Circuit Relations for Real Stabilizers: Towards TOF $+$ H

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Outline

Graphical calculi and completeness

- The graphical calculus for PROPs models circuit computation.
- Finite presentations of different fragments of computing are studied.
- Complete presentation is a strict †-symmetric monoidal faithful functor.
- For quantum computing: ZX-calculus, ZH-calculus, ZW-calculus.
- For reversible computing: CNOT and TOF.

The category CNOT

Consider the PROP generated by cnot, $|1\rangle$, $\langle 1|$:
The controlled not gate, cnot, takes bits:

$$|b_1, b_2\rangle \mapsto |b_1, b_1 \oplus b_2\rangle$$

cnot is drawn as:

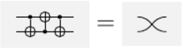
$|1\rangle$ is preparing 1 and $\langle 1|$ is postselecting 1:

The not gate and $|0\rangle$, $\langle 0|$ are derived:

This category has a finite, complete presentation in terms of *circuit identities*, CNOT [CCS18]:

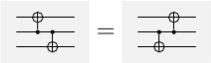
The identities of CNOT

[CNOT.1]

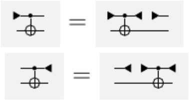


[CNOT.2]

[CNOT.3]



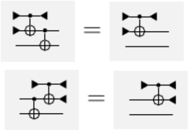
[CNOT.4]



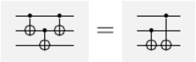
[CNOT.5]

[CNOT.6]

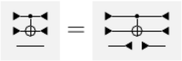
[CNOT.7]



[CNOT.8]



[CNOT.9]

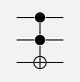


The category TOF

Consider the PROP generated by tof, $|1\rangle$, $\langle 1|$:
The Toffoli gate, tof, takes bits:

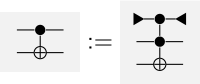
$$|b_1, b_2, b_3\rangle \mapsto |b_1, b_1 \cdot b_2 \oplus b_1\rangle$$

tof is drawn as:



$|1\rangle$ is preparing 1 and $\langle 1|$ is postselecting 1:

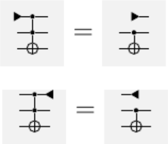
The cnot gate is derived:



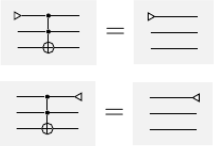
This category has a finite, complete presentation, TOF [CC19]:

The identities of TOF

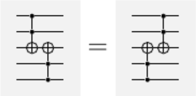
[TOF.1]



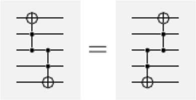
[TOF.2]



[TOF.3]



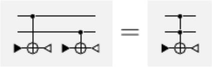
[TOF.4]



[TOF.5]

[TOF.6]

[TOF.7]

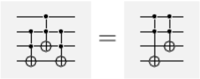


The identities of TOF

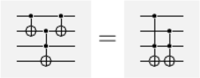
[TOF.8]

[TOF.9]

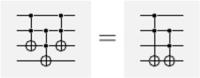
[TOF.10]



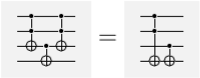
[TOF.11]



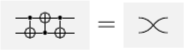
[TOF.12]



[TOF.13]

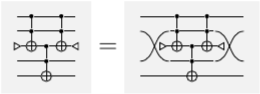


[TOF.14]



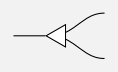
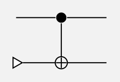
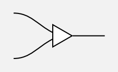
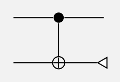
[TOF.15]

[TOF.16]



Both CNOT and TOF have concrete equivalent categories.
In particular they are discrete inverse categories.

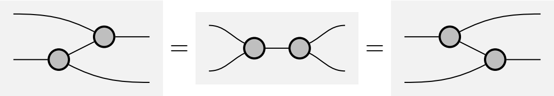
That means that they have a total copying map generated by:

 $:=$  and  $:=$ 

Frobenius algebras

A **Frobenius algebra** is a monoid-comonoid pair:

Satisfying the *Frobenius law*:

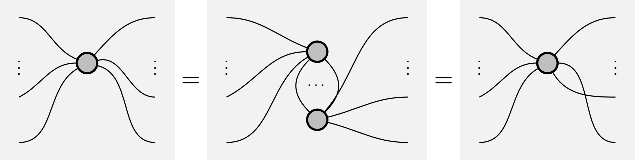


Frobenius algebras

A Frobenius algebra is **commutative** if:

And **special** if:

Connected components of Frobenius algebras can be uniquely represented by spiders:



Frobenius algebras are bases in FdHilb

Theorem ([CPV13])

Orthonormal bases $\{|i\rangle\}_{i\in B}$ *in* FdHilb *are in one-to-one correspondence with special, commutative* $\dagger$*-Frobenius algebras:*

$$\sum_{i\in B}|i\rangle \qquad \sum_{i\in B}|i\rangle\langle i,i| \qquad \sum_{i\in B}\langle i| \qquad \sum_{i\in B}|i,i\rangle\langle i|$$

Therefore, we can consider the Frobenius algebras associated to the eigenbasis of quantum observables.
For example, consider the Hermetian matrices:

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

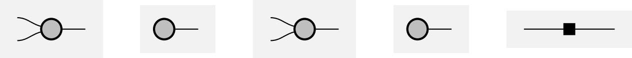
X and Z have spectra:

$$X_+ = |+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle) \qquad X_- = |-\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$$

$$Z_+ = |0\rangle, \qquad Z_- = |1\rangle$$

ZX_π

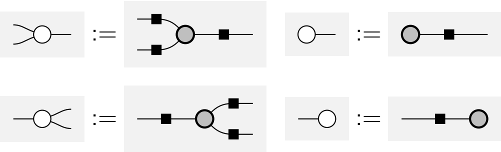
The **phase-free ZX-calculus**, ZX_π, [DP13] is the PROP generated by the Z Frobenius algebra and Hadamard gate:



The Hadamard gate is a self-inverse change of basis matrix so that:

$$X_+H = Z_+H \qquad X_-H = Z_-H \qquad Z_+H = X_+H \qquad Z_-H = X_-H$$

The Frobenius algebra for X is therefore given by conjugation.



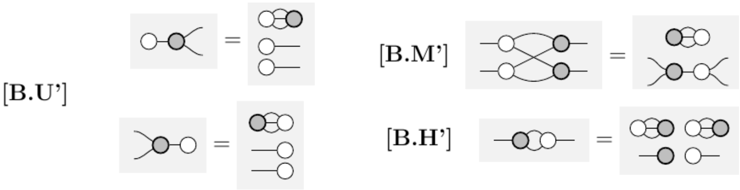
ZX_π

ZX_π has a finite presentation:

The first identity is that *the axioms of a special †-Frobenius algebra hold for Z*.

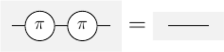
The Frobenius algebras associated to the Z and X observables are strongly complimentary.

They form a Hopf algebra up to an invertible scalar:

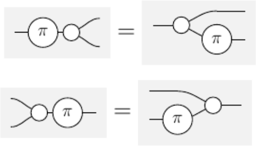


This corresponds to the bases being mutually unbiased [CD11].

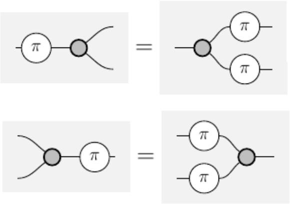
[PP]



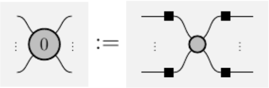
[PH]



[PI]

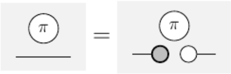


[S3]

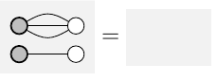


[H2]

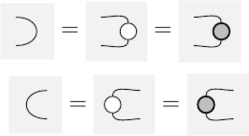
[ZO]



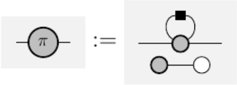
[IV]



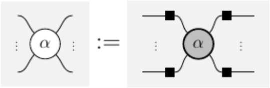
[L]



[S1]



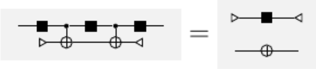
[S2]

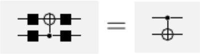


CNOT + H

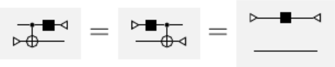
Consider the extension of CNOT with the Hadamard gate (and $\sqrt{2}$):

[H.I] 

[H.L] 

[H.F] 

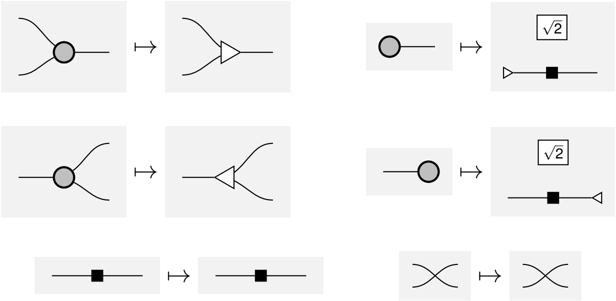
[H.Z] 

[H.U] 

[H.S] 

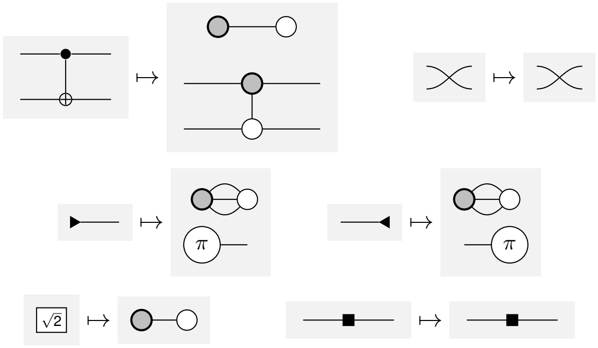
$\mathrm{ZX}_\pi \to \mathrm{CNOT} + H$

Consider $G : \mathrm{ZX}_\pi \to \mathrm{CNOT} + H$, sending:



CNOT $+ H \to$ ZX$_\pi$

Consider F : CNOT $+ H \to$ ZX$_\pi$, sending:



CNOT is isomorphic to ZX_π

Proposition

$F : \mathrm{CNOT} + H \to \mathrm{ZX}_\pi$ and $G : \mathrm{ZX}_\pi \to \mathrm{CNOT} + H$ are $\dagger$-preserving symmetric monoidal functors.

Theorem

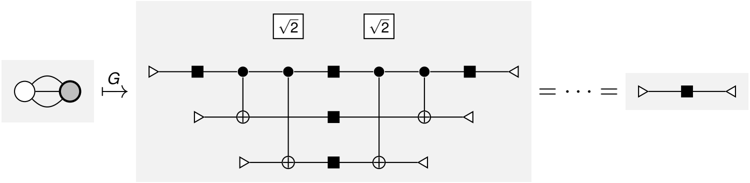
$F : \mathrm{CNOT} + H \to \mathrm{ZX}_\pi$ and $G : \mathrm{ZX}_\pi \to \mathrm{CNOT} + H$ are inverses.

This implies that $\mathrm{CNOT} + H$ is complete...

Theorem ([DP13])

ZX_π is complete for real stabilizer circuits.

We can also remove the scalar $\sqrt{2}$ by being careful.


G
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Stabilizer circuits and universality

There is a caveat:

Theorem ([Got98])

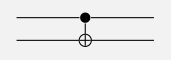
Stabilizer circuits can be simulated in polynomial time on a classical probabilistic computer.

However,

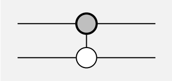
Theorem ([Aha03])

The Toffoli and Hadamard gates, together are an approximately universal gate set for quantum computing.

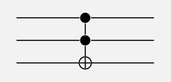
Is there a presentation in terms of the Toffoli gate and H?



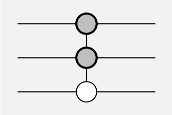
$\sim$



But

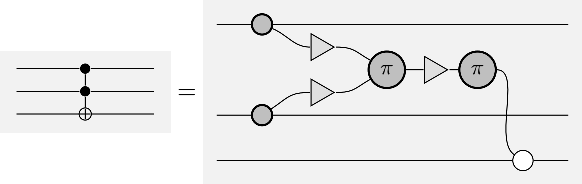


$\not\sim$

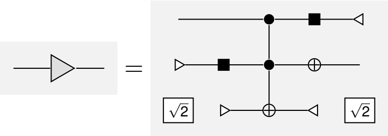


The Toffoli gate in ZX_π

The Toffoli gate has the following representation in ZX_π [NW18]:

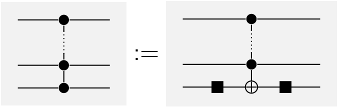


However, the Triangle has the following representation in CNOT $+$ H [Vil18]:

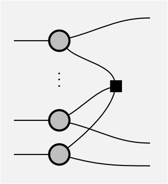


The ZH-calculus

The controlled-Z gate can be represented with Toffoli gate and Hadamard:

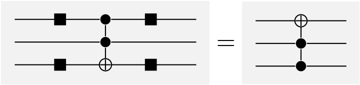


In the ZH calculus controlled Z-gates are given by:



Axiom **[H.F]** of CNOT $+$ H generalizes to Toffoli gates:

[H.F']



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