Nominal PROPs

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23rd of May 2019

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Partially Monoidal Categories

Monoidal categories are models of resources

In some models partiality arises naturally

Example: Memory allocation

Example: Simultaneous substitutions

2-Dimensional Calculus of Simultaneous Substitutions

horizontal/sequential composition: $[a{\mapsto}b]\,;[b{\mapsto}c] = [a{\mapsto}c]$

vertical/parallel composition: $[a{\mapsto}b] \oplus [c{\mapsto}d] = [a{\mapsto}b, c{\mapsto}d]$

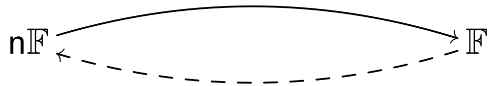
$\oplus$ is partial since the following is not allowed: $[a{\mapsto}b] \oplus [a{\mapsto}c]$

semantics: functions $f : \{a, c\} \to \{b, d\}$

Semantics of Simultaneous Substitutions

The category $\mathsf{n}\mathbb{F}$ of finite subsets of a countably infinite set $\mathcal{N}$ of ‘names’ or ‘variables’.

$\mathsf{n}\mathbb{F}$ is equivalent to the category $\mathbb{F}$ of finite cardinals with all functions.



So why do we care of representing $\mathsf{n}\mathbb{F}$ as opposed to $\mathbb{F}$?

- Syntax is not invariant under isomorphism, see variables vs de Bruin indices in λ-calculus.

- $\mathsf{n}\mathbb{F}$ has more structure, namely that of a **nominal category**, and this structure is not preserved by the equivalence.

- in other words: $\mathbb{F} \dashrightarrow \mathsf{n}\mathbb{F}$ is not an internal functor in the category Nom of **nominal sets**.

Internal Monoidal Categories

What is the relevant structure of $\mathsf{n}\mathbb{F}$?

It is an **internal monoidal category** in $(\mathsf{Nom}, 1, *)$ where $*$ is the so-called separated product of nominal sets.

To make this precise we need to show that we can extend the monoidal operation

$$* : \mathsf{Nom} \times \mathsf{Nom} \to \mathsf{Nom}$$

to an operation

$$* : \mathsf{Cat}(\mathsf{Nom}) \times \mathsf{Cat}(\mathsf{Nom}) \to \mathsf{Cat}(\mathsf{Nom})$$

on internal categories in Nom.

In the following we generalise from Nom to $\mathcal{V}$ and only assume that $(\mathcal{V}, I, \otimes)$ is a monoidal category with finite limits in which I is the terminal object.

Internal Monoidal Categories

pull back the internal category

$$(\mathbb{C}_1 \times \mathbb{D}_1, \mathbb{C}_0 \times \mathbb{D}_0)$$

along

$$\mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$$

$$\begin{array}{ccc} (\mathbb{C} \otimes \mathbb{D})_1 & \dashrightarrow & \mathbb{C}_1 \times \mathbb{D}_1 \\ {\scriptstyle dom} \downarrow \downarrow {\scriptstyle cod} & & {\scriptstyle dom} \downarrow \downarrow {\scriptstyle cod} \\ C_0 \otimes \mathbb{D}_0 & \longrightarrow & \mathbb{C}_0 \times \mathbb{D}_0 \end{array} \tag{1}$$

Lifting $\mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$ to $\mathbb{C} \otimes \mathbb{D} \to \mathbb{C} \times \mathbb{D}$ has a universal property

Lemma 1: The forgetful functor $\mathsf{Cat}(\mathcal{V}) \to \mathcal{V}$ is a fibration.

Where: $\mathsf{Cat}(\mathcal{V})$ is the category of internal catgories in $\mathcal{V}$.

Internal Monoidal Categories, cont'd

But we need more, namely that

$$\mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0 \qquad \text{and} \qquad \mathbb{C} \otimes \mathbb{D} \to \mathbb{C} \times \mathbb{D}$$

are *natural transformations*.

Hence, we extend the previous lemma to functor categories:

Lemma 2: If $P : \mathcal{E} \to \mathcal{B}$ is a fibration, then $P^{\mathcal{A}} : \mathcal{E}^{\mathcal{A}} \to \mathcal{B}^{\mathcal{A}}$ is a fibration.

Theorem: Let $(\mathcal{V}, 1, \otimes)$ be a (symmetric) monoidal category with finite limits in which the monoidal unit is the terminal object. $(\mathsf{Cat}(\mathcal{V}), 1, \otimes)$ inherits from $(\mathcal{V}, 1, \otimes)$ the structure of a (symmetric) monoidal category with finite limits in which the monoidal unit is the terminal object,

Internal Monoidal Categories, cont'd

Definition: A strict internal monoidal category $\mathbb{C}$ is a monoid $(\mathbb{C}, \varnothing, \odot)$ in $(\mathsf{Cat}(\mathcal{V}), 1, \otimes)$.

Example: The category $\mathsf{n}\mathbb{F}$ of finite subsets of a set $\mathcal{N}$ of names is an internal monoidal category in $(\mathsf{Nom}, 1, *)$, where

$$* : \mathsf{Cat}(\mathsf{Nom}) \times \mathsf{Cat}(\mathsf{Nom}) \to \mathsf{Cat}(\mathsf{Nom})$$

$$\uplus : \mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$$

$\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F}$ has objects: pairs of disjoint sets

arrows: pairs of functions with disjoint domains and disjoint codomains

$\uplus$ is disjoint union, partial wrt to $\mathsf{n}\mathbb{F} \times \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$ but total wrt $\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$

Nominal PROPs

Definition: A nominal PROP is strict internal monoidal category in $(\mathtt{Nom}, 1, *)$ which has finite subsets of $\mathcal{N}$ as objects (supported by themselves) and all bijections as arrows. A morphism of nominal PROPs is an internal strict monoidal functor that preserves bijections.

Equivalence of PROPs and nominal PROPs

Definition/Proposition: For any PROP $\mathcal{S}$, there is an nPROP

$$NOM(\mathcal{S})$$

that has for all arrows $f : \underline{n} \to \underline{m}$ of $\mathcal{S}$, and for all lists $\boldsymbol{a} = [a_1, \ldots a_n]$ and $\boldsymbol{b} = [b_1, \ldots b_m]$ arrows $[\boldsymbol{a}\rangle f \langle \boldsymbol{b}]$. These arrows are subject to equations

$$[\boldsymbol{a}\rangle f \,; g\langle \boldsymbol{c}] = [\boldsymbol{a}\rangle f \langle \boldsymbol{b}] \,; [\boldsymbol{b}\rangle g\langle \boldsymbol{c}] \tag{NOM-1}$$

$$[\boldsymbol{a} +\!\!\!+\, \boldsymbol{c}\rangle f \oplus g\langle \boldsymbol{b} +\!\!\!+\, \boldsymbol{d}] = [\boldsymbol{a}\rangle f \langle \boldsymbol{b}] \uplus [\boldsymbol{c}\rangle g\langle \boldsymbol{d}] \tag{NOM-2}$$

$$[\boldsymbol{a}\rangle id\langle \boldsymbol{b}] = [\boldsymbol{a}|\boldsymbol{b}] \tag{NOM-3}$$

$$[\boldsymbol{a}\rangle \, \langle \boldsymbol{b}|\boldsymbol{b}'\rangle \,; f \, \langle \boldsymbol{c}] \; = [\boldsymbol{a}|\boldsymbol{b}] \,; [\boldsymbol{b}'\rangle f \langle \boldsymbol{c}] \tag{NOM-4}$$

$$[\boldsymbol{a}\rangle \, f \,; \langle \boldsymbol{b}|\boldsymbol{b}'\rangle \, \langle \boldsymbol{c}] \; = [\boldsymbol{a}\rangle f \langle \boldsymbol{b}] \,; [\boldsymbol{b}'|\boldsymbol{c}] \tag{NOM-5}$$

Equivalence of PROPs and nominal PROPs, cont'd

Definition/Proposition: For any nPROP $\mathcal{T}$ there is a PROP

$$ORD(\mathcal{T})$$

that has for all arrows $f : A \to B$ of $\mathcal{T}$, and for all lists $\boldsymbol{a} = [a_1, \ldots a_n]$ and $\boldsymbol{b} = [b_1, \ldots b_m]$ arrows $\langle \boldsymbol{a}] f [\boldsymbol{b}\rangle$. These arrows are subject to equations

$$\langle \boldsymbol{a}]\, f \,;\, g\, [\boldsymbol{c}\rangle = \langle \boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle \,;\, \langle \boldsymbol{b}]\, g\, [\boldsymbol{c}\rangle \tag{ORD-1}$$

$$\langle \boldsymbol{a}_f +\!\!+\, \boldsymbol{a}_g]\, f \uplus g\, [\boldsymbol{b}_f +\!\!+\, \boldsymbol{b}_g\rangle = \langle \boldsymbol{a}_f]\, f\, [\boldsymbol{b}_f\rangle \oplus \langle \boldsymbol{a}_g]\, g\, [\boldsymbol{b}_g\rangle \tag{ORD-2}$$

$$\langle \boldsymbol{a}]\, id\, [\boldsymbol{a}\rangle = id \tag{ORD-3}$$

$$\langle \boldsymbol{a}]\, [\boldsymbol{a}'|\boldsymbol{b}] \,;\, f\, [\boldsymbol{c}\rangle = \langle \boldsymbol{a}|\boldsymbol{a}'\rangle \,;\, \langle \boldsymbol{b}]\, f\, [\boldsymbol{c}\rangle \tag{ORD-4}$$

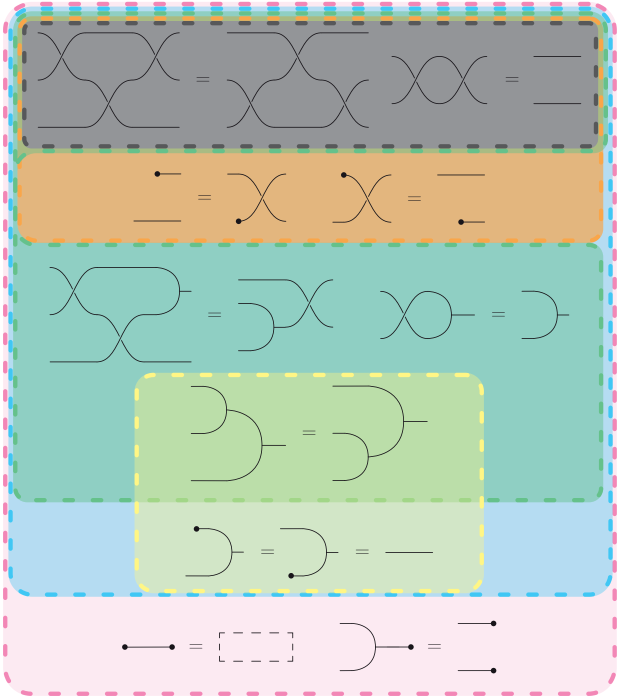
$$\langle \boldsymbol{a}]\, f \,;\, [\boldsymbol{b}|\boldsymbol{c}]\, [\boldsymbol{c}'\rangle = \langle \boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle \,;\, \langle \boldsymbol{c}|\boldsymbol{c}'\rangle \tag{ORD-5}$$

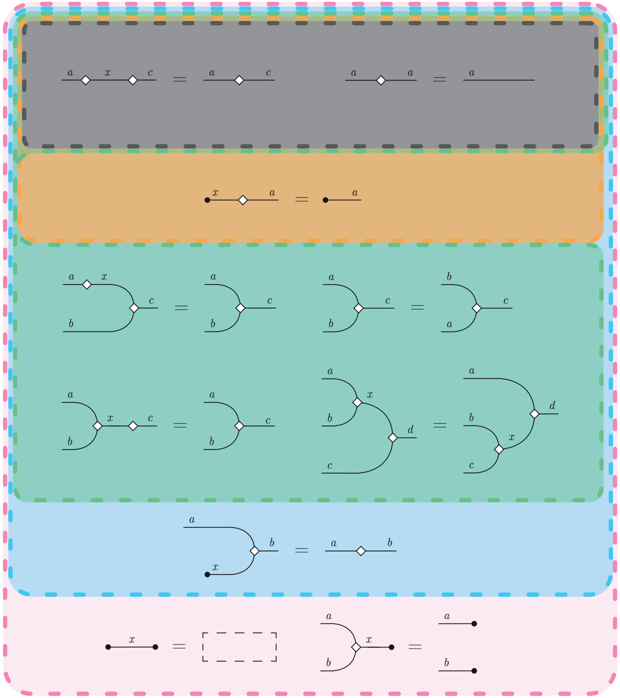
Equivalence of PROPs and nominal PROPs, cont'd

Theorem: The categories PROP and nPROP are equivalent.

Remark: The interesting part of the proof is to show how commutativity of $\uplus$ in nPROPs and naturality of symmetries in PROPs correspond to each other.

Equivalence of PROPs and nominal PROPs, cont'd





Equivalence of PROPs and nominal PROPs, cont'd