Duality for Algebras of the Connected Planar Wiring Diagrams Operad

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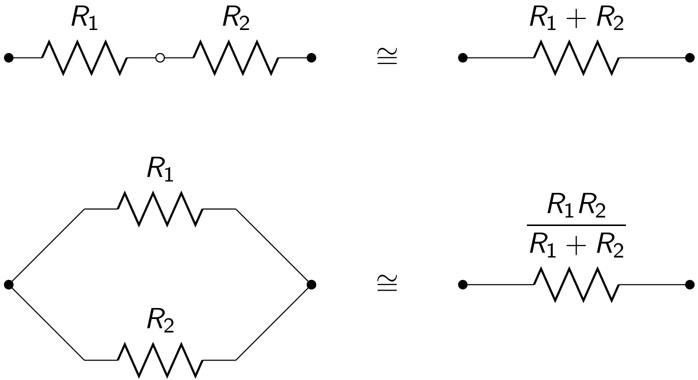
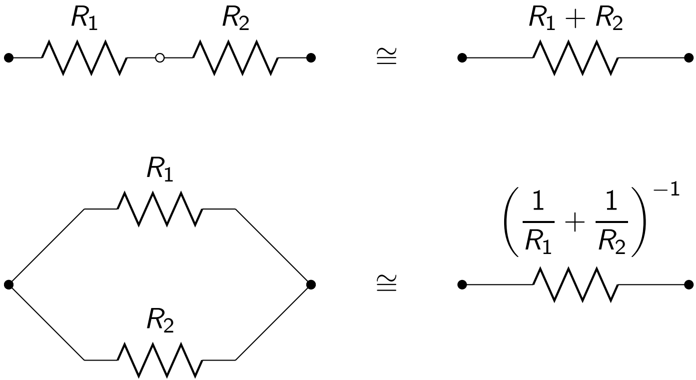
Combining Resistance

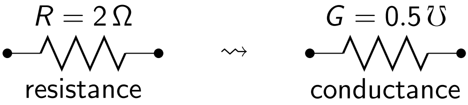


Combining Resistance

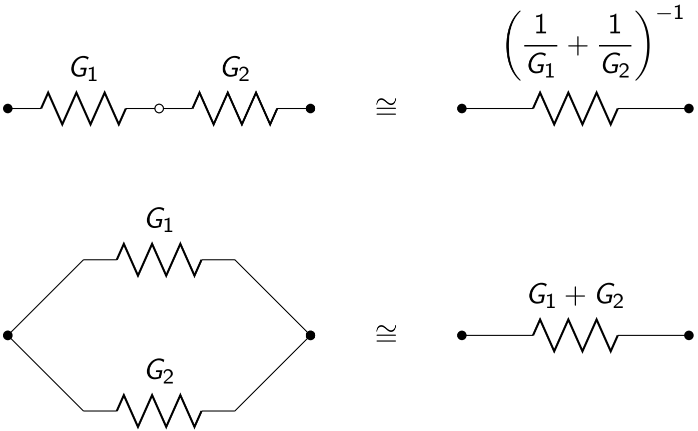


Conductance is Inverse Resistance

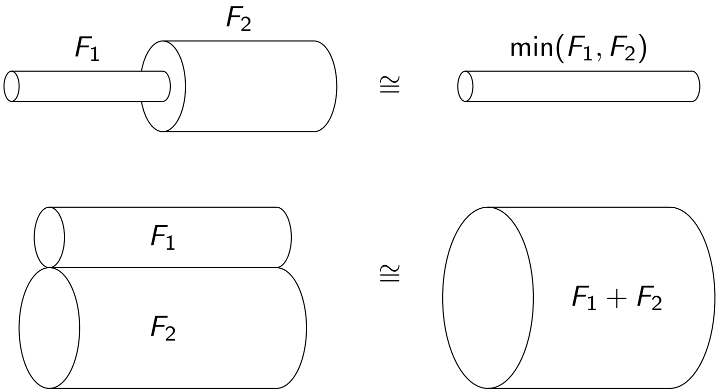
$$G = 1/R$$



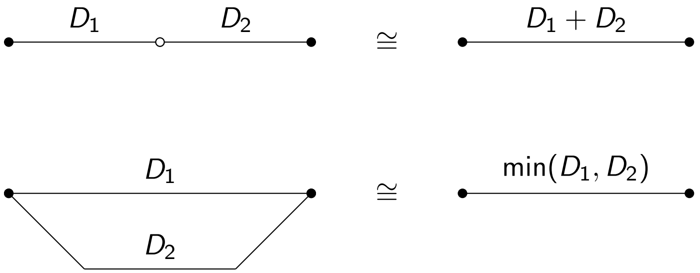
Combining Conductance



Combining Maximum Flow Rates



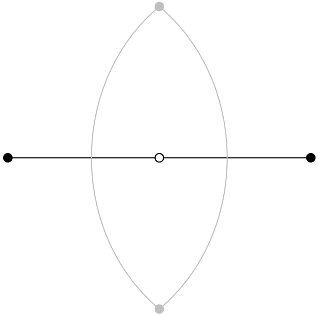
Combining Minimum Path Lengths



Series and parallel formulas

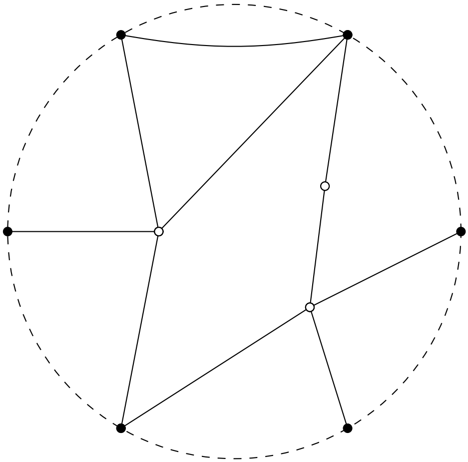
	Series	Parallel
Resistance	$R_1 + R_2$	$(R_1^{-1} + R_2^{-1})^{-1}$
Conductance	$(G_1^{-1} + G_2^{-1})^{-1}$	$G_1 + G_2$
Max Flow Rate	$\min(F_1, F_2)$	$F_1 + F_2$
Min Path Length	$D_1 + D_2$	$\min(D_1, D_2)$

Series and parallel connections are dual

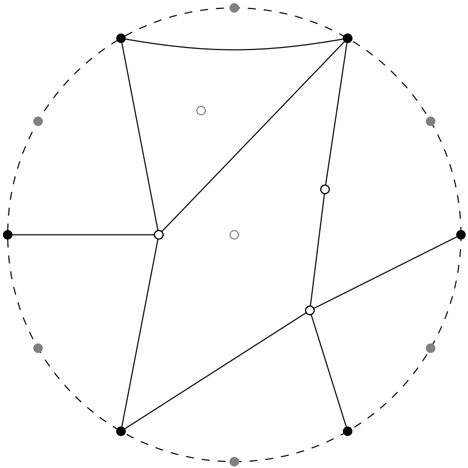


Not quite the usual “dual graph.”

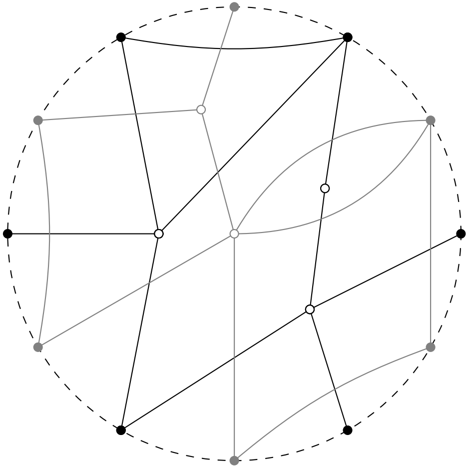
Connected circular planar graphs



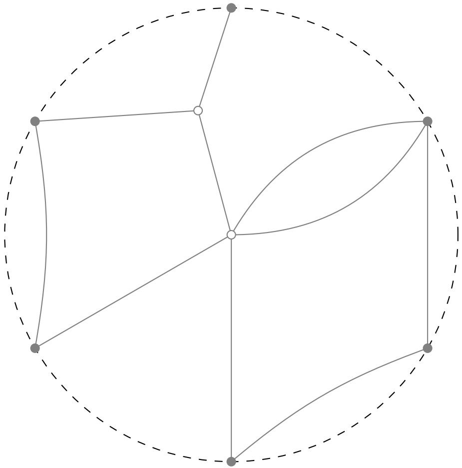
Placing the dual vertices



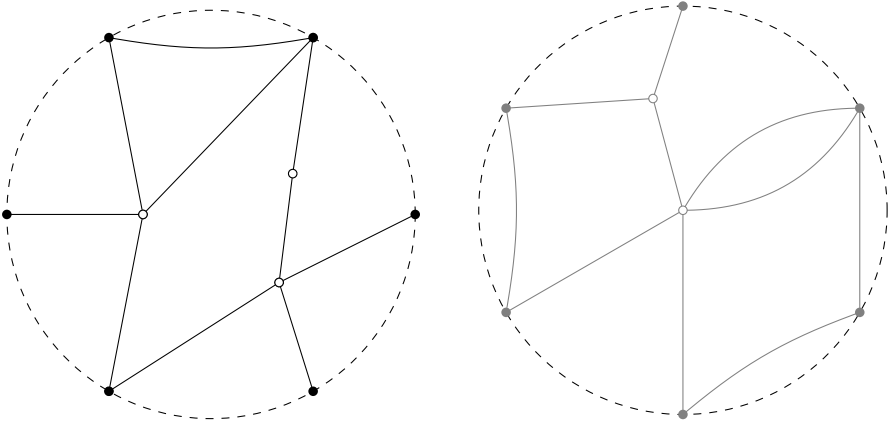
Placing the dual edges



The dual graph

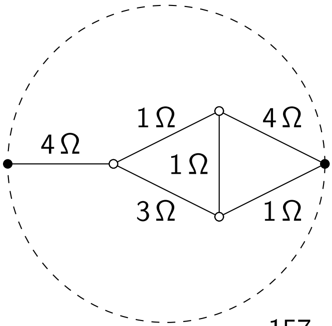


Dual Connected Circular Planar Graphs



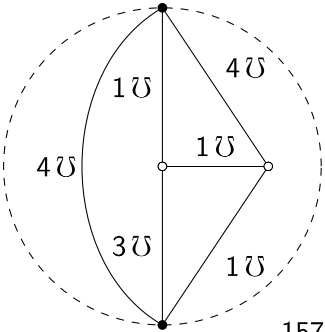
Resistance and conductance are “dual”

Resistances



$$\text{Effective resistance} = \frac{157}{29}\,\Omega$$

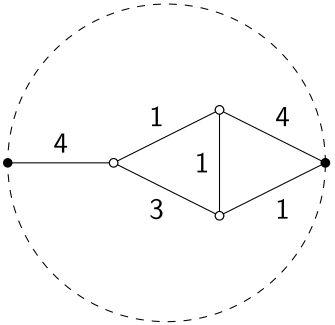
Conductances



$$\text{Effective conductance} = \frac{157}{29}\,\mho$$

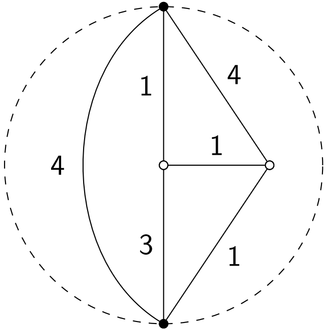
Max flow rate and min path length are "dual"

Max flow rates



Overall max flow rate = 3

Edge lengths



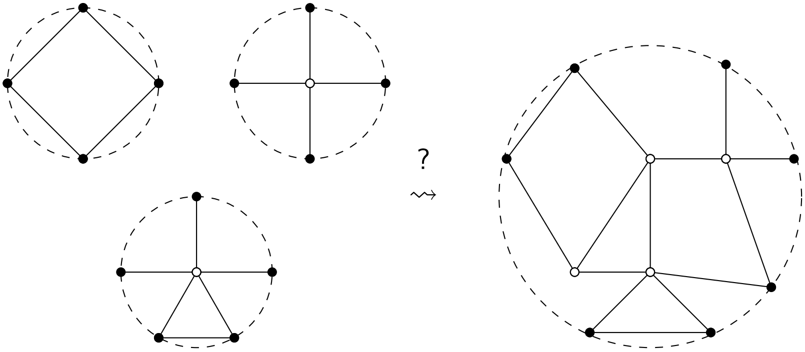
Min path length = 3

Main theorem

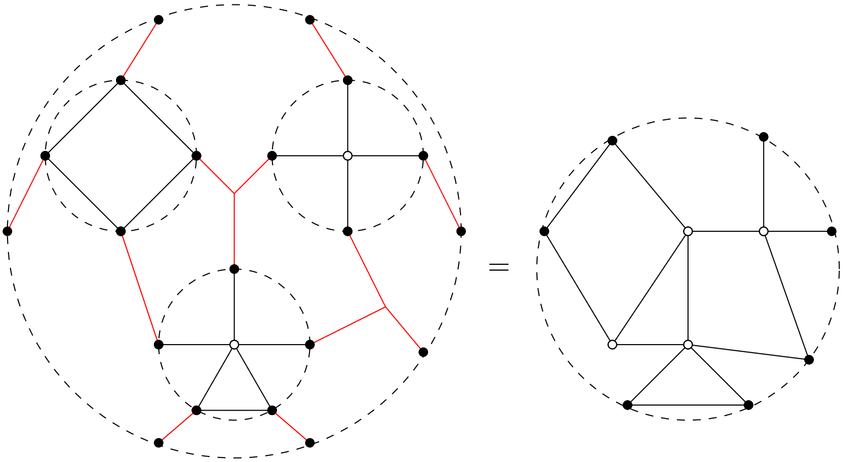
Theorem (B.—, 2019)

- *(Connected circular planar) graphs form an algebra of the operad $\mathcal{P}lan$ of "connected planar wiring diagrams."*
- *"Max flow rate," "min path length," "effective resistance," and "effective conductance" are all morphisms between $\mathcal{P}lan$-algebras.*
- *$\mathcal{P}lan$ has a duality automorphism giving every algebra a "dual" algebra.*
- *The algebra of graphs is isomorphic to its dual algebra, and the isomorphism sends a graph to its dual graph.*
- *The dual of the "max flow rate" morphism is "min path length," and the dual of "effective resistance" is "effective conductance."*

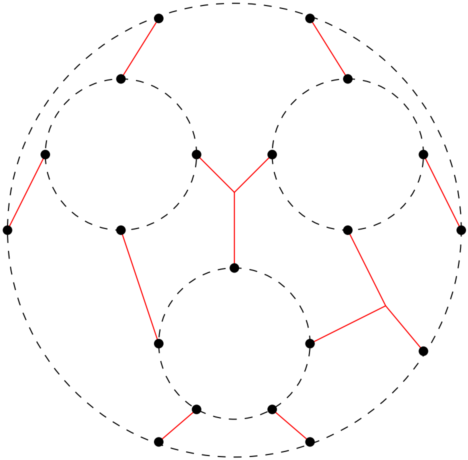
Gluing together circular planar graphs



Gluing with a planar wiring diagram

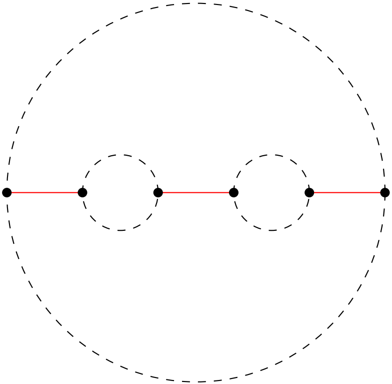


A connected planar wiring diagram

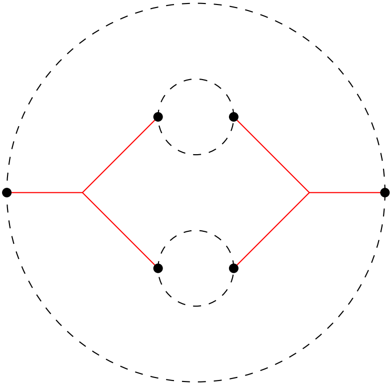


This is a *morphism* in the operad $\mathcal{P}lan$ of *connected planar wiring diagrams*.

Series and parallel wiring diagrams



Series



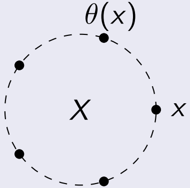
Parallel

The operad of connected planar wiring diagrams

Definition (B.—, 2019)

The (symmetric, coloured) operad $\mathcal{P}lan$:

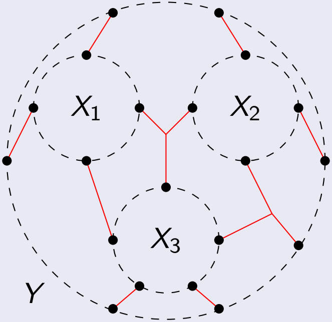
- objects are *circularly ordered finite sets* (X, θ).



The operad of connected planar wiring diagrams

Definition (continued)

- morphisms from $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ to (Y, φ) are planar wiring diagrams with the X_i on the inside and Y on the outside:



Every "cable" has ≤ 1 element of Y.

Every "face" has ≤ 1 arc of outer circle.

Lemma: This really does define an operad!

Relationships to other operads

$\mathcal{P}lan$ has:

- A forgetful map to Spivak's operad of all wiring diagrams. $\mathcal{P}lan$ inherits several algebras from there, like flows and potentials. However, Spivak's operad does not have a duality automorphism.
- An inclusion map to Jones's "planar algebras" operad. $\mathcal{P}lan$ inherits its duality automorphism Jones's "1-click" automorphism, but Jones's operad has too many morphisms for circular planar graphs to be an algebra.

$\mathcal{P}lan$-algebras

A $\mathcal{P}lan$-algebra $\mathcal{A}$ assigns:

- to each circularly ordered set (X, θ) a set $\mathcal{A}(X, \theta)$, and
- to each morphism $(X_1, \theta_1), \ldots, (X_n, \theta_n) \to (Y, \varphi)$ a function

$$\prod_{i=1}^{n} \mathcal{A}(X_i, \theta_i) \to \mathcal{A}(Y, \varphi).$$

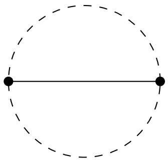
These describe *what* may be inserted into the slots of a wiring diagram and *how* they glue together.

$$\mathcal{P}lan \xrightarrow{\mathcal{A}} Op(\mathrm{Set})$$

Example $\mathcal{P}lan$ algebras: $\mathcal{G}$ and $\mathcal{G}_{(0,\infty)}$

- $\mathcal{G}(X, \theta) =$ set of connected circular planar graphs with boundary vertices (X, θ).
- $\mathcal{G}_{(0,\infty)}(X, \theta)$: same, but with edges weighted by positive real numbers.

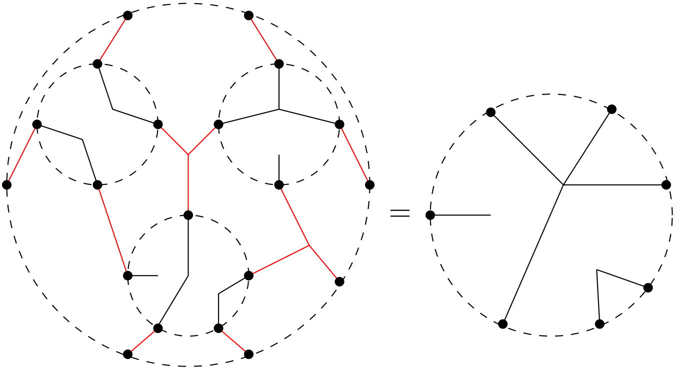
Lemma: $\mathcal{G}$ is generated by the single element



satisfying a single relation. That makes it easy to describe algebra morphisms out of $\mathcal{G}$!

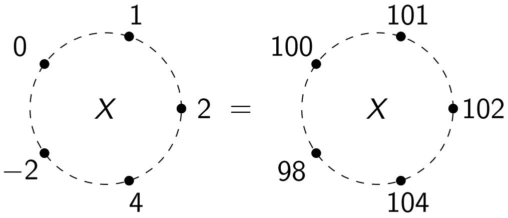
Example $\mathcal{P}lan$ algebra: Π

- $\Pi(X,\theta)$ = set of planar (noncrossing) partitions of (X,θ).



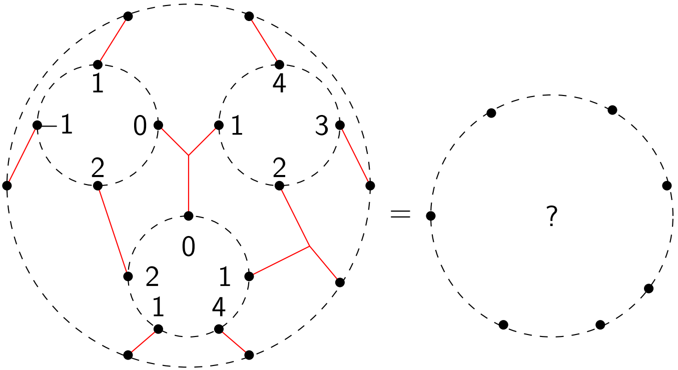
Example $\mathcal{P}lan$-algebra: potential sets

- A *potential* on (X, θ) is a function $X \to \mathbb{R}$ up to overall additive constant:

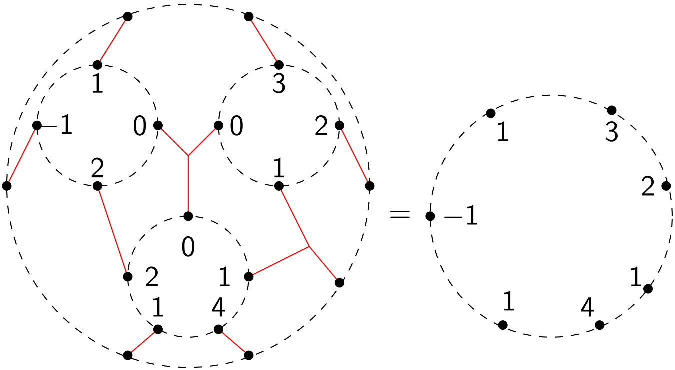


$\mathcal{V}(X, \theta)$ is the set of potentials on X.

Gluing compatible potentials



Gluing compatible potentials



Gluing potential sets

Not all potentials can be glued, so $\mathcal{V}$ is only a *relational* $\mathcal{P}$*lan*-algebra. But

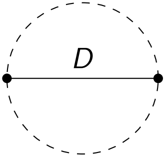
$$\mathcal{P}(\mathcal{V}) : (X, \theta) \mapsto \text{the set of subsets of } \mathcal{V}(X, \theta)$$

is an actual $\mathcal{P}$*lan*-algebra!

Send a collection of potential sets to the set of potentials obtained by gluing compatible members.

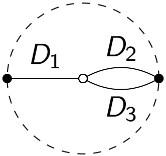
Min path length: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V})$

Weights on a graph $\rightsquigarrow$ distances $\rightsquigarrow$ potential set:



$\rightsquigarrow \{(a,b) \in \mathbb{R}^2 \mid |a-b| \leq D\}$

Min path length: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V})$

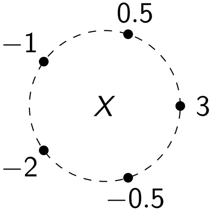


$$\rightsquigarrow \{(a,b) \in \mathbb{R}^2 \mid \exists x \in \mathbb{R} \text{ such that } |a-x| \le D_1, |x-b| \le D_2, |x-b| \le D_3\}$$
$$= \{(a,b) \in \mathbb{R}^2 \mid |a-b| \le D_1 + \min(D_2, D_3)\}$$

"Min path length" is an algebra morphism $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V})$!

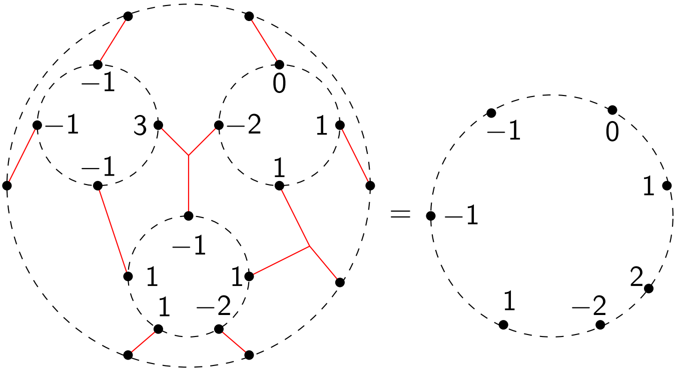
Example $\mathcal{P}lan$-algebras: Flow Sets

- The algebra $\mathcal{P}(\mathcal{F})$ of *flow sets*: A *flow* on (X, θ) is a sum-zero function $X \to \mathbb{R}$:



$\mathcal{F}(X, \theta) =$the set of flows on X.

Gluing compatible flows



Gluing flow sets

Not all flows can be glued, so $\mathcal{F}$ is only a *relational* $\mathcal{P}$*lan*-algebra. But

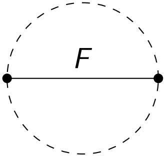
$$\mathcal{P}(\mathcal{F}) : (X, \theta) \mapsto \text{the set of subsets of } \mathcal{F}(X, \theta)$$

is an actual $\mathcal{P}$*lan*-algebra!

Send a collection of flow sets to the set of flows obtained by gluing compatible members.

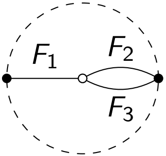
Max flow rate: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{F})$

Weights on a graph $\rightsquigarrow$ possible flows:



$\rightsquigarrow \{(a, -a) \in \mathbb{R}^2 \mid |a| \leq F\}$

Max flow rate: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{F})$



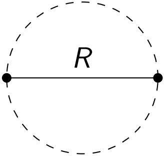
$$\rightsquigarrow \{(a,b) \in \mathbb{R}^2 \mid \exists x, y, z \in \mathbb{R} \text{ such that } |x| \le F_1,\ |y| \le F_2,\ |z| \le F_3,\ x = a,\ y + z = b,\ x + y + z = 0\}$$

$$= \{(a, -a) \in \mathbb{R}^2 \mid |a| \le \min(F_1, F_2 + F_3)\}$$

"Max flow rate" is a $\mathcal{P}lan$-algebra morphism $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{F})$!

Resistance: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V} \times \mathcal{F})$

Weights on a graph $\rightsquigarrow$ voltage-current relationships:



$$\rightsquigarrow \left\{ \left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \begin{bmatrix} I \\ -I \end{bmatrix} \right) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; (V_1 - V_2) = IR \right\}$$

In general, send a weighted graph to the set of pairs (boundary voltages, induced boundary currents). "Effective resistance" is a $\mathcal{P}lan$-algebra morphism $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V} \times \mathcal{F})$!

Duality automorphism of $\mathcal{P}lan$

$\mathcal{P}lan$ has a "duality" automorphism

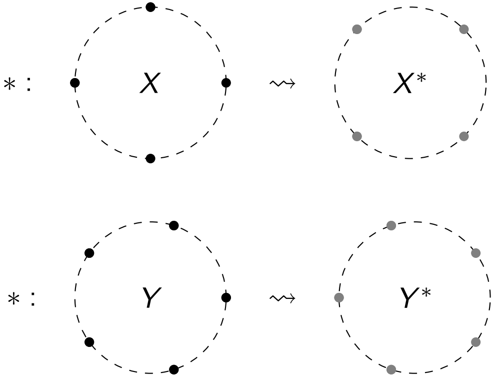
$$* : \mathcal{P}lan \to \mathcal{P}lan$$

Compose with any algebra $\mathcal{A} : \mathcal{P}lan \to \mathcal{O}p(\text{Set})$ to get a "dual" algebra

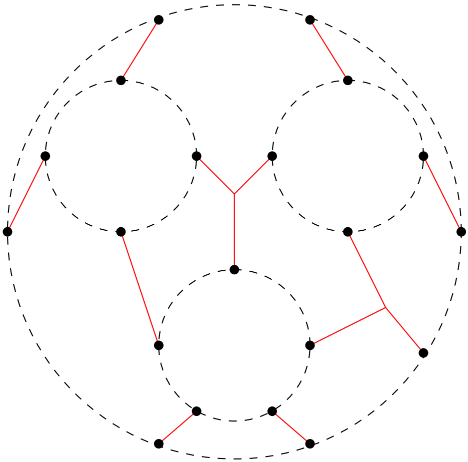
$$\mathcal{A}^* : \mathcal{P}lan \xrightarrow{*} \mathcal{P}lan \xrightarrow{\mathcal{A}} \mathcal{O}p(\text{Set}).$$

$$\mathcal{A}^{**} \cong \mathcal{A}.$$

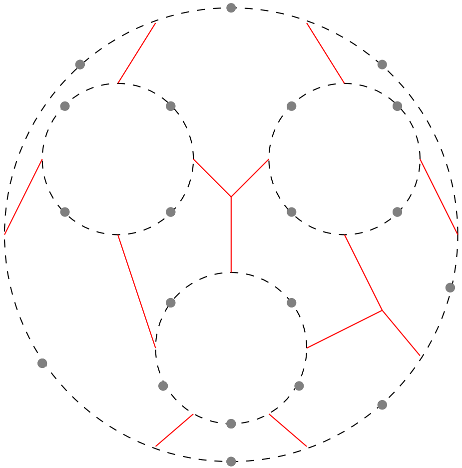
Duality automorphism of $\mathcal{P}lan$: on objects



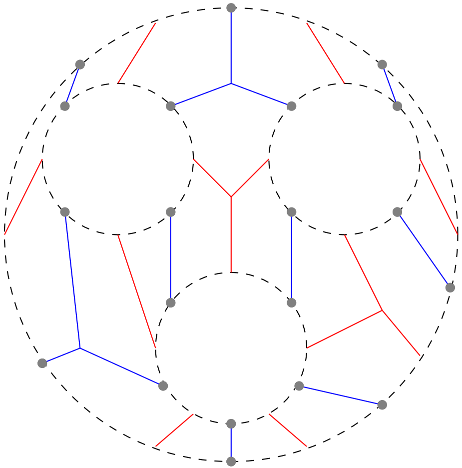
Duality automorphism of $\mathcal{P}lan$: on morphisms



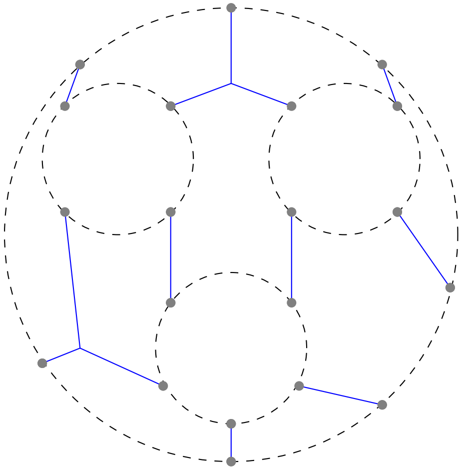
Duality automorphism of $\mathcal{P}lan$: on morphisms



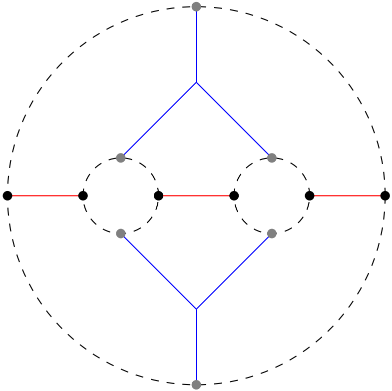
Duality automorphism of $\mathcal{P}lan$: on morphisms



Duality automorphism of $\mathcal{P}lan$: on morphisms

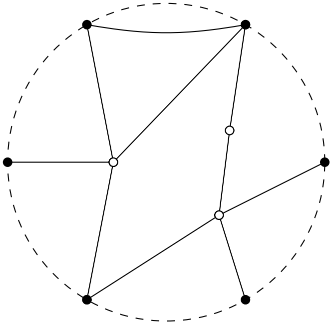


Series and parallel wiring diagrams are dual

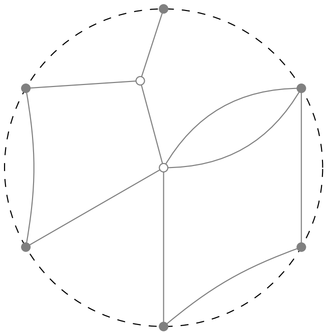


Dual of $\mathcal{G}$ is $\mathcal{G}$

$\mathcal{G}$ is isomorphic to its own dual: the isomorphism sends a graph to its dual.



$\rightsquigarrow$



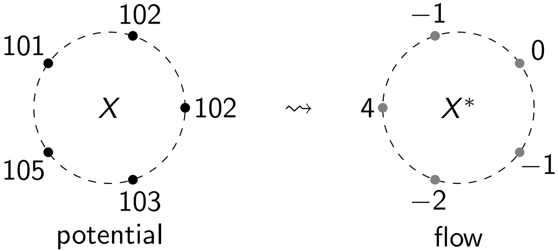
The same holds for $\mathcal{G}_{(0,\infty)}$.

Dual of $\mathcal{V}$ is $\mathcal{F}$

Theorem (B.—, 2019)

The dual of the algebra of potentials is the algebra of flows: $\mathcal{V}^* \cong \mathcal{F}$.

Take successive differences of potential values to obtain a flow:



Corollary: $\mathcal{P}(\mathcal{V})^* \cong \mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{V} \times \mathcal{F})^* \cong \mathcal{P}(\mathcal{V} \times \mathcal{F})$ as well.

Dual of min path length is max flow rate

Theorem (B.—, 2019)

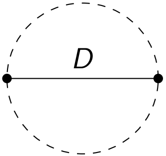
The dual of the "min path length" morphism is the "max flow rate" morphism: the square

$$\text{(min path length)}^* \quad \begin{array}{ccc} \mathcal{G}^*_{(0,\infty)} & \cong & \mathcal{G}_{(0,\infty)} \\ \downarrow & & \downarrow \\ \mathcal{P}(\mathcal{V})^* & \cong & \mathcal{P}(\mathcal{F}) \end{array} \quad \text{(max flow)}$$

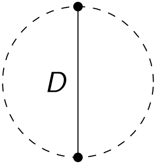
commutes.

Dual of min path length is max flow rate

Proof sketch: only have to check for single weighted edges!



$\leftrightarrow$



$\rightsquigarrow$ $\qquad$ $\rightsquigarrow$

$$\{(a,b) \mid |a-b| \leq D\} \quad \leftrightarrow \quad \{(a,-a) : |a| \leq D\} = \{(a-b, b-a) : |a-b| \leq D\}$$

Dual of resistance is conductance

Theorem (B.—, 2019)

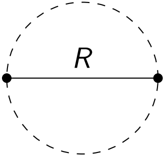
The dual of the "effective resistance" morphism is "effective 1/resistance": the square

$$\begin{array}{ccccc} & \mathcal{G}^*_{(0,\infty)} & \cong & \mathcal{G}_{(0,\infty)} & \\ (\textit{effective resistance})^* & \downarrow & & \downarrow & (\textit{effective resistance}) \\ & \mathcal{P}(\mathcal{V} \times \mathcal{F})^* & \cong & \mathcal{P}(\mathcal{V} \times \mathcal{F}) & \end{array}$$

commutes, where the isomorphism $\mathcal{G}^*_{(0,\infty)} \cong \mathcal{G}_{(0,\infty)}$ *dualizes the graph and inverts the edge weights.*

Dual of resistance is conductance

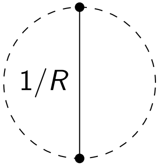
Proof sketch:



$$\rightsquigarrow \left\{ \left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \begin{bmatrix} I \\ -I \end{bmatrix} \right) \;\middle|\; (V_1 - V_2) = IR \right\}$$

$$\leftrightarrow \left\{ \left(\begin{bmatrix} c + I \\ c \end{bmatrix}, \begin{bmatrix} V_1 - V_2 \\ V_2 - V_1 \end{bmatrix} \right) \;\middle|\; (V_1 - V_2) = IR \right\}$$

$$=$$



$$\rightsquigarrow \left\{ \left(\begin{bmatrix} V_1' \\ V_2' \end{bmatrix}, \begin{bmatrix} I' \\ -I' \end{bmatrix} \right) \;\middle|\; (V_1' - V_2') = I'/R \right\}$$

Thank you!