The support is a morphism of monads

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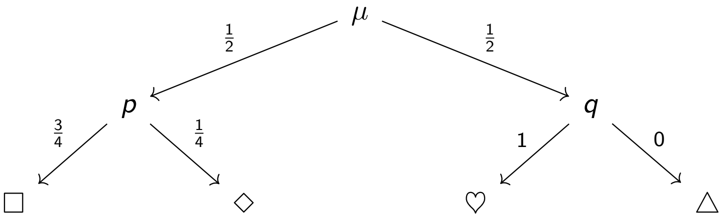
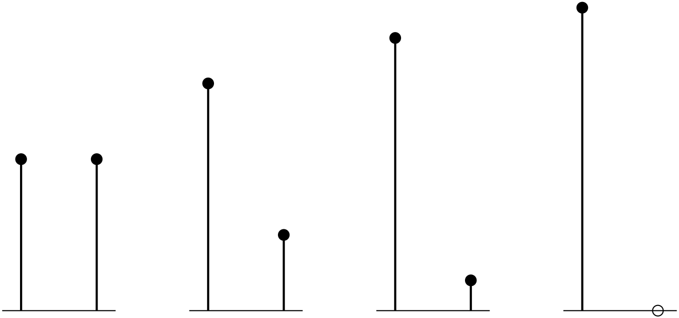
A simple example



$$\mathrm{supp}(\mu) = \mathrm{supp}\left(\frac{1}{2} \cdot p + \frac{1}{2} \cdot q\right) = \mathrm{supp}(p) \cup \mathrm{supp}(q)$$

Another simple example

Consider the sequence of probability vectors $\left\{1 - \frac{1}{i+1}, \frac{1}{i+1}\right\}_{i \in \mathbb{N}}$.
Both entries are positive for every $i \in \mathbb{N}$.



$i = 1$ $i = 3$ $i = 9$ $i \to \infty$

The support discontinuously shrinks: Lower semicontinuity in the order of set inclusion.

The question

Probability $\rightarrow$ possibility: A morphism from a monad of probabilistic powerspaces to a monad of (possibilistic) powerspaces?

Applications: Denotational semantics, Dynamical systems, ...

This can also help us to better understand abstract notions of convexity...

Main problem: How to encode the lower semicontinuity of the support?

We work in the category Top of *topological spaces and continuous maps*.

The hyperspace

Let X be a topological space.

Definition

Let $A \subseteq X$. We set $\mathrm{Hit}(A) := \{C \subseteq X : C \text{ is closed and } C \cap A \neq \varnothing\}$.

Definition (Hyperspace)

The hyperspace of X is the set $HX := \{C \subseteq X : C \text{ is closed}\}$ equipped with the *lower Vietoris topology* with subbasis: $\{\mathrm{Hit}(U) : U \subseteq X \text{ is open}\}$.

Duality theory for H

Theorem

There is an isomorphism of complete lattices between HX and Scott-continuous functionals $\phi : \mathcal{O}(X) \to S$ with the following two properties.

1. *Strictness:* $\phi(\varnothing) = 0$.
2. *Modularity:* $\phi(U \cap V) \vee \phi(U \cup V) = \phi(U) \vee \phi(V)$.

(S denotes the Sierpinski space.)

We adopt functional-analytic coupling notation.

$$\langle C, U \rangle := \begin{cases} 1 \text{ if } C \text{ hits } U \\ 0 \text{ otherwise} \end{cases}$$

The H-monad

H : Top $\to$ Top is a functor: $X \mapsto HX$, $f \mapsto f_\sharp$ where $f_\sharp(C) = \mathrm{cl} f(C)$.

Definition (Unit)

The map $\sigma : X \to HX$ where $\sigma(x) \in HX$ fulfills

$$\langle \sigma(x), U \rangle \equiv \begin{cases} 1 \text{ if } x \in U \\ 0 \text{ otherwise} \end{cases}$$

for every open $U \subseteq X$.

Definition (Multiplication)

The map $\mathcal{U} : HHX \to HX$ where for $\mathcal{C} \in HHX$ we have

$$\langle \mathcal{U}\mathcal{C}, U \rangle \equiv \langle \mathcal{C}, \mathsf{Hit}(U) \rangle$$

for every open $U \subseteq X$.

$$\begin{array}{ccc} HX & \xrightarrow{\sigma} & HHX \\ & \searrow\!\!\!= & \downarrow \mathcal{U} \\ & & HX \end{array} \qquad \begin{array}{ccc} HX & \xrightarrow{\sigma_\sharp} & HHX \\ & \searrow\!\!\!= & \downarrow \mathcal{U} \\ & & HX \end{array} \qquad \begin{array}{ccc} HHHX & \xrightarrow{\mathcal{U}_\sharp} & HHX \\ \downarrow \mathcal{U} & & \downarrow \mathcal{U} \\ HHX & \xrightarrow{\mathcal{U}} & HX \end{array}$$

The triple $(H, \sigma, \mathcal{U})$ is a monad on Top.

(In fact, it is even a 2-monad.)

H-algebras

Theorem (Schalk 1993)

The category of H-algebras consists of complete lattices equipped with a sober topology whose specialization preorder equals the respective order. The structure maps are given by the join. The algebra-morphisms are continuous join-preserving maps.

(Recall: The algebras of the powerset monad on the category of sets are complete semilattices.)

Continuous subprobability valuations

Definition

A continuous map $\nu : \mathcal{O}(X) \to [0,1]$ that satisfies the following four conditions.

1. Monotonicity: $U \subseteq V$ implies $\nu(U) \leq \nu(V)$.
2. Strictness: $\nu(\varnothing) = 0$.
3. Modularity: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$.
4. Scott-continuity:
$$\nu\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigvee_{\alpha \in A} \nu(U_\alpha)$$
for any directed increasing net $(U_\alpha)_{\alpha \in A}$.

The space VX

Let X be a topological space.

Definition

We define the space VX to be the set of continuous subprobability valuations on X equipped with the topology for which the sets of the following form are a subbasis,

$$\theta(U, r) := \{\nu \ : \ \nu(U) > r\}$$

for some open $U \subseteq X$ and some $r \in [0, 1)$.

(This is very similar to the *extended probabilistic powerdomain*.)

Duality theory for V

We denote the *lower integral* of the lower semicontinuous function $f : X \to [0,1]$ with respect to the valuation ν by $\langle \nu, f \rangle$.

Theorem

There is a bijection between continuous valuations on the topological space X and Scott-continuous functionals $L(X) \to [0,1]$ with the following two properties.

1. *Strictness:* $\langle v, 0 \rangle = 0$.
2. *Modularity:* $\langle v, f \wedge g \rangle + \langle v, f \vee g \rangle = \langle v, f \rangle + \langle v, g \rangle$.

($L(X)$ denotes the set of lower semicontinuous functions $X \to [0,1]$.)

The V monad

V : Top $\to$ Top is a functor: $X \mapsto VX$, $f \mapsto f_*$, the pushforward operation.

Definition (Unit)

The map $\delta : X \to VX$ where $x \mapsto \delta_x$ where δ_x is the point-mass valuation characterized by

$$\langle \delta_x, g \rangle \equiv g(x)$$

for every lower semicontinuous $g : X \to [0, 1]$.

Definition (Multiplication)

The map $\mathcal{E} : VVX \to VX$ where for $\xi \in VVX$ we have

$$\langle \mathcal{E}\xi, g \rangle \equiv \langle \xi, \langle -, g \rangle \rangle$$

for every lower semicontinuous $g : X \to [0, 1]$.

(Note that the map $\langle -, g \rangle : VX \to [0, 1]$ is itself lower semicontinuous.)

$$
\begin{array}{ccc}
VX & \xrightarrow{\delta} & VVX \\
& \searrow^{=} & \downarrow \mathcal{E} \\
& & VX
\end{array}
\qquad
\begin{array}{ccc}
VX & \xrightarrow{\delta_*} & VVX \\
& \searrow^{=} & \downarrow \mathcal{E} \\
& & VX
\end{array}
\qquad
\begin{array}{ccc}
VVVX & \xrightarrow{\mathcal{E}_*} & VVX \\
\downarrow \mathcal{E} & & \downarrow \mathcal{E} \\
VVX & \xrightarrow{\mathcal{E}} & VX
\end{array}
$$

The triple $(V, \delta, \mathcal{E})$ is a monad on Top.

(In fact, it is even a 2-monad.)

V-algebras

"Probability-type" monads have "convex-type" algebras.

Definition (Category of convex spaces)

A set A with a map $c : [0,1] \times A \times A \to A$ fulfilling

1. Unitality: $c(0, x, y) = y$,
2. Idempotency: $c(\lambda, x, x) = x$,
3. Parametric commutativity: $c(\lambda, x, y) = c(1 - \lambda, y, x)$,
4. Parametric associativity: $c(\lambda, c(\mu, x, y), z) = c(\lambda\mu, x, c(\nu, y, z))$,

$$\nu = \begin{cases} \frac{\lambda(1-\mu)}{1-\lambda\mu} \text{ if } \lambda, \mu \neq 1 \\ \text{otherwise arbitrary in } [0,1]. \end{cases}$$

Theorem

Every V-algebra is a convex space and every morphism of V-algebras is a map that preserves the convex structure (an affine map).

(Compare: Goubault-Larrecq and Jia 2019, Arxiv-preprint.)

The idea is simple: Let (A, a) be a V-algebra, set

$$c(\lambda, x, y) := a\Big(\lambda \cdot \delta_x + (1 - \lambda) \cdot \delta_y\Big).$$

The support is a morphism in Top

Definition (Support of a valuation)

Let $\nu \in VX$. The support is defined by

$$\langle \mathrm{supp}(\nu), U \rangle := \begin{cases} 1 & \text{if } \nu(U) > 0 \\ 0 \text{ otherwise.} \end{cases}$$

The support is a continuous map $\mathrm{supp} : VX \to HX$ since

$$\mathrm{supp}^{-1}(\mathsf{Hit}(U)) = \theta(U, 0).$$

The support is a natural transformation

$$\begin{array}{ccc} VX & \xrightarrow{\text{supp}} & HX \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_\sharp} \\ VY & \xrightarrow{\text{supp}} & HY \end{array}$$

Proof:

$$\begin{aligned} \langle \mathrm{supp}(f_* p), U \rangle &= \langle (f_* p)(U) > 0 \rangle \\ &= \langle p(f^{-1}(U)) > 0 \rangle \\ &= \langle \mathrm{supp}(p), f^{-1}(U) \rangle \\ &= \langle f_\sharp(\mathrm{supp}(p)), U \rangle. \end{aligned}$$

The support is a morphism of monads

$$
\begin{array}{ccc}
 & & VX \\
 & \overset{\delta}{\nearrow} & \\
X & & \Big\downarrow \mathrm{supp} \\
 & \underset{\sigma}{\searrow} & \\
 & & HX
\end{array}
\qquad\qquad
\begin{array}{ccc}
VVX & \xrightarrow{\mathcal{E}} & VX \\
\Big\downarrow \mathrm{supp} & & \\
HVX & & \Big\downarrow \mathrm{supp} \\
\Big\downarrow \mathrm{supp}_\sharp & & \\
HHX & \xrightarrow{\mathcal{U}} & HX
\end{array}
$$

Theorem

The support induces a morphism of monads $\mathrm{supp} : (V, \delta, \mathcal{E}) \to (H, \sigma, \mathcal{U})$.

Theorem

Every H-algebra is also a V-algebra.

It is a standard result that a morphism of monads induces a pullback functor between the respective categories of algebras.

Here: Let (A, a) be an H-algebra, then

$$(A, a) \longmapsto (A, a \circ \mathrm{supp})$$

yields a V-algebra with structure map

$$\nu \longmapsto \bigvee \mathrm{supp}(\nu).$$

The case of Borel probability measures

The functor $P : \mathrm{Top} \to \mathrm{Top}$ that assigns to a space X the set PX of τ-smooth Borel probability-measures with the A(lexandrov)-topology generates a submonad of V.

For Tikhonov spaces, the P-construction is equivalent to assigning the weak topology. This includes all spaces usually studied in measure theory.

We still have $\mathrm{supp} : P \to H$.

This is the most general Borel-probability monad that we are aware of.

Conclusions

- A natural appearance of exotic convex spaces as V-algebras mediated by supp.
- Clear connection between probabilistic and possibilistic representations of systems, in denotational semantics, dynamical systems, entropy-theory, ...
- supp is induced by a morphism of effect monoids, general constructions are forthcoming.
- We work on a generalization to the category of locales.

Preliminary paper: `http://www.mis.mpg.de/publications/preprints/2019/prepr2019-33.html`

Some literature

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