Scalars in a Tangent Category

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Motivation

Classical differential geometry

Everything is defined in terms of the ring $\mathcal{R}$.

1. Every manifold is locally $\mathcal{R}^n$
2. Tangent vectors: equivalence classes of curves $\mathcal{R} \to M$.

Synthetic Differential Geometry (SDG)

Still the case.

1. $\mathcal{R}$ has infinitesimals $D = [d : R | d^2 = 0]$.
2. Must satsify *Kock-Lawvere* axiom:
   $\nu : R \times R \to [D, R] := (a, b) \mapsto \lambda d.ad + b$
   is an iso
3. Tangent vectors are $D \to M$.

Tangent Categories

Definition (Rosický, Cockett-Cruttwell)

Abstract setting for differential geometry: only the behaviour of the *tangent bundle* is axiomatized.

- No ring object.
- Tangent vector addition - but no tangent vector subtraction.

This captures examples from computer science:

- REL: differential structure, no subtraction or ring of scalars.
- The classifying category for the $\partial - \lambda$ calculus.

Modular: Add a structure and see how it fits:

- A class of “submersions”: *display* tangent categories
- Solutions to ODEs: *curve* objects

Goal

Sort out “what does a scalar rig look like in a tangent category”

- What universal properties should it satisfy?
- Does it make “calculus” better behaved?

Result: The *Kock-Lawvere* axiom without infinitesimals.

Tangent categories are *enriched* categories - extend those results.

- Embed a tangent category into one with a scalar unit.
- Show that differential objects are (enriched) sketchable.

Tangent Categories

A *tangent category* is a category $\mathbb{X}$ equipped with an endofunctor T on $\mathbb{X}$ and natural transformations

$$p : T \Rightarrow id, 0 : id \Rightarrow T, + : T_p\times_p T \Rightarrow T, \ell : T \Rightarrow T^2, c : T^2 \Rightarrow T^2$$

All pullback powers of p exist and are preserved by T
$(p_M, +_M, 0_M)$ is an additive bundle
$(\ell, 0)$, (c, id) are additive.

ℓ is *universal*: $T(M)_p\times_p T(M) \xrightarrow{\mu} T^2(M) \overset{T(p)}{\underset{pp0}{\rightrightarrows}} T(M)$

(where $\mu = \langle \pi_0 \ell, \pi_1 0\rangle T(+)$.)
c is a symmetry $cc = id$ and $T(c)cT(c) = cT(c)c$
Symmetric cosemigroup: $\ell c = \ell$, $\ell T(\ell) = \ell\ell$ and $cT(c)\ell = T(\ell)c$

Differential Objects

Definition

(V, σ_V, ξ_V) is a commutative monoid object such that.

1. Biproduct in CMon

$$\begin{array}{ccccc} V & & = & & V \\ & \searrow^{\lambda} & & \nearrow^{\hat{p}} & \\ & & T(V) & & \\ & \nearrow^{0} & & \searrow^{p} & \\ V & & = & & V \end{array}$$

2. Compatible addition

$$\begin{array}{ccc} V & \xrightarrow{!_V} & 1 \\ {\scriptstyle 0_V}\downarrow & & \downarrow{\scriptstyle \xi} \\ T(V) & \xrightarrow[\hat{p}]{} & V \end{array} \qquad \begin{array}{ccc} T_2 V & \xrightarrow{(\hat{p}\pi_0, \hat{p}\pi_1)} & V \times V \\ \downarrow{\scriptstyle +_V} & & \downarrow{\scriptstyle \sigma_V} \\ T(V) & \xrightarrow[\hat{p}]{} & V \end{array}$$

3. Compatible lift:

$$\begin{array}{ccc} TV & \xrightarrow{l} & T^2 V \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\ V & \xleftarrow[\hat{p}]{} & TV \end{array}$$

Differential objects ii

Theorem (Cockett and Cruttwell)

The full subcategory of differential objects is cartesian differential category.

Set $\nu = (\lambda \times 0)T(\sigma_V)$

The derivative:
$$\begin{array}{ccc} V \times V & \xrightarrow{D[f]} & W \\ {\scriptstyle \nu}\downarrow & & \uparrow{\scriptstyle \hat{p}} \\ T(V) & \xrightarrow{T(f)} & T(W) \end{array}$$

Linear in V:
$$\begin{array}{ccc} T(V \times X) & \xrightarrow{T(f)} & T(W) \\ {\scriptstyle \lambda_V \times 0_X}\uparrow & & \uparrow{\scriptstyle \lambda_W} \\ V \times X & \xrightarrow{f} & W \end{array}$$

Note: λ is universal $\quad V \xrightarrow{\lambda} T(V) \overset{p}{\underset{!\xi}{\rightrightarrows}} V$

Linear Classifier

Under a very mild assumption, we can add a universal ring object to a tangent category.

Definition (Blute-Cockett-Seeley)

A *Scalar Unit* is a differential object with a point $1 \xrightarrow{u} R$ with the universal property that for all

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \langle 1,u \rangle}\downarrow & \nearrow_{\exists ! \hat{f} \text{ linear in } R} & \\ V \times R & & \end{array}$$

f (multi)-linear in $V \Rightarrow \hat{f}$ is (multi)-linear in V

Consequences of a Linear Point Classifier

- The unit object is a commutative rig R.

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle\langle 1,u\rangle}\big\downarrow & \nearrow_{\cdot} & \\ R \times R & & \end{array}$$

- Every differential object is an R-module.

$$\begin{array}{ccc} V & = & V \\ {\scriptstyle\langle 1,u\rangle}\big\downarrow & \nearrow_{\cdot} & \\ V \times R & & \end{array}$$

- Every linear map preserves the R-module action (persistence).

Rewriting the lift

Every R-module has the map λ^R:

$$V \xrightarrow{\langle 1,u\rangle} V \times R \xrightarrow{0\times\lambda} T(V \times R) \xrightarrow{T(\cdot)} T(V)$$

In SDG: $v \mapsto \lambda d.vd$.

Lemma

For a differential object, λ^R *satisfies the equalizer*

$$V \xrightarrow{\lambda^R_V} T(V) \underset{!\xi}{\overset{p}{\rightrightarrows}} V$$

so $\lambda_V = \lambda^R_V$.

Corollary

Homogenous morphisms of differential objects are linear.

KL-Modules

Set $\nu^T := \; V \times V \xrightarrow{\lambda^R \times 0} T(V) \times T(V) \xrightarrow{T(\sigma)} T(V)$

Definition (*Kock-Lawvere R*-module)

V is a KL-module if there is an R-module map $\hat{p}$ making

$$((\lambda^R \times 0)T(\sigma))^{-1} = \langle \hat{p}, p_V \rangle$$

- The category of KL-modules is equivalent to the category of differential objects.
- In a locally presentable tangent category, KL-modules is a full reflective subcategory of R-modules.
- If $\mathbb{X}$ has negatives, then KL-modules are a completion of R-modules.

New Questions

The notion of a scalar unit allows one to use the simpler definition of KL-modules.

If R is a ring, KL-modules are a completion of R-modules - is there a sketch of KL-modules?

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Move to *enriched category theory*.

Weil Algebras, Microlinear Weil Spaces

Definition (The category Weil)

The full subcategory of $\pi : W \to R$ in unital $R\mathsf{Alg}/R$ so that:

- $\ker(\pi)$ is a nilpotent ideal
- $U(W) = R^n$.

e.g. the dual numbers $R[x]/x^2$

A *Microlinear Weil Space* is a presheaf Weil $\to$ Set preserving connected limits. Call the category of microlinear weil spaces $\mathcal{E}$.

Theorem

The category of microlinear weil spaces is

- *Locally finitely presentable*
- *A coherently closed tangent category*
- *Has a scalar unit* $R = [y(R[x]/x^2), y(R[x]/x^2)]$

Microlinear Weil Spaces

Recall that in a $\mathcal{E}$-category $\mathcal{C}$

- Power: $\mathcal{E}(w, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, w \pitchfork Y)$
- Copower: $\mathcal{E}(w \bullet X, Y) \cong \mathcal{E}(w, \mathcal{C}(X, Y))$

Theorem (Garner, Leung)

A tangent category is equivalently a category enriched in $\mathcal{E}$ with powers by representables.

$$T(X) := y(R[x]/x^2) \pitchfork X$$

Observation

The tangent bundle is now a weighted limit.

Units in Presheaf Categories

Theorem

The enriched Yoneda embedding preserves differential objects.

Theorem

The enriched presheaf category of a tangent category has a representable unit:

$$1 \bullet R \cong [1 \bullet y(x^2), 1 \bullet y(x^2)]$$

Observation

Every differential objects is a KL-module in the presheaf category.

KL-modules as sketches

Define the $\mathcal{E}$-sketch KLMod.

- Objects: $n \in \mathbb{N}$
- Hom-Objects: $[n, m] = R^{n \times m}$
- Composition: Matrix multiplication

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  - $0_n : 1 \to R^{n \times 2n}$ picks out the matrix $\begin{bmatrix} 0 \\ I \end{bmatrix}$
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  We derive:
  - λ_n picks out $\begin{bmatrix} I \\ 0 \end{bmatrix}$
  - $\hat{p}$ picks out $\begin{bmatrix} I & 0 \end{bmatrix}$

  ($I = n \times n$ identity matrix)
- Powers on homs: $\Delta_{n,m} : R^{n \times m} \to R^{2n \times 2m}$

Conclusions and Future Work

We used the notion of a scalar unit to simplify Differential Objects and find a $\mathcal{E}$-sketch.

Opens the door for sketch theory to be applied in tangent categories

- Gabriel-Ulmer duality: a free KL-module construction?
- Differential Bundles
- *Involution Algebroids* a sketch for Lie theory (Joint work with Matthew Burke)

Sketches can be interpreted as *Abstract Data Types*

- $\mathcal{E}$-sketches as ∂-ADTs in differential programming?

References

- Adamek, Jiri, Jiri Rosicky, et al. (1994). *Locally presentable and accessible categories*. Vol. 189. Cambridge University Press.
- Blute, R, J Robin B Cockett, and Robert AG Seely (2015). "Cartesian differential storage categories". In: *Theory and Applications of Categories* 30.18, pp. 620–686.
- Cockett, J Robin B and Geoff SH Cruttwell (2014). "Differential structure, tangent structure, and SDG". In: *Applied Categorical Structures* 22.2, pp. 331–417.
- Cockett, JRB and GSH Cruttwell (2017). "Connections in tangent categories". In: *Theory and Applications of Categories* 32.26, pp. 835–888.
- Garner, Richard (2018). "An embedding theorem for tangent categories". In: *Advances in Mathematics* 323, pp. 668–687.
- Leung, Poon (2017). "Classifying tangent structures using Weil algebras". In: *Theory and Applications of Categories* 32.9, pp. 286–337.
- Rosicky, Jiri (1984). "Abstract tangent functors". In: *Diagrammes* 12, JR1–JR11.