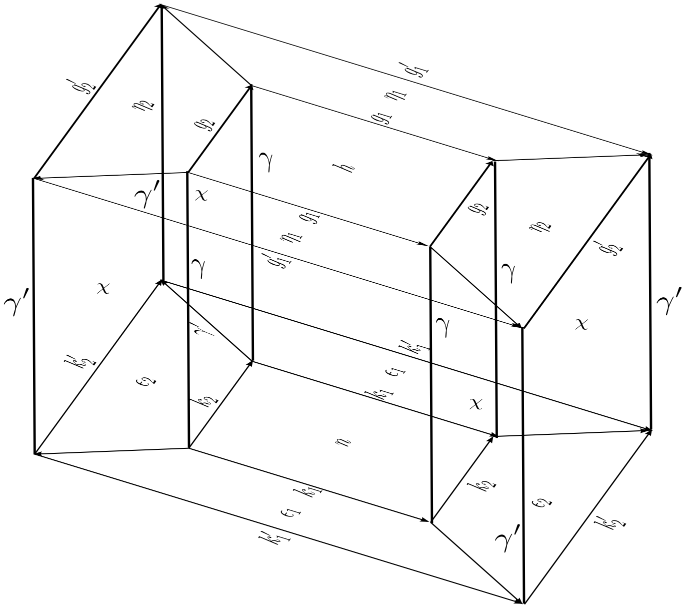


(1)

This is a 2-group, and it acts on **Conn**.



Vertical Morphism in **Conn**$/\!\!/$**Gauge**$(T^2, \mathcal{D})$ (From action of G^V)



Square - Gauge Modification on $(T^2, \mathcal{D})$ (from action of $(G \ltimes H)^V$)

Preview

In our forthcoming work (MP3) we will prove something about this double groupoid:

$$\mathbf{Conn} /\!\!/ \mathbf{Gauge} \cong Hom_{\Box}(\Pi_2(M), \mathcal{G}) \tag{2}$$

Where, for bicategories **A** and **(**B), $Hom_{\Box}(\mathbf{A}, \mathbf{B})$ is a double category with:

- **Objects**: 2-functors from **A** to **B**
- **Vertical Morphisms**: *Strict* natural transformations between 2-functors
- **Horizontal Morphisms**: *Costrict Pseudo*natural transformations between 2-functors
- **Squares**: Modifications $M : s_2 \circ c_F \Rightarrow c_G \circ s_1$:

$$\begin{array}{ccc} F_1 & \xrightarrow{c_1} & G_1 \\ {\scriptstyle s_F}\downarrow & \swarrow\!\!\!\swarrow_M & \downarrow{\scriptstyle s_G} \\ F_2 & \xrightarrow[c_2]{} & G_2 \end{array} \tag{3}$$