Don't try this at home: No-Go Theorems for Distributive Laws

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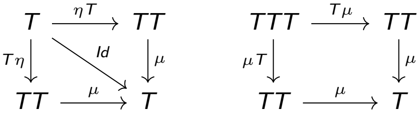
Overview

- Introduction
  - Monads
  - Distributive laws
- Previously broken distributive laws
  - Plotkin's counterexample
- General No-Go theorems: the algebraic approach
  - Generalized Plotkin
  - And more...
- Results

Monads

A monad is a triple $\langle T, \eta, \mu \rangle$, with T an endofunctor and $\eta : 1 \Rightarrow T$, $\mu : TT \Rightarrow T$ natural transformations, such that:



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\begin{array}{ccc}
T & \xrightarrow{\eta T} & TT \\
{\scriptstyle T\eta}\downarrow & \overset{Id}{\searrow} & \downarrow{\scriptstyle \mu} \\
TT & \xrightarrow{\mu} & T
\end{array}
\qquad\qquad
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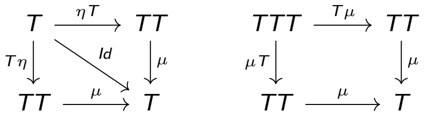

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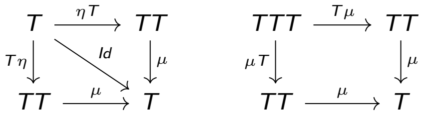

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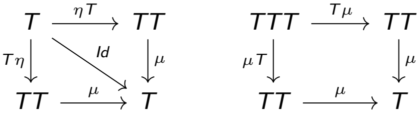

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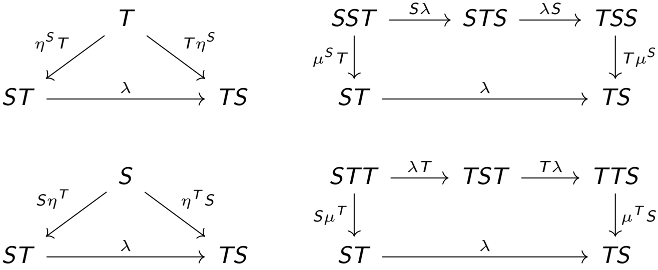
- More examples: Multiset, Exception, Reader, Writer, ...

Composing Monads with Distributive Laws

We can compose monads with the help of a distributive law - *Beck 1969*

$$\langle TS, \eta^T \eta^S, \mu^T \mu^S \cdot T\lambda S \rangle$$

Where $\lambda : ST \to TS$ is a natural transformation satisfying the following axioms.



Examples

There is a distributive law $LP \Rightarrow PL$. It works like the famous 'times over plus' distributivity:

$$(a + b) * c = a * b + a * c$$
$$[\{a, b\}, \{c\}] \mapsto \{[a, c], [b, c]\}$$

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Some general results:

- If T is a commutative monad, and S a monad defined by linear equations, then there is a distributive law $ST \Rightarrow TS$ - *Manes and Mulry 2007*.
- There are variations on the above theorem for affine and relevant monads - *Dahlqvist, Parlant and Silva 2018*.

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  - Several examples of mistakes in the literature.
  - According to Bonsangue, Hansen, Kurz, and Rot:
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- What to do now? Is there a distributive law at all?
- Our goal: to find general principles that tell us when no distributive law exists.

Previous Results

- No distributive law $DP \Rightarrow PD$ *- Plotkin / Varacca and Winskel 2005*
- No distributive law $PD \Rightarrow DP$ *- Varacca 2003, without proof*
- No monad structure on PD *- Dahlqvist and Neves 2018*
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What is so special about powerset?

Plotkin's Proof

Main idea is to chase a specially chosen element:

$$\Xi = \{a, b\} +^{\frac{1}{2}} \{c, d\} \in DP(X)$$

round the naturality diagram:

$$\begin{array}{ccc} \Xi & \xrightarrow{\lambda_X} & ? \\ {\scriptstyle DP(f)}\downarrow & & \downarrow{\scriptstyle PD(f)} \\ ? & \xrightarrow[\lambda_X]{} & ? \end{array}$$

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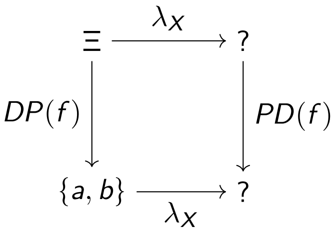

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- 3 facts together get contradiction.

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- "Going up": easy to take inverse image.

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Idea: try an algebraic perspective!

A Quick Reminder

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  - Signature Σ and a set of variables give *terms*.
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  Theory of complete semilattices: $\Sigma = \{\bot^{(0)}, \vee^{(2)}\}$

$$\bot \vee x = x \qquad \text{(left unit)}$$

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$$x \vee y = y \vee x \qquad \text{(commutativity)}$$

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- Monads arise from free/forgetful adjunction between Set and category of (Σ, E)-algebras.

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  - Need additional assumptions about variables to get useful conclusions.

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Let $\mathbb{P}$ and $\mathbb{V}$ be algebraic theories such that:

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5. *For all v' in $\mathbb{V}$, if $v' = y$ then $\mathrm{var}(v') = \{y\}$*

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$\Longrightarrow$ Many No-Go theorems:

- P has idempotent term, V has idempotent term. (Plotkin)
- P has idempotent term, V has term with units.
- P has term with units, V has idempotent term.
- P has term with units, V has term with units. (List monad!!)

Monads combinations we've seen before

Is there a distributive law of type Row$\circ$Column $\Rightarrow$ Column$\circ$Row?

Previous results:

	L	M	P	D
L		Y	Y	
M		Y	Y	
P			N	N
D			N	

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M	N	Y	Y	N
P	N	N	N	N
D		N	N	N

Extended Boom Hierarchy

		associative	commutative	idempotent
Tree	Tr	N	N	N
Idempotent tree	I	N	N	Y
Commutative tree	C	N	Y	N
Commutative Idempotent tree	CI	N	Y	Y
List	L	Y	N	N
Associative Idempotent tree	AI	Y	N	Y
Multiset	M	Y	Y	N
Powerset	P	Y	Y	Y

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$+$ all non-empty-versions: without constants.

256 combinations of monads.

Results

Is there a distributive law of type Row∘Column ⇒ Column∘Row?
Row and Column both without units:

	Tr^+	I^+	C^+	CI^+	L^+	AI^+	M^+	P^+
Tr^+	Y						Y	Y
I^+		N		N		N		N
C^+	Y		Y?				Y	Y
CI^+		N		N		N		N
L^+	Y				Y		Y	Y
AI^+		N		N		N		N
M^+	Y						Y	Y
P^+		N		N		N		N

More Results

Is there a distributive law of type Row∘Column ⇒ Column∘Row?
Row with unit, Column without unit:

	Tr^+	I^+	C^+	CI^+	L^+	AI^+	M^+	P^+
Tr	Y	N		N		N	Y	Y
I		N		N		N		N
C	Y	N		N		N	Y	Y
CI		N		N		N		N
L	Y	N		N	Y	N	Y	Y
AI		N		N		N		N
M	Y	N		N		N	Y	Y
P		N		N		N		N

Even More Results

Is there a distributive law of type Row∘Column ⇒ Column∘Row?
Row without unit, Column with unit:

	Tr	I	C	CI	L	AI	M	P
Tr^+							Y	Y
I^+		N	N	N		N	N	N
C^+							Y	Y
CI^+		N	N	N		N	N	N
L^+							Y	Y
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M^+							Y	Y
P^+		N	N	N		N	N	N

Final slide with results

Is there a distributive law of type Row∘Column ⇒ Column∘Row?
Row and Column both have units:

	Tr	I	C	CI	L	AI	M	P
Tr	N	N	N	N	N	N	Y	Y
I	N	N	N	N	N	N	N	N
C	N	N	N	N	N	N	Y	Y
CI	N	N	N	N	N	N	N	N
L	N	N	N	N	N	N	Y	Y
AI	N	N	N	N	N	N	N	N
M	N	N	N	N	N	N	Y	Y
P	N	N	N	N	N	N	N	N