Monoidal Grothendieck Construction

Joe Moeller Christina Vasilakopoulou

University of California, Riverside

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Motivation

$k \in \text{Ring}$

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$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

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$$\mathsf{Mod}_{all}???$$

Motivation

Grothendieck: Yes!

- objects (k, M), where $M \in \mathrm{Mod}_k$
- maps $(f, g)\colon (k, M) \to (k', M')$ where $f\colon k \to k'$ and $g\colon M \to f^*(M')$

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$$f \colon k \to k' \rightsquigarrow f^* \colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

Motivation

$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

$$\mathsf{Mod} \colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

$$f \colon k \to k' \rightsquigarrow f^* \colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

Motivation

Given

$$\mathrm{Mod}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{Cat}$$

we defined Mod_{all} to have

- objects (k, M), where $M \in \mathrm{Mod}_k$
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Given

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

we define $\int \mathcal{F}$ to have

- objects (x, a), where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- maps $(f, g)\colon (x, a) \to (x', a')$ where $f\colon x \to x'$ and $g\colon a \to \mathcal{F}f(a')$

Indexed Categories

2-category $\mathsf{ICat}(\mathcal{X})$:

- an **indexed category** is a pseudofunctor $\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$.
- an **indexed functor** is a pseudonatural transformation $\alpha\colon \mathcal{F} \Rightarrow G$
- an **indexed natural transformation** is a modification $m\colon \alpha \Rrightarrow \beta$

Fibrations

$$
\begin{array}{ccccc}
\mathcal{A} & & & & b \\
\downarrow{\scriptstyle P} & & & & \\
\mathcal{X} & & x & \xrightarrow{f} & y
\end{array}
$$

Fibrations

cartesian lift

$$\begin{array}{ccc} \mathcal{A} & \quad & a \xrightarrow{\phi} b \\ \Big\downarrow P & & \\ \mathcal{X} & & x \xrightarrow{f} y \end{array}$$

Fibrations

pullback

$$\begin{array}{ccc} \mathcal{A} & & f^*(b) \xrightarrow{\ \phi\ } b \\ \Big\downarrow P & & \\ \mathcal{X} & & x \xrightarrow{\ f\ } y \end{array}$$

Fibrations

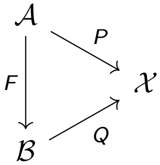
reindexing functor

$$\mathcal{A}_y \xrightarrow{f^*} \mathcal{A}_x$$

Fibrations

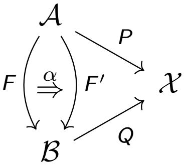
2-category $\mathrm{Fib}(\mathcal{X})$:

- an object is a fibration $P\colon \mathcal{A} \to \mathcal{X}$
- a 1-morphism is a functor F



which preserves cartesian liftings

- a 2-morphism is a natural transformation



The Grothendieck Construction

For indexed category $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$, $\int \mathcal{F}$ is naturally fibred over $\mathcal{X}$:

$$\begin{aligned} P_{\mathcal{F}}\colon \int \mathcal{F} &\to \mathcal{X} \\ (x, a) &\mapsto x \\ (f, k) &\mapsto f \end{aligned}$$

2-Equivalence

Theorem

The Grothendieck construction gives an equivalence:

$$\mathsf{ICat}(\mathcal{X}) \cong \mathsf{Fib}(\mathcal{X})$$

Fibre-wise Monoidal Grothendieck Construction

A **(fibre-wise) monoidal indexed category** is

- a pseudofunctor $\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}$

Let $f\mathsf{MonICat}(\mathcal{X})$ denote the 2-category of fibre-wise monoidal indexed categories

Fibre-wise Monoidal Grothendieck Construction

A **(fibre-wise) monoidal fibration** is

- fibration $P\colon \mathcal{A} \to \mathcal{X}$
- the fibres $\mathcal{A}_x$ are monoidal
- the reindexing functors are monoidal

Let $f\mathsf{MonFib}(\mathcal{X})$ denote the 2-category of fibre-wise monoidal fibrations.

Fibre-wise Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$f\mathsf{MonFib}(\mathcal{X}) \simeq f\mathsf{MonICat}(\mathcal{X})$$

Global Monoidal Grothendieck Construction

A **(global) monoidal indexed category** is

- an indexed category $\mathcal{F}\colon \mathcal{X}^{\text{op}} \to \mathsf{Cat}$
- $\mathcal{X}$ is monoidal
- $\mathcal{F}$ is lax monoidal $(\mathcal{F}, \phi)\colon (\mathcal{X}^{\text{op}}, \otimes) \to (\mathsf{Cat}, \times)$

Let $g\mathsf{MonICat}$ denote the 2-category of global monoidal indexed categories.

Global Monoidal Grothendieck Construction

A **(global) monoidal fibration** is a fibration $P\colon \mathcal{A} \to \mathcal{X}$

- $\mathcal{A}$ and $\mathcal{X}$ are monoidal
- P is a strict monoidal functor
- $\otimes_{\mathcal{A}}$ preserves cartesian liftings.

Let $g\mathsf{MonFib}(\mathcal{X})$ denote the 2-category of global monoidal fibrations.

Global Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$g\mathsf{MonFib}(\mathcal{X}) \simeq g\mathsf{MonICat}(\mathcal{X})$$

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi) \colon (\mathcal{X}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\phi \colon \mathcal{F}x \times \mathcal{F}y \to \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

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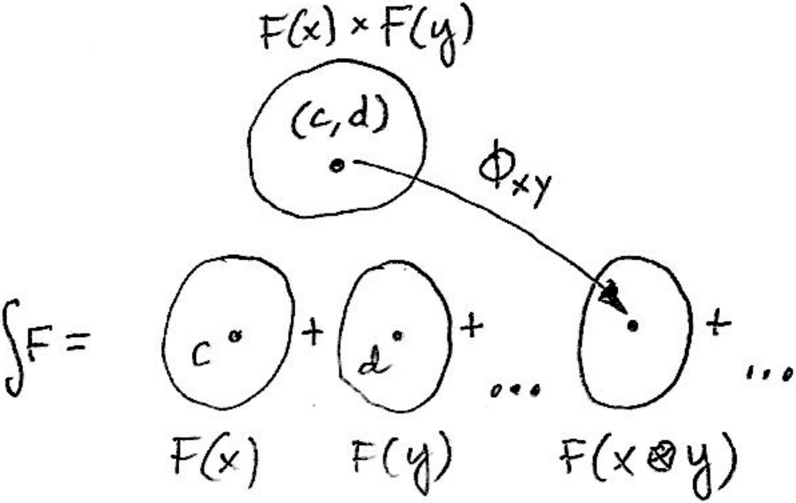
$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$



Credit: Bartosz Milewski

Shulman's Monoidal Grothendieck Construction

Theorem (Shulman)

If $\mathcal{X}$ is cartesian monoidal, then

$$g\mathsf{MonFib}(\mathcal{X}) \simeq f\mathsf{MonICat}(\mathcal{X})$$

Cartesian Case

Theorem (Vasilakopoulou, M)

If $\mathcal{X}$ is a cartesian monoidal category, then

$$\begin{array}{ccc} g\mathsf{MonFib}(\mathcal{X}) & \xrightarrow{\simeq} & g\mathsf{MonICat}(\mathcal{X}) \\ \wr\big\downarrow & & \big\downarrow\wr \\ f\mathsf{MonFib}(\mathcal{X}) & \xrightarrow{\simeq} & f\mathsf{MonICat}(\mathcal{X}) \end{array}$$

Example: Modules

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$
$$(f\colon k \to k') \mapsto (f^*\colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k)$$

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$\int \mathsf{Mod}$:

$$(k, M) \xrightarrow{(f,g)} (k', N)$$

Example: Modules

$$(\mathsf{Mod}, \mu) \colon (\mathsf{Ring}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\begin{aligned} \mu \colon \mathsf{Mod}_k \times \mathsf{Mod}_{k'} &\to \mathsf{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

Example: Modules

$$(\mathsf{Mod}, \mu) \colon (\mathsf{Ring}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\begin{aligned} \mu \colon \mathsf{Mod}_k \times \mathsf{Mod}_{k'} &\to \mathsf{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

$(\int \mathsf{Mod}, \otimes_\mu)$:

$$(k, M) \otimes_\mu (k', N) = (k \otimes k', M \otimes_{\mathbb{Z}} N)$$

Applications

- Zunino & Turaev modules
- (Co)modules of (co)algebras
- Operad of wiring diagrams/discrete dynamical systems
- Structured cospans
- Catalysts in Petri nets

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