Bisimulation maps in presheaf categories

Harsh Beohar[1] and Sebastian Küpper[2]

[1]University of Duisburg-Essen

harsh.beohar@uni-due.de

[2]FernUniversität in Hagen

sebastian.kuepper@fernuni-hagen.de

Motivation

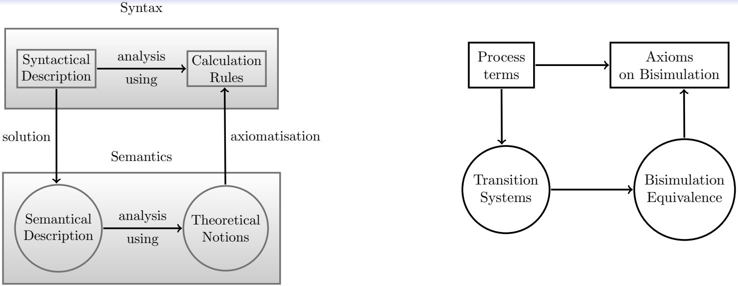

Figure: Mathematical Modelling (Cuijpers 2004).

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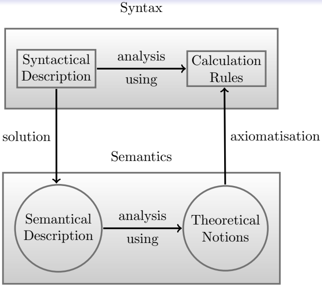

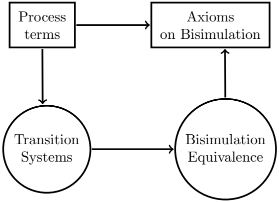

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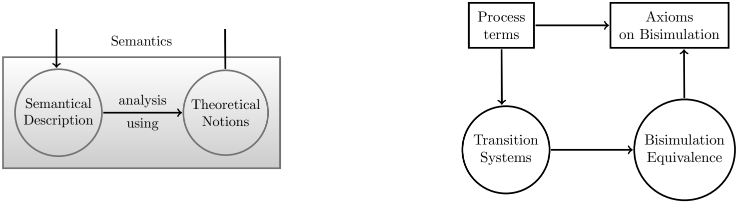

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Objectives

- Find an "abstract" semantic universe to study behaviour of dynamical systems.
- What is refinement of behaviour?
- Can we study bisimulations in an abstract way?

In this talk...

- Presheaves as semantic models of dynamical system.

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- Refinement of behaviour (simulation) as presheaf morphism.
- Bisimulation maps are special presheaf morphisms.

$\mathbf{PSh}(A^\star)$	strong bisimulation
$\mathbf{PSh}(A^\infty)$	$\forall$-fair bisimulation
$\mathbf{PSh}(A^\star_{\neq\tau})$	branching bisimulation

Recall that

- $A^\star$ is the poset of finite words.
- A^ω is the poset of infinite words.
- $A^\infty = A^\star \cup A^\omega$.

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What about observation?

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- For each $t \in \mathbf{T}$, $\mathcal{O}(t)$ is a set of 'plausible' observations.

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The existence of a hypothetical 'observer' $\mathcal{O}$.

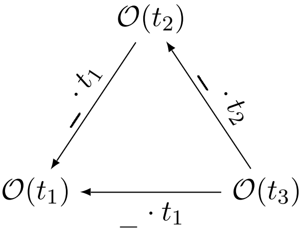
- For each $t \in \mathbf{T}$, $\mathcal{O}(t)$ is a set of 'plausible' observations.
- *Earlier observations can be deduced from the later observations*, i.e.,

  for $t \preceq t'$ there is a 'restriction' $\mathcal{O}(t) \xleftarrow{\cdot t} \mathcal{O}(t')$

  intuition: if x is observed at t' then $x \cdot t$ was observed at t.

The restriction $\cdot$ satisfies the following axioms:

1. if $t = t'$ then $\_ \cdot t$ is an identity on $\mathcal{O}(t)$.
2. if $t_1 \preceq t_2 \preceq t_3$ then the triangle commutes:



In hindsight

$\mathcal{O}$ is a presheaf, i.e., a functor $\mathbf{T}^{\mathsf{op}} \longrightarrow \mathbf{Set}$.

Example

Labelled transition system (LTS)

- LTS is a triple $(X, A, \rightarrow)$, where $\rightarrow \subseteq X \times A \times X$ is often called the transition relation.

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Labelled transition system (LTS)

- LTS is a triple $(X, A, \to)$, where $\to \subseteq X \times A \times X$ is often called the transition relation.
- $\mathbf{T}$ is the set of natural numbers $\mathbb{N}$ ordered by $\leq$.
- For a given alphabet A, we define a presheaf $\mathcal{A} \in \mathbf{PSh}(\mathbb{N})$:

$$\mathcal{A}(n) = \{\sigma \in A^\star \mid |\sigma| = n\} \qquad \text{(for every } n \in \mathbb{N}\text{)},$$

together with the restriction on $\mathcal{A}$ given by

$$\sigma \cdot n = \sigma|_n \quad \text{(for every } \sigma \in \mathcal{A}(n') \text{ and } n \leq n'\text{)}.$$

Towards a formal definition

Intuitively

For a fixed $\mathbf{T}$ and $\mathcal{O} \in \mathbf{PSh}(\mathbf{T})$, a dynamical system describes:

- What are the *runs* (aka trajectories/executions) of the system?
- What is the observation associated with each run?

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  - *Model as a presheaf map, i.e., a family* $F(t) \xrightarrow{\alpha_t} \mathcal{O}(t)$

$$\begin{array}{ccccc} t' & & F(t') & \xrightarrow{\alpha_{t'}} & \mathcal{O}(t') \\ \geq & & {\_ \cdot t} \downarrow & & \downarrow {\_ \cdot t} \\ t & & F(t) & \xrightarrow{\alpha_t} & \mathcal{O}(t) \end{array}$$

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Categorical definition

Dynamical systems are objects in $\mathbf{PSh}(\mathbf{T})/\mathcal{O}$.

‘Conceptual’ Simplification

Idea

Time can be made inherent with observation.

Definition (Category of Elements)

Given a presheaf $F \in \mathbf{PSh}(\mathbf{T})$, define $\mathbb{E}(F)$

- Elements are tuples (x, t) with $x \in F(t)$ and $t \in \mathbf{T}$.
- $(x, t) \preceq (x', t') \iff t \preceq t' \wedge x' \cdot t = x$.

Theorem

For any $F \in \mathbf{PSh}(\mathbf{T})$ *we have* $\mathbf{PSh}(\mathbf{T})/F \cong \mathbf{PSh}(\mathbb{E}(F))$.

Application

Recall the presheaf $\mathcal{A} \in \mathbf{PSh}(\mathbb{N})$ induced by an alphabet A.

1. Elements are (σ, n) with $\sigma \in \mathcal{A}(n)$.
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Clearly, $\mathbb{E}(\mathcal{A}) \cong A^\star$. Thus, presheaves over $A^\star$ are suitable for LTSs (cf. Winskel and his colleagues). I.e.,

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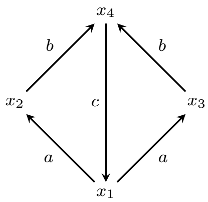

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$$[\![X]\!](\varepsilon) = \Big\{\{\varepsilon \mapsto x_i\} \mid i \in \{1,2,3,4\}\Big\}$$

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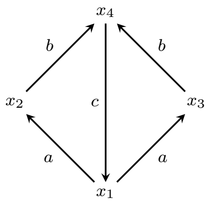

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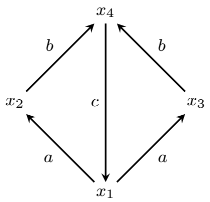

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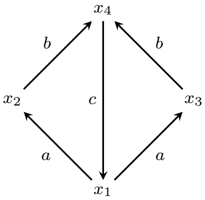

SEMANTICS OF LTSS

OBJECTS

Given an LTS $(X, A, \to)$, then we define $[\![X]\!] \in \mathbf{PSh}(A^\star)$:

$$[\![X]\!](\sigma) = \left\{ \downarrow \sigma \xrightarrow{p} X \mid \forall_{\sigma',a} \left(\sigma' a \preceq \sigma \implies p(\sigma') \xrightarrow{a} p(\sigma' a) \right) \right\}$$

$$p \cdot \sigma = p|_{\downarrow\sigma} \quad \left(\text{for any } \sigma \preceq \sigma' \text{ and } p \in [\![X]\!](\sigma')\right).$$

MORPHISMS?

Given two LTSs $(X, A, \to), (Y, A, \to)$ and a function $X \xrightarrow{f} Y$, then we have a family of functions (for $\sigma \in A^\star$):

$$[\![X]\!](\sigma) \xrightarrow{[\![f]\!]_\sigma} [\![Y]\!](\sigma)$$

Semantics of LTSs

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Morphisms?

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$$[\![X]\!](\sigma) \xrightarrow{[\![f]\!]_\sigma} [\![Y]\!](\sigma) \qquad p \mapsto f \circ p$$

When is $[\![f]\!]$ a presheaf map?

Simulation maps

Theorem (Winskel et al.)

Given a simulation function $X \xrightarrow{f} Y$*, i.e.,*

$$\forall_{x,x',a}\ x \xrightarrow{a} x' \implies f(x) \xrightarrow{a} f(x'),$$

then $[\![f]\!]$ *is a presheaf map, i.e., the following square commutes*

$$\begin{array}{ccc}
[\![X]\!](\sigma') & \xrightarrow{\quad [\![f]\!]_{\sigma'} \quad} & [\![Y]\!](\sigma') \\
{\scriptstyle _ \cdot \sigma}\big\downarrow & & \big\downarrow{\scriptstyle _ \cdot \sigma \text{ (for } \sigma \preceq \sigma' \text{)}} \\
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Conversely, a presheaf map $[\![f]\!]$ *implies that* f *is a simulation map.*

Bisimulation maps

Definition

A map $F \xrightarrow{f} G \in \mathbf{PSh}(\mathbf{C})$ is a *bisimulation* iff for every commutative square with a mono $P \xhookrightarrow{g} Q$ (each g_C is injective) and maps m, n

$$\begin{array}{ccc} Q & \xrightarrow{n} & G \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f} \\ P & \xrightarrow[m]{} & F \end{array}$$

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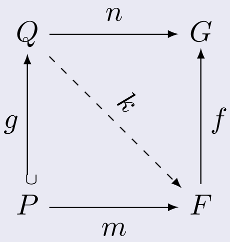

there exists a map $Q \xrightarrow{k} F$ such that the two triangles commute.

Bisimulation maps

Theorem (Complete refinement)

Every bisimulation map $F \xrightarrow{f} G \in \mathbf{PSh}(\mathbf{C})$ *is a retract, i.e.,*

$$\begin{array}{ccc} & F & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\ G & \xrightarrow[\mathit{id}_G]{} & G \end{array}$$

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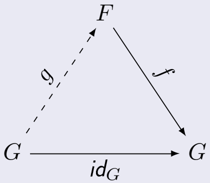

Theorem

Given a simulation function $X \xrightarrow{f} Y$, *then* $[\![f]\!]$ *is a bisimulation map in* $\mathbf{PSh}(A^\star)$ *iff the function* f *is a surjection satisfying:*

$$\forall_{x \in X, y \in Y}\ \left(f(x) \xrightarrow{a} y \implies \exists_{x' \in X}\ (x \xrightarrow{a} x' \wedge f(x') = y)\right).$$

Concluding remarks

- Presheaves are suitable for modelling runs of dynamical systems. To define semantics to a category of models $\mathbf{M}$:
  - Fix a notion of time $\mathbf{T}$ and observer $\mathcal{O} \in \mathbf{PSh}(\mathbf{T})$.
  - Simplify using the category of elements $\mathbb{E}(\mathcal{O})$.
  - Define a 'semantics' functor $\mathbf{M} \xrightarrow{[\![\_]\!]} \mathbf{PSh}(\mathbb{E}(\mathcal{O}))$.
- Presheaves morphisms encodes refinement of behaviour.
- Bisimulation maps:

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- Future work: presheaf semantics of hybrid systems.

Thank You

Fairness

Syntax

- $(X, A, \rightarrow, \mathsf{Fair}_X)$ where Fair_X is a predicate on infinite executions.
- An infinite execution $\downarrow \sigma \xrightarrow{p} X \cup \{\Omega\}$ with $\sigma \in A^\omega$ s.t.
  - $\forall_{\sigma',a} \left(\sigma' a \preceq \sigma \implies p(\sigma') \xrightarrow{a} p(\sigma' a)\right)$.
  - $p(\sigma) = \Omega$.

Semantics

- Time:
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Just like earlier, we have $\mathbb{E}(\mathcal{O}) \cong A^\infty$.

Fair simulation maps

Definition

Given a function $X \xrightarrow{f} Y$, then a chaos preserving extension $X \cup \{\Omega\} \xrightarrow{f_\Omega} Y \cup \{\Omega\}$ ( ) of f is a *fair simulation* iff

1. $\forall_{x,x',a}\; x \xrightarrow{a} x' \implies f(x) \xrightarrow{a} f(x')$.
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Henceforth, we do not distinguish f_Ω, f.

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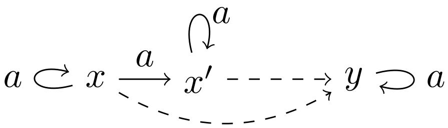

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Theorem

Given a fair simulation function $X \xrightarrow{f} Y$ then $[\![X]\!] \xrightarrow{[\![f]\!]} [\![Y]\!]$ is a presheaf map in $\mathbf{PSh}(A^\infty)$, where $[\![X]\!](\sigma)$ is the set of fair executions whose trace is $\sigma \in A^\infty$.

Fair bisimulation maps

Definition

A *fair bisimulation* $X \cup \{\Omega\} \xrightarrow{f} Y \cup \{\Omega\}$ is a fair simulation s.t.

1. f is surjective and $\forall_{x \in X, y \in Y} \left(f(x) \xrightarrow{a} y \implies \exists_{x' \in X} \; (x \xrightarrow{a} x' \wedge f(x') = y) \right)$.
2. for any increasing sequence of finite executions $(p_i)_{i \in \mathbb{N}}$:

$$f \circ \bigsqcup_{i \in \mathbb{N}} p_i \approx \bigsqcup_{i \in \mathbb{N}} f \circ p_i .$$

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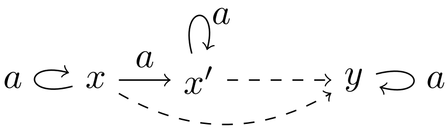

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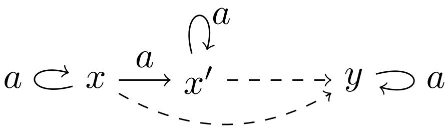

There is a sequence $p_0, p_1, p_2, \cdots$

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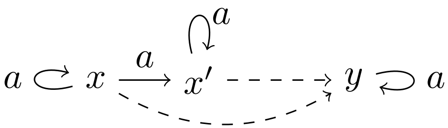

There is a sequence $p_0, p_1, p_2, \cdots$ such that $\bigsqcup_{i\in\mathbb{N}} f \circ p_i$ exists; however $\bigsqcup_{i\in\mathbb{N}} p_i$ does not exists.

Fair bisimulation maps

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Theorem

A fair simulation function f is a fair bisimulation if, and only if, the underlying map $[\![f]\!]$ is a bisimulation map in $\mathbf{PSh}(A^\infty)$.

Fair bisimulation maps

Definition

A *fair bisimulation* $X \cup \{\Omega\} \xrightarrow{f} Y \cup \{\Omega\}$ is a fair simulation s.t.

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Theorem

Two states x and x' are related by a $\forall$-fair bisimulation relation iff there is a fair bisimulation function f such that $f(x) = f(x')$.

∀-fair bisimulation relation

Definition

A *∀-fair bisimulation* on $(X, A, \to, \mathsf{Fair}_X)$ is an equivalence relation $\mathcal{R} \subseteq X \times X$ satisfying the following transfer properties:

1. $\forall_{x,y,x',a} \left((x \xrightarrow{a} x' \wedge x\mathcal{R}y) \implies \exists_{y'} \; (y \xrightarrow{a} y' \wedge x'\mathcal{R}y')\right)$, and
2. $\forall_{p,q} \left((p =_{\mathcal{R}} q \wedge p \in \mathsf{Fair}_X) \implies q \in \mathsf{Fair}_X\right)$.

Here, $p =_{\mathcal{R}} q \iff \mathsf{dom}(q) = \mathsf{dom}(p) \wedge \forall_{\sigma \in \mathsf{dom}(p) \cap A^\star} \; p(\sigma)\mathcal{R}q(\sigma)$.

Theorem

Two states x and x' are related by a ∀-fair bisimulation relation iff there is a fair bisimulation function f such that $f(x) = f(x')$.

$\forall$-fair bisimulation relation

Definition

A $\forall$-*fair bisimulation* on $(X, A, \rightarrow, \mathsf{Fair}_X)$ is an equivalence relation $\mathcal{R} \subseteq X \times X$ satisfying the following transfer properties:

1. $\forall_{x,y,x',a} \left((x \xrightarrow{a} x' \wedge x\mathcal{R}y) \implies \exists_{y'} \left(y \xrightarrow{a} y' \wedge x'\mathcal{R}y'\right)\right)$, and
2. $\forall_{p,q} \left((p =_{\mathcal{R}} q \wedge p \in \mathsf{Fair}_X) \implies q \in \mathsf{Fair}_X\right)$.

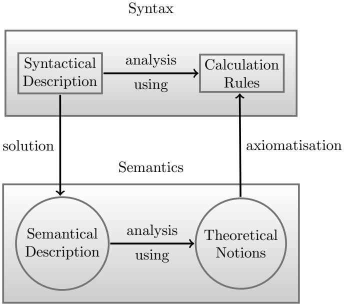
Here, $p =_{\mathcal{R}} q \iff \mathsf{dom}(q) = \mathsf{dom}(p) \wedge \forall_{\sigma \in \mathsf{dom}(p) \cap A^{\star}} \, p(\sigma)\mathcal{R}q(\sigma)$.

Remark

Every $\forall$-fair bisimulation relation is an equivalence (only symmetric requirement is assumed in the temporal logic literature) is not superfluous because these *relations are not closed under union and relational composition*.

Future work

Finding models for process theories



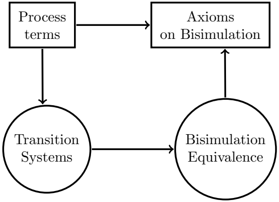


Figure: Mathematical Modelling (Cuijpers 2004).

Future Work

- For any geometric theory $\mathbb{T}$ there is a classifying topos $\mathbf{B}(\mathbb{T})$:

$$\mathsf{Geom}(\mathbf{E}, \mathbf{B}(\mathbb{T})) \cong \mathbb{T}\text{-mod}(\mathbf{E}).$$

- There is a universal model living in $U \in \mathbb{T}\text{-mod}(\mathbf{B}(\mathbb{T}))$.
- If $\mathbb{T} = \mathbb{T}_{\mathsf{BSP}}$, then $U = (\mathcal{C}(\mathsf{BSP})/\vdash, [0]_\vdash, [1]_\vdash, +_\vdash, \cdots)$.
- In process algebra, 'term model' means $(\mathcal{C}(\mathsf{BSP})/\Leftrightarrow, [0]_\Leftrightarrow, [1]_\Leftrightarrow, +_\Leftrightarrow, \cdots)$ isomorphic to U.

$x + y = y + x$	A1	$x + 0 = x$	A6
$(x + y) + z = x + (y + z)$	A2	$0 \cdot x = 0$	A7
$x + x = x$	A3	$x \cdot 1 = x$	A8
$(x + y) \cdot z = x \cdot z + y \cdot z$	A4	$1 \cdot x = x$	A9
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	A5	$a.x \cdot y = a.(x \cdot y)$	A10

Figure: Basic Sequential Processes (cf. Baeten et al. 2010)