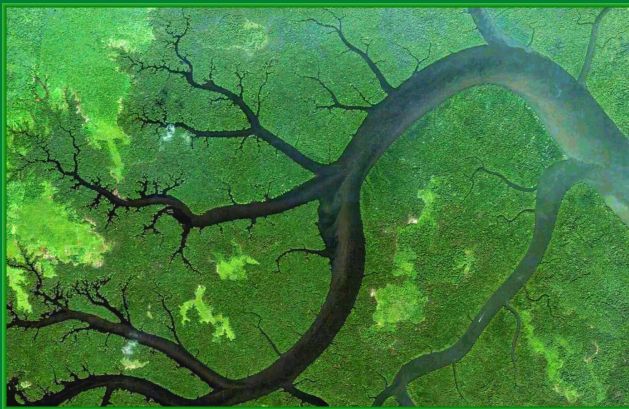
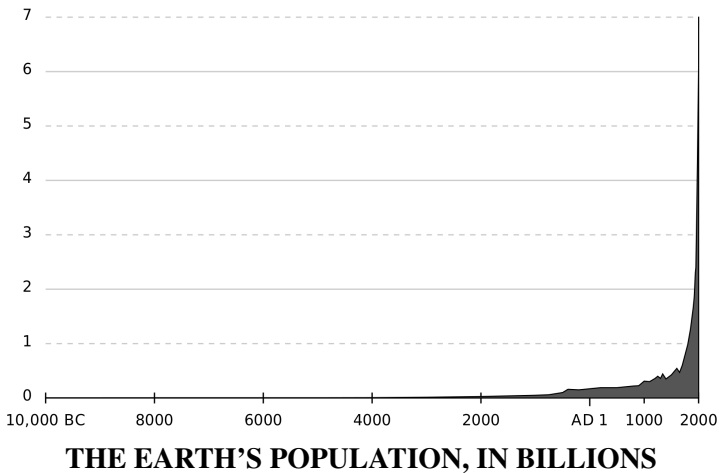


Props in Network Theory

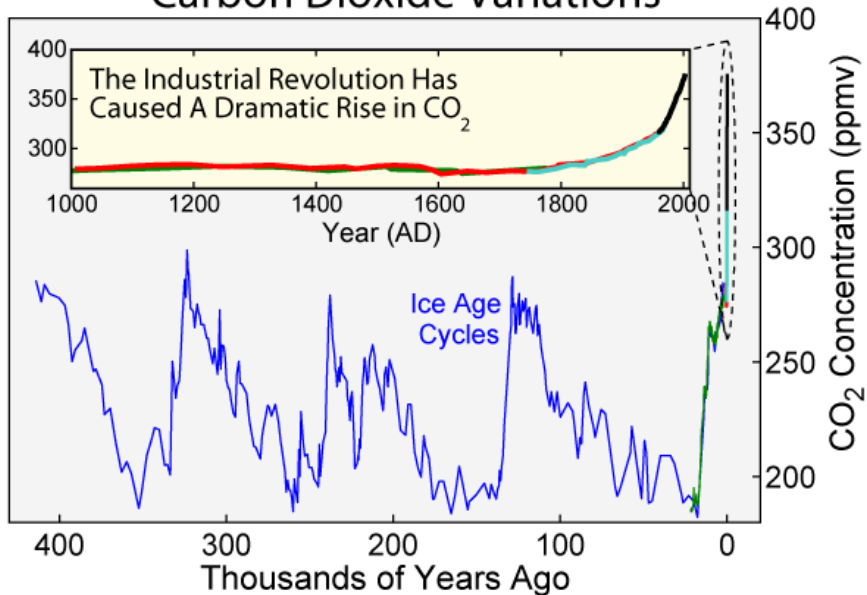


John Baez
SYCO4, Chapman University
22 May 2019

We have left the Holocene and entered a new epoch, the [Anthropocene](#), in which the biosphere is rapidly changing due to human activities.



Carbon Dioxide Variations



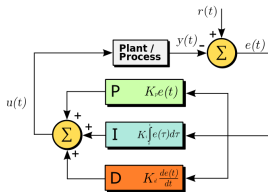
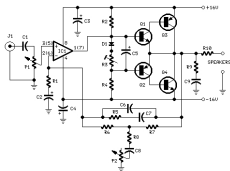
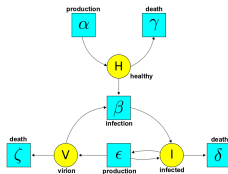
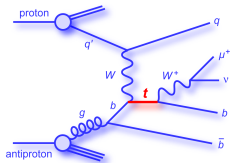
Climate change is not an isolated ‘problem’ of the sort routinely ‘solved’ by existing human institutions. It is part of a shift from the exponential growth phase of human impact on the biosphere to a new, uncharted phase.

- ▶ About 1/4 of all chemical energy produced by plants is now used by humans.
- ▶ Humans now take more nitrogen from the atmosphere and convert it into nitrates than all other processes combined.
- ▶ 8-9 times as much phosphorus is flowing into oceans than the natural background rate.
- ▶ The rate of species going extinct is 100-1000 times the usual background rate.
- ▶ Populations of ocean fish have declined 90% since 1950.

So, we can expect that in this century, scientists, engineers and mathematicians will be increasingly focused on *biology*, *ecology* and *complex networked systems* — just as the last century was dominated by physics.

What can category theorists contribute?

One thing category theorists can do: *understand networks*.



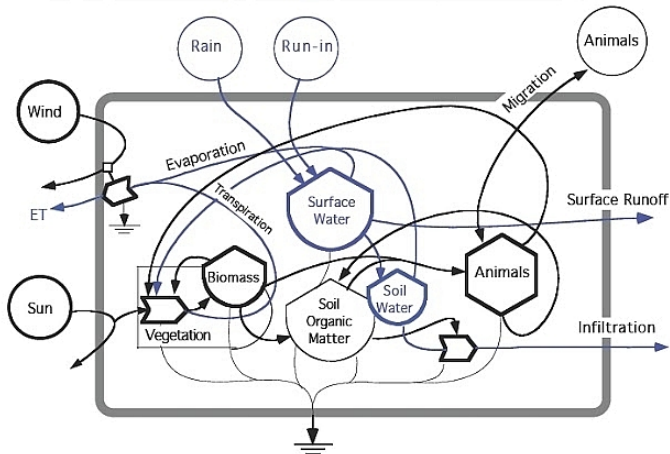
We need a good general theory of these. It will require category theory.

To understand ecosystems, ultimately will be to understand networks. — B. C. Patten and M. Witkamp



I believe biology proceeds at a *higher level of abstraction* than physics, so it calls for new mathematics.

Back in the 1950's, Howard Odum introduced an **Energy Systems Language** for ecology:

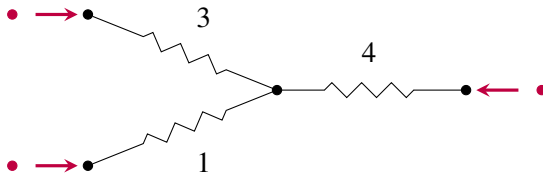


Maybe we are finally ready to develop these ideas.

The dream: each different kind of network or open system should be a morphism in a different symmetric monoidal category.

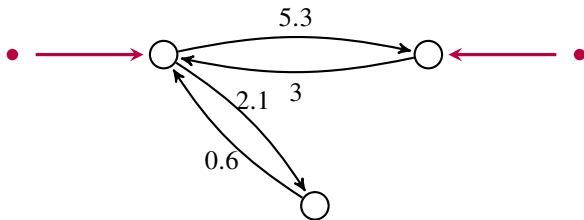
Some examples:

- ▶ ResCirc, where morphisms are circuits of resistors with inputs and outputs:



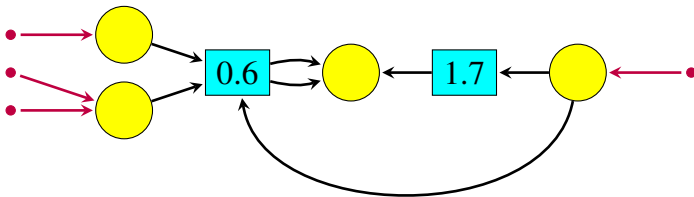
These, and many variants, are important in electrical engineering.

- ▶ Markov, where morphisms are **open Markov processes**:



These help us model stochastic processes: technically, they describe continuous-time finite-state Markov chains with inflows and outflows.

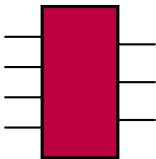
- ▶ RxNet, where morphisms are **open reaction networks with rates**:



Also known as **open Petri nets with rates**, these are used in chemistry, population biology, epidemiology etc. to describe changing populations of interacting entities.

All these examples can be seen as **props**: strict symmetric monoidal categories whose objects are natural numbers, with addition as tensor product.

A morphism $f: 4 \rightarrow 3$ in a prop can be drawn this way:



FinCospan

Steve Lack,

Composing PROPs

$\text{FinCospan} \longrightarrow \text{FinCorel}$

Brandon Coya & Brendan Fong,
*Corelations are the prop for extraspecial
commutative Frobenius monoids*

$\text{Circ} \longrightarrow \text{FinCospan} \longrightarrow \text{FinCorel}$

R. Rosebrugh, N. Sabadini & R. F. C. Walters

*Generic commutative separable algebras and cospan
of graphs*

Circ \longrightarrow FinCospan \longrightarrow FinCorel \longrightarrow LagRel

JB & Brendan Fong,
*A compositional framework for passive
linear circuits*

JB, Brandon Coya & Franciscus Rebro,
Props in network theory

Circ \longrightarrow FinCospan \longrightarrow FinCorel \longrightarrow LagRel

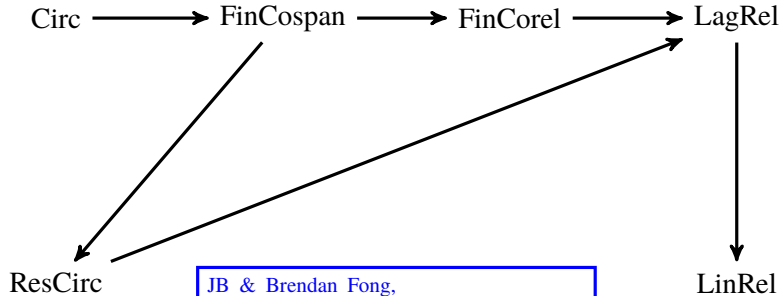
Filippo Bonchi, Pawel Sobocinski &
Fabio Zanasi,

Interacting Hopf algebras

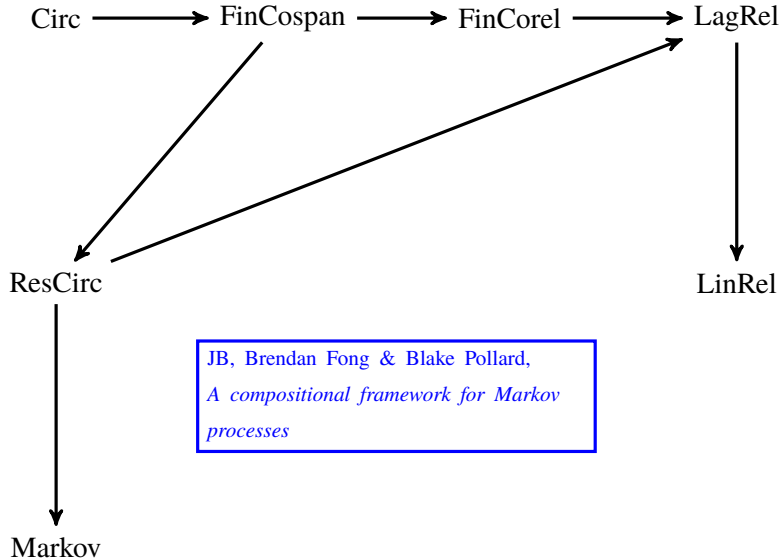
JB & Jason Erbele,

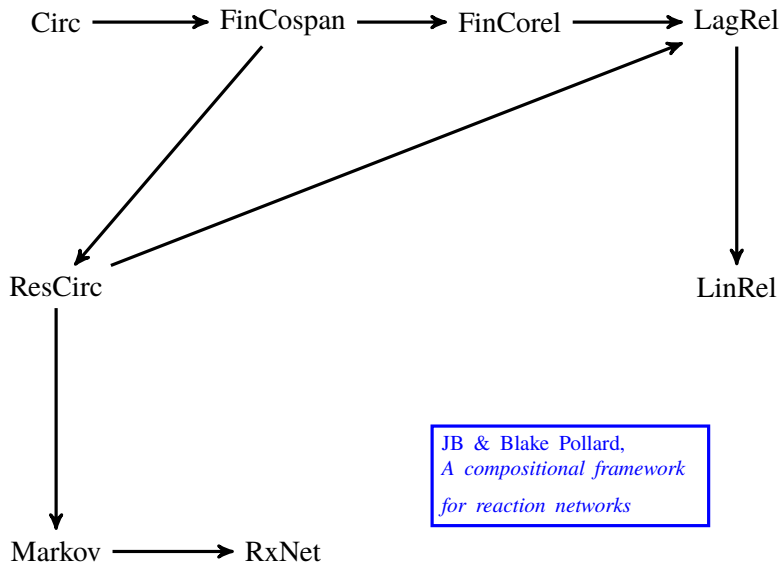
Categories in control

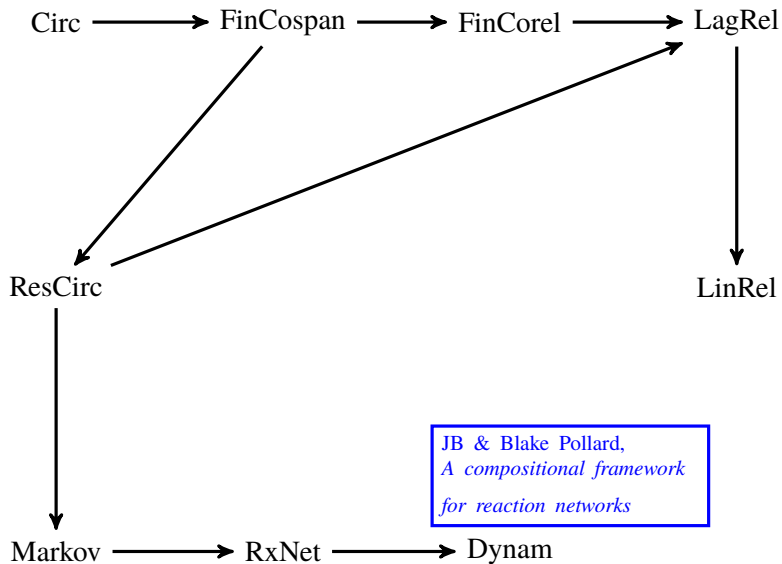
LinRel

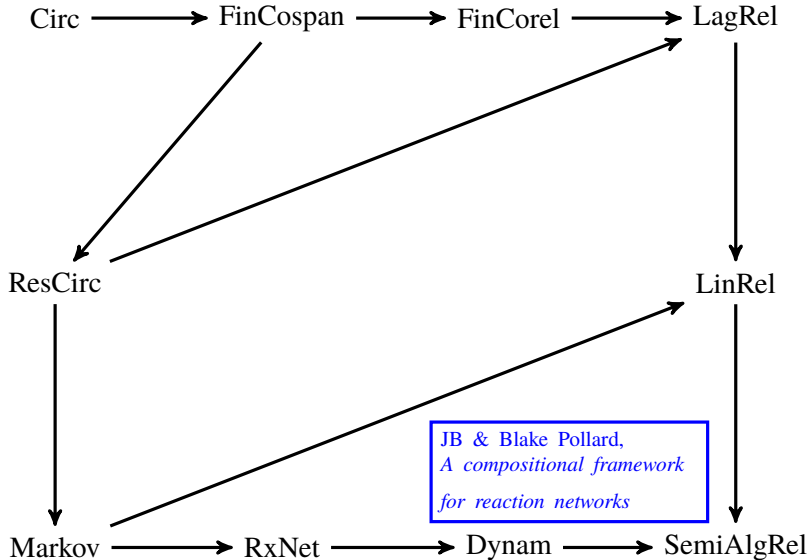


JB & Brendan Fong,
*A compositional framework for passive
linear circuits*









Let's look at a little piece of this picture:

$$\text{Circ} \xrightarrow{G} \text{FinCospan} \xrightarrow{H} \text{FinCorel} \xrightarrow{K} \text{LagRel}$$

The composite sends any circuit made just of purely conductive wires

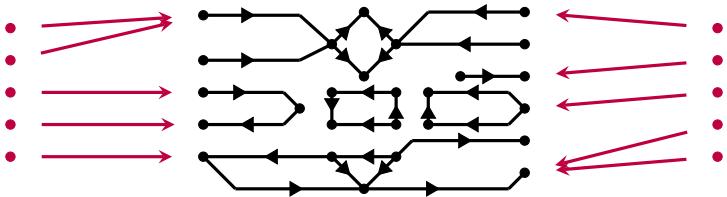
$$f: m \rightarrow n$$

to the linear relation

$$KHG(f) \subseteq \mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$$

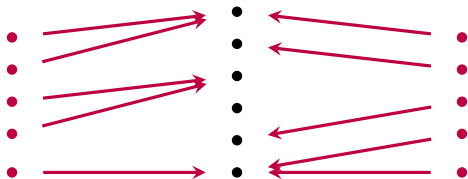
that this circuit establishes between the potentials and currents at its inputs and outputs.

In the prop Circ , a morphism looks like this:



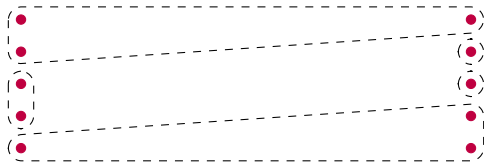
We can use such a morphism to describe an electrical circuit made of purely conductive wires.

In the prop FinCospan, a morphism looks like this:



We can use such a morphism to say which inputs and outputs lie in which connected component of our circuit.

In the prop FinCorel, a morphism looks like this:



Here a morphism $f: m \rightarrow n$ is a **corelation**: a partition of the set $m + n$. We can use such a morphism to say which inputs and outputs are connected to which others by wires.

In the prop LagRel, a morphism $L: m \rightarrow n$ is a **Lagrangian linear relation**

$$L \subseteq \mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$$

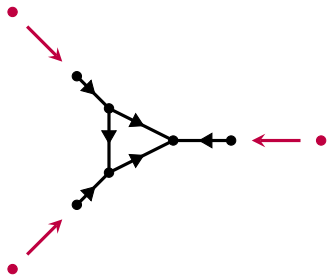
that is, a linear subspace of dimension $m + n$ such that

$$\omega(v, w) = 0 \text{ for all } v, w \in L.$$

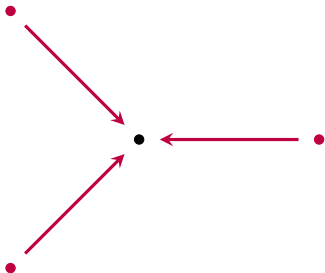
Here ω is a well-known bilinear form on $\mathbb{R}^{2m} \oplus \mathbb{R}^{2n}$, called a “symplectic structure”.

Remarkably, any circuit made of purely conductive wires establishes a linear relation between the potentials and currents at its inputs and its outputs that is *Lagrangian*!

A morphism $f: 2 \rightarrow 1$ in Circ:

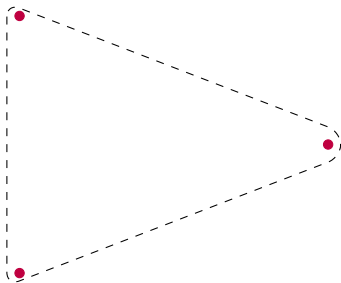


The morphism $G(f): 2 \rightarrow 1$ in $\mathbf{FinCospan}$:



$$\mathbf{Circ} \xrightarrow{G} \mathbf{FinCospan}$$

The morphism $HG(f): 2 \rightarrow 1$ in $\mathbf{FinCorel}$:



$$\mathbf{Circ} \xrightarrow{G} \mathbf{FinCospan} \xrightarrow{H} \mathbf{FinCorel}$$

The morphism $L = KHG(f): 2 \rightarrow 1$ in LagRel :

$$L = \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) : \phi_1 = \phi_2 = \phi_3, I_1 + I_2 = I_3\}$$

$$(\phi_1, I_1) \bullet$$

$$\bullet (\phi_3, I_3)$$

$$(\phi_2, I_2) \bullet$$

$$\text{Circ} \xrightarrow{G} \text{FinCospan} \xrightarrow{H} \text{FinCorel} \xrightarrow{K} \text{LagRel}$$

In working on these issues, three questions come up:

- ▶ When is a symmetric monoidal category equivalent to a prop?
- ▶ What exactly is a map between props?
- ▶ How can you present a prop using generators and relations?

Answers can be found here:

- ▶ John Baez, Brendan Coya and Franciscus Rebro, [Props in network theory](#), arXiv:1707.08321.

We start with the 2-category SymMonCat , where:

- ▶ objects are symmetric monoidal categories,
- ▶ morphisms are symmetric monoidal functors,
- ▶ 2-morphisms are monoidal natural transformations.

We often prefer to think about the category PROP , where:

- ▶ object are props: strict symmetric monoidal categories with natural numbers as objects and addition as tensor product,
- ▶ morphisms are strict symmetric monoidal functors sending 1 to 1.

This is evil, but convenient. When can we get away with it?

Theorem. $C \in \text{SymMonCat}$ is equivalent to a prop iff there is an object $x \in C$ such that every object of C is isomorphic to

$$x^{\otimes n} = x \otimes (x \otimes (x \otimes \cdots))$$

for some $n \in \mathbb{N}$.

Theorem. Suppose $F: C \rightarrow D$ is a symmetric monoidal functor between props. Then F is isomorphic, in SymMonCat , to a *strict* symmetric monoidal functor $G: C \rightarrow D$.

If $F(1) = 1$, G is a morphism of props.

We all “know” how to describe props using generators and relations.
 For example, the prop for commutative monoids can be presented with
 two generators:

$$\begin{array}{cc}
 \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \text{---} \bullet & \bullet \text{---} \\
 \mu: 2 \rightarrow 1 & \iota: 0 \rightarrow 1
 \end{array}$$

and three relations:

$$\begin{array}{ccc}
 \begin{array}{c} \text{ } \\ \text{ } \end{array} \text{---} \bullet & = & \begin{array}{c} \text{ } \\ \text{ } \end{array} \text{---} \bullet \\
 \text{(associativity)} & & \text{(unitality)} \\
 \begin{array}{c} \bullet \\ \text{---} \end{array} \text{---} \bullet & = & \text{---} \\
 \text{(commutativity)} & &
 \end{array}$$

But what are we really doing here?

There is a forgetful functor from props to signatures:

$$U: \mathbf{PROP} \rightarrow \mathbf{Set}^{\mathbb{N} \times \mathbb{N}}$$

A **signature** just gives a set $\text{hom}(m, n)$ for each $(m, n) \in \mathbb{N} \times \mathbb{N}$.

Theorem. The forgetful functor U is **monadic**, meaning that it has a left adjoint

$$F: \mathbf{Set}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbf{PROP}$$

and \mathbf{PROP} is equivalent to the category of algebras of the resulting monad $UF: \mathbf{Set}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbf{Set}^{\mathbb{N} \times \mathbb{N}}$.

Everything one wants to do with generators and relations follows from $U: \mathbf{PROP} \rightarrow \mathbf{Set}^{\mathbb{N} \times \mathbb{N}}$ being monadic.

For example:

Corollary. Any prop T is a coequalizer

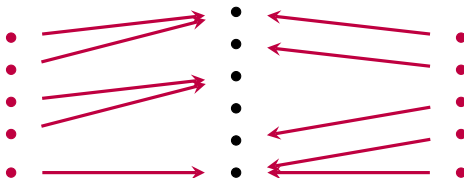
$$F(R) \rightrightarrows F(G) \longrightarrow T$$

for some signatures G, R .

We call elements of G **generators** and elements of R **relations**.

Example. The symmetric monoidal category where

- ▶ objects are finite sets
- ▶ morphisms are isomorphism classes of cospans of finite sets:



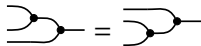
- ▶ the tensor product is disjoint union

is equivalent to a prop, FinCospan .

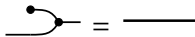
Theorem (Lack). The prop $\mathbf{FinCospan}$ has generators



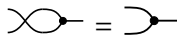
and relations:



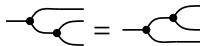
associativity



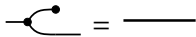
unitality



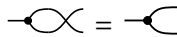
commutativity



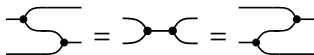
coassociativity



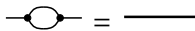
counitality



cocommutativity



Frobenius law



special law

Thus, for any strict symmetric monoidal category C , there's a 1-1 correspondence between:

- ▶ strict symmetric monoidal functors $F : \text{FinCospan} \rightarrow C$

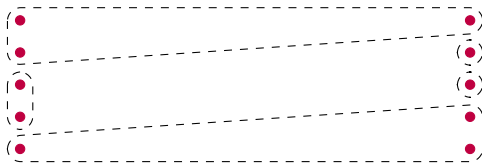
and

- ▶ special commutative Frobenius monoids in C .

We summarize this by saying FinCospan is “the prop for special commutative Frobenius monoids”.

Example. The symmetric monoidal category where:

- ▶ objects are finite sets,
- ▶ morphisms are corelations:



- ▶ the tensor product is disjoint union

is equivalent to a prop, $\mathbf{FinCorel}$.

Theorem (Coya, Fong). The prop FinCorel has the same generators as FinCospan:



and all the same relations, together with one more:

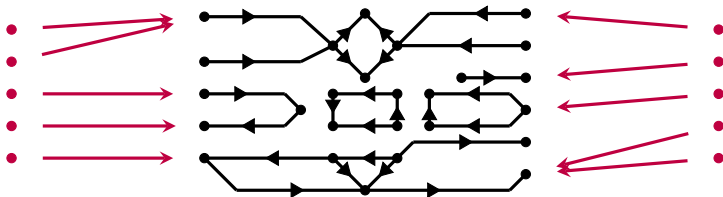
$$\bullet \text{---} \bullet =$$

extra law

Thus, FinCorel is the prop for extraspecial commutative Frobenius monoids.

Example. The symmetric monoidal category where:

- ▶ objects are finite sets,
- ▶ morphisms are circuits made solely of wires:



- ▶ the tensor product is disjoint union

is equivalent to a prop, Circ.

Theorem (Rosebrugh, Sabadani, Walters). The prop **Circ** has all the same generators and relations as **Cospan**, together with one additional generator $f: 1 \rightarrow 1$.

Thus, **Circ** is the prop for special commutative Frobenius monoids X equipped with a morphism $f: X \rightarrow X$.

In applications to electrical circuits, this morphism describes a *purely conductive wire*:



We can now understand these maps of props:

$$\text{Circ} \xrightarrow{G} \text{FinCospan} \xrightarrow{H} \text{FinCorel} \xrightarrow{K} \text{LagRel}$$

using generators and relations:

- ▶ Circ is the prop for special commutative Frobenius monoids with endomorphism f .
- ▶ FinCospan is the prop for special commutative Frobenius monoids. G sends f to the identity.
- ▶ FinCorel is the prop for extraspecial commutative Frobenius monoids. H does the obvious thing.
- ▶ K sends the extraspecial commutative Frobenius monoid $1 \in \text{FinCorel}$ to $\mathbb{R}^2 \in \text{LagRel}$, which becomes an extraspecial commutative Frobenius monoid by ‘duplicating potentials and adding currents’. For example



gets sent to the Lagrangian relation

$$L = \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) : \phi_1 = \phi_2 = \phi_3, I_1 + I_2 = I_3\} \subseteq \mathbb{R}^4 \oplus \mathbb{R}^2.$$

This is just the tip of the iceberg. Many fields of science and engineering use networks. A unified theory of networks will:

- ▶ reveal and clarify the mathematics underlying these fields,
- ▶ help integrate these fields,
- ▶ enhance interoperability of human-designed systems,
- ▶ focus attention on *open* systems: systems with inflows and outflows.



References

- ▶ J. Baez, B. Coya and F. Rebro, Props in network theory. Available as [arXiv:1707.08321](#).
- ▶ J. Baez, J. Erbele, Categories in control, *Theory Appl. Categ.* **30** (2015), 836–881. Available at <http://www.tac.mta.ca/tac/volumes/30/24/30-24abs.html>.
- ▶ J. Baez, B. Fong, A compositional framework for passive linear circuits. Available at [arXiv:1504.05625](#).
- ▶ J. Baez, B. Fong and B. S. Pollard, A compositional framework for Markov processes, *Jour. Math. Phys.* **57** (2016), 033301. Available at [arXiv:1508.06448](#).
- ▶ F. Bonchi, P. Sobociński, F. Zanasi, Interacting Hopf algebras, *J. Pure Appl. Alg.* **221** (2017), 144–184. Available as [arXiv:1403.7048](#).

- ▶ B. Coya, B. Fong, Correlations are the prop for extraspecial commutative Frobenius monoids, *Theory Appl. Categ.* **32** (2017), 380–395. Available at <http://www.tac.mta.ca/tac/volumes/32/11/32-11abs.html>.
- ▶ S. Lack, Composing PROPs, *Theory Appl. Categ.* **13** (2004), 147–163. Available at <http://www.tac.mta.ca/tac/volumes/13/9/13-09abs.html>.
- ▶ R. Rosebrugh, N. Sabadini, R. F. C. Walters, Generic commutative separable algebras and cospans of graphs, *Theory Appl. Categ.* **15** (2005), 164–177. Available at <http://www.tac.mta.ca/tac/volumes/15/6/15-06abs.html>.