From parametricity to modularity and back in correspondence theory: preliminary considerations

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The phenomenon of correspondence

$$\mathcal{F}, w \Vdash \Diamond\Diamond p \to \Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall y, z(xRy \& yRz \to xRz)[w]$$

The phenomenon of correspondence

$$\mathcal{F}, w \Vdash \Diamond\Diamond p \to \Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall y, z(xRy \& yRz \to xRz)[w]$$

(⇒) Assume wRy and yRz. To show: $w \in R^{-1}[z]$.
Consider the minimal valuation making the antecedent true at w:

$$V^*(p) = \{z\}.$$

If wRy and yRz then $\mathcal{F}, V^*, w \Vdash \Diamond\Diamond p$. Hence, $\mathcal{F}, V^*, w \Vdash \Diamond p$, i.e.

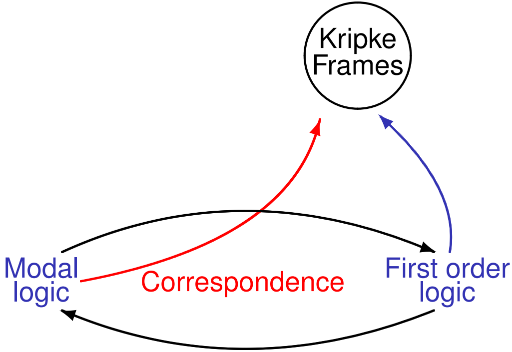
$$w \in [\![\Diamond p]\!]_{V^*} = R^{-1}[V^*(p)] = R^{-1}[z].$$

Correspondence theory

- gives syntactic conditions for modal formulas to have a first order correspondent (e.g. Sahlqvist formulas)
- Computes algorithmically the first order correspondent of these formulas
- Benefits: These formulas generate logics that are strongly complete w.r.t. first-order definable classes of frames.

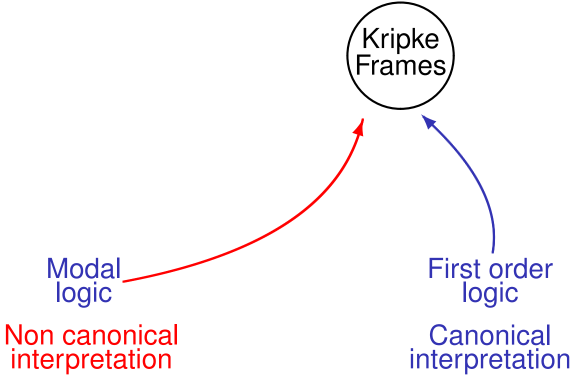
Correspondence via Duality

Correspondence theory arises semantically:



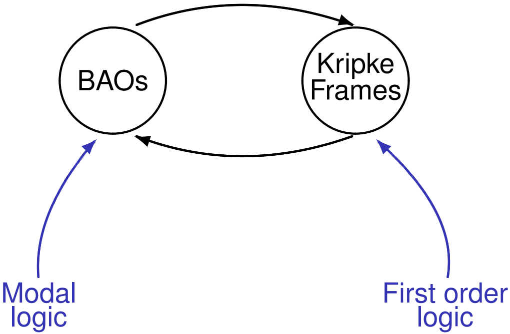
Correspondence via Duality

An asymmetry:



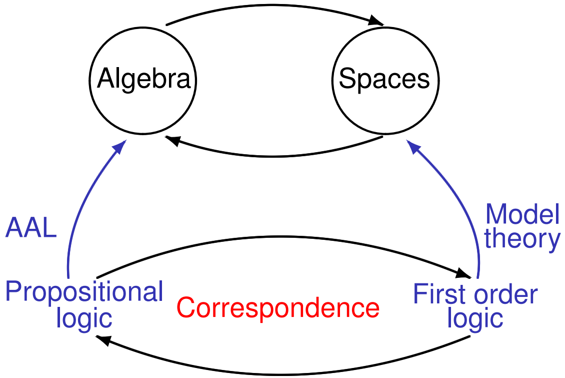
Correspondence via Duality

Symmetry re-established via **duality**:



Correspondence via Duality

Correspondence available not just for modal logic:



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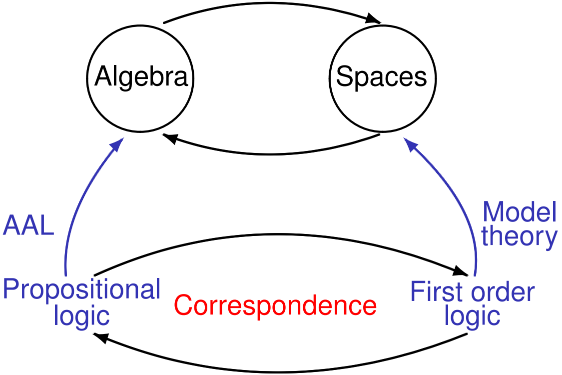


- specific correspondences as logical reflections of dualities
- dual characterizations as instances of **unified correspondence**

Unified correspondence

Display calculi [GMPTZ18]

Hybrid logics [CR17]

MV-logics [BCM19]

Normal (D)LE-logics [CP12, CP19]

Polarity-based and graph-based semantics [CFPPW]

Mu-calculi [CFPS15, CGP14, CC17]

Sahlqvist via translation [CPZ19]

Regular DLE-logics Kripke frames with impossible worlds [PSZ17a]

Constructive canonicity [CP16, CCPZ]

Finite lattices and monotone ML [FPS]

Jónsson-style vs Sambin-style canonicity [PSZ17b]

Canonicity via pseudo-correspondence [CPSZ]

Main tools of unified correspondence

Parametric Sahlqvist theory

- Definition of **Sahlqvist** formulas/sequents for **all** signatures of normal (D)LE-logics
- in terms of the order-theoretic properties of the algebraic interpretation of logical connectives

The algorithm ALBA (also **parametric**)

- computes the first-order correspondent of normal DLE-terms/inequalities.
- reduction steps sound on complex algebras of relational structures (perfect normal DLEs)

Normal DLE-logics

(D)LE: (Distributive) Lattice Expansions: $\mathbb{A} = (\mathbb{L}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$
(distributive) lattice signature + operations of any finite arity.
Additional operations partitioned in families $f \in \mathcal{F}$ and $g \in \mathcal{G}$.
Normality: In each coordinate,

- f-type operations *preserve* finite **joins** or *reverse* finite **meets**;
- g-type operations *preserve* finite **meets** or *reverse* finite **joins**.

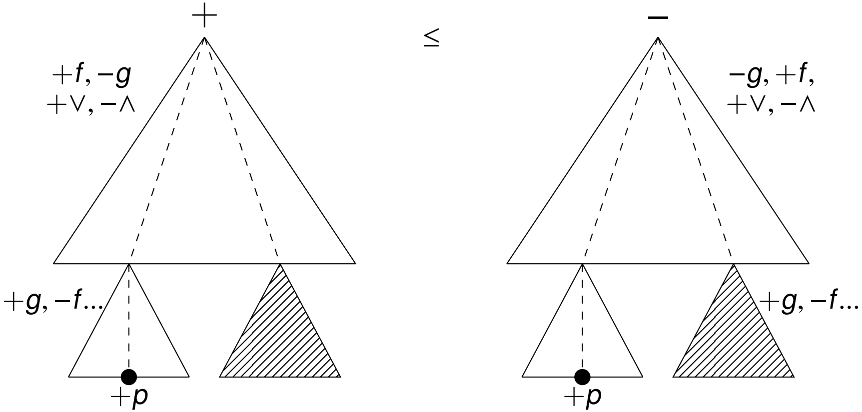
Examples

- Distributive Modal Logic: $\mathcal{F} := \{\Diamond, \lhd\}$ and $\mathcal{G} := \{\Box, \rhd\}$
- Bi-intuitionistic modal logic: $\mathcal{F} := \{\Diamond, \succ\!\!-\}$ and $\mathcal{G} := \{\Box, \rightarrow\}$
- Full Lambek calculus: $\mathcal{F} := \{\circ\}$ and $\mathcal{G} := \{/, \backslash\}$
- Lambek-Grishin calculus: $\mathcal{F} := \{\circ, /_{\oplus}, \backslash_{\oplus}\}$ and $\mathcal{G} := \{\oplus, /_{\circ}, \backslash_{\circ}\}$
- ...

Relational/complex algebra semantics

- f-type operations have residuals $f_i^{\sharp}$ in each coordinate i;
- g-type operations have residuals $g_h^{\flat}$ in each coordinate h.

Inductive inequalities



Examples: reflexivity and transitivity

$$
\begin{array}{lll}
 & \forall p[\Box p \leq p] & \\
\text{iff} & \forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \Box p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \text{(generators)} \\
\text{iff} & \forall p \forall \mathbf{j} \forall \mathbf{m}[(\blacklozenge \mathbf{j} \leq p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \text{(adjunction)} \\
\text{iff} & \forall \mathbf{j} \forall \mathbf{m}[\blacklozenge \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}] & \text{(Ackermann)} \\
\text{iff} & \forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] & \text{(Inv. Ackermann)}
\end{array}
$$

Examples: reflexivity and transitivity

	$\forall p[\Box p \leq p]$	
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iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\blacklozenge \mathbf{j} \leq p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]$	(adjunction)
iff	$\forall \mathbf{j} \forall \mathbf{m}[\blacklozenge \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]$	(Ackermann)
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	$\forall p[\Diamond\Diamond p \leq \Diamond p]$	
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Modularity: One reduction, many translations! On Kripke frames:

$$\forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \rightsquigarrow \forall x[\Delta[\{x\}] \subseteq R[\{x\}]] \quad \text{i.e.} \quad \Delta \subseteq R$$

$$\forall \mathbf{j}[\Diamond\Diamond \mathbf{j} \leq \Diamond \mathbf{j}] \rightsquigarrow \forall x[R^{-1}[R^{-1}[\{x\}]] \subseteq R^{-1}[\{x\}]] \quad \text{i.e.} \quad R \,;\, R \subseteq R$$

But how about more general semantic contexts?

Questions

Conceptual questions

- can we connect the **meaning** of the first-order correspondents in the ‘Boolean contexts’ to the meaning of those in other contexts?
- can we characterize or capture (or even define) **meaning-preservation** across contexts?

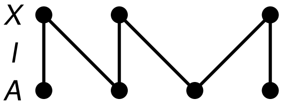
Technical questions

- is there an **automated** way to syntactically generate fist-order correspondents from those on the Boolean context?
- more broadly, is there an **automated** way to syntactically generate fist-order correspondents relative to a more general semantic context from those relative to a more restricted context?

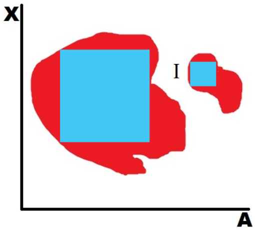
Case Studies

CS1: Polarity-based semantics of LE-logics

Formal contexts (A, X, I) are abstract representations of databases:

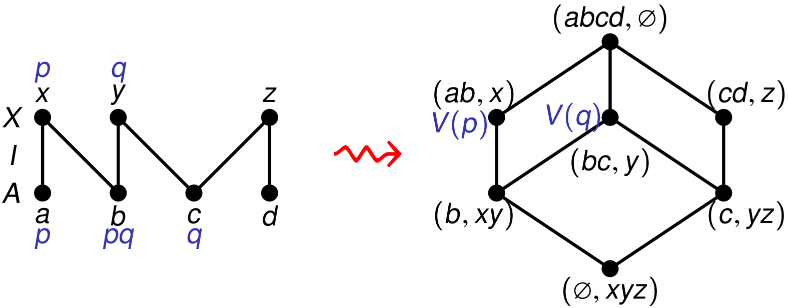


A: set of *Objects*
X: set of *Features*
$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Formal concepts:
"rectangles" maximally contained in I

Formulas as formal concepts



Let $\mathbb{P} = (A, X, I)$ and $\mathbb{P}^+$ be the complex algebra of $\mathbb{P}$.
Models: $\mathbb{M} := (\mathbb{P}, V)$ with $V : Prop \to \mathbb{P}^+$

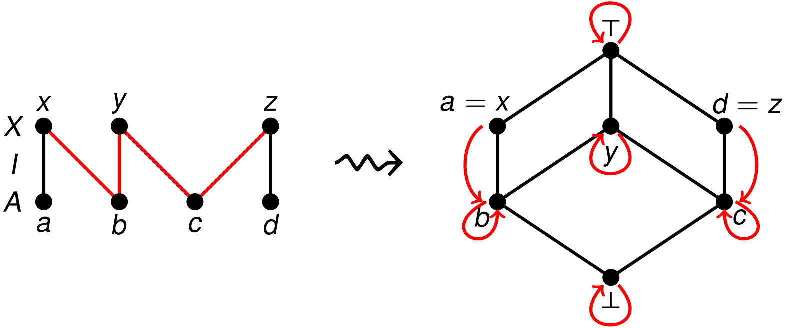
$$V(p) := ([\![p]\!], (\![p]\!))$$

membership:	$\mathbb{M}, a \Vdash p$	iff	$a \in [\![p]\!]_{\mathbb{M}}$
description:	$\mathbb{M}, x \succ p$	iff	$x \in (\![p]\!)_{\mathbb{M}}$

Semantics of modal formulas

Enriched formal contexts: $\mathbb{F} = (A, X, I, \{R_i \mid i \in Agents\})$

$R_i \subseteq A \times X$ and $\forall a((R^{\uparrow}[a])^{\downarrow\uparrow} = R^{\uparrow}[a])$ and $\forall x((R^{\downarrow}[x])^{\uparrow\downarrow} = R^{\downarrow}[x])$



$\Box_i\varphi$: concept φ **according to** agent i

$$V(\Box_i\varphi) = \Box_i V(\varphi) = (R_i^{\downarrow}[(\![\varphi]\!)], (R_i^{\downarrow}[(\![\varphi]\!)])^{\uparrow})$$

$\mathbb{M}, a \Vdash \Box_i\varphi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \varphi$, then aR_ix

$\mathbb{M}, x \succ \Box_i\varphi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box_i\varphi$, then aIx

Epistemic interpretation

Reflexivity aka Factivity

$\forall p[\Box p \leq p]$

iff $\forall \boldsymbol{j}[\boldsymbol{j} \leq \blacklozenge \boldsymbol{j}]$

iff $\forall a[a^{\uparrow\downarrow} \subseteq R^{\downarrow}[a^{\uparrow}]]$

iff $\forall a[a \in R^{\downarrow}[a^{\uparrow}]]$ ($R^{\downarrow}[a^{\uparrow}]$ Galois-stable)

iff $R_i \subseteq I$ Agent *i*'s attributions are factually correct!

Transitivity aka Positive introspection

$\forall p[\Box p \leq \Box\Box p]$

iff $\forall \boldsymbol{m}[\Box \boldsymbol{m} \leq \Box\Box \boldsymbol{m}]$

iff $\forall x[R^{\downarrow}[x^{\downarrow\uparrow}] \subseteq R^{\downarrow}[(R^{\downarrow}[x^{\downarrow\uparrow}])^{\uparrow}]]$

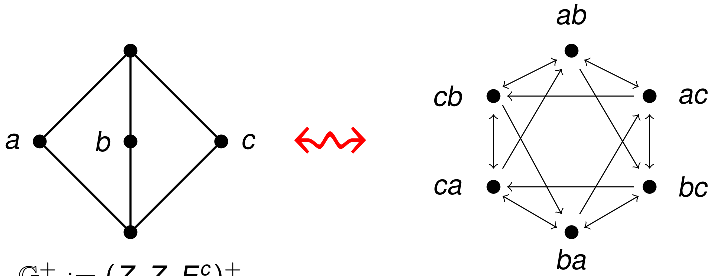
iff $\forall x[R^{\downarrow}[x] \subseteq R^{\downarrow}[(R^{\downarrow}[x])^{\uparrow}]]$ ($R^{\downarrow}[a^{\uparrow}]$ Galois-stable)

iff $R \subseteq R; R$

If agent *i* recognizes object *a* as an *x*-object, then *i* must also attribute to *a* all the features shared by *x*-objects according to *i*.

CS2: Graph-based semantics of LE-logics

One-sorted structures $\mathbb{G} = (Z, E)$, with E reflexive:



$\mathbb{G}^+ := (Z, Z, E^c)^+$

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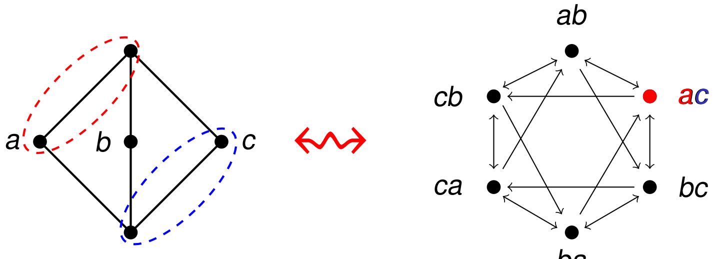

$\mathbb{G}^+ := (Z, Z, E^c)^+$

Representation. States: maximally disjoint filter-ideal pairs (F, I);

$$(F, I)\ E\ (F', I') \quad \text{iff} \quad F \cap I' = \varnothing$$

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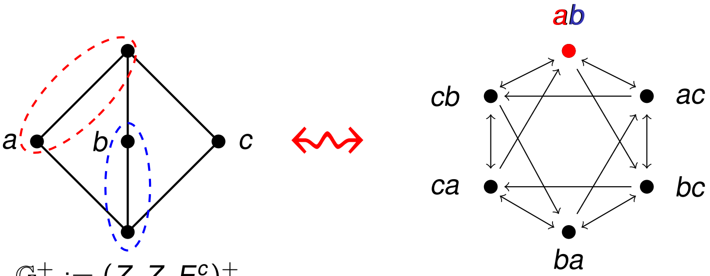

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Representation. States: maximally disjoint filter-ideal pairs (F, I);

$$(F, I)\ E\ (F', I') \quad \text{iff} \quad F \cap I' = \varnothing$$

$$\mathbb{M}, z \Vdash \Box\psi \quad \text{iff} \quad \forall z'[zR_{\Box}z' \Rightarrow \mathbb{M}, z' \not\succ \psi]$$

$$\mathbb{M}, z \succ \Box\psi \quad \text{iff} \quad \forall z'[z'Ez \Rightarrow \mathbb{M}, z' \not\Vdash \Box\psi]$$

Modelling informational entropy

Informational entropy: an inherent boundary to knowability, due e.g. to perceptual, theoretical, evidential or linguistic limits.

Reflexivity as E-reflexivity

$$
\begin{array}{ll}
 & \forall p[\Box p \leq p] \\
\text{iff} & \forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \\
\text{iff} & \forall z[z^{[10]} \subseteq R^{[0]}[z^{[1]}]] \\
\text{iff} & E \subseteq R
\end{array}
$$

the agent correctly recognizes inherent indistinguishability

Transitivity as E-transitivity

$$
\begin{array}{ll}
 & \forall p[\Box p \leq \Box\Box p] \\
\text{iff} & \forall \mathbf{j}[\blacklozenge\blacklozenge \mathbf{j} \leq \blacklozenge \mathbf{j}] \\
\text{iff} & \forall z[R^{[0]}[(R^{[0]}[z^{[01]}])^{[1]}]] \subseteq R^{[0]}[z^{[01]}]] \\
\text{iff} & R \circ_E R \subseteq R
\end{array}
$$

E-compositions of $R, S \subseteq Z \times Z$:

$$x(R \circ_E S)a \quad \text{iff} \quad \exists b(xRb \;\&\; E^{(1)}[b] \subseteq S^{(0)}[a]).$$

$$a(R \diamond_E S)x \quad \text{iff} \quad \exists y(aRy \;\&\; E^{(0)}[y] \subseteq S^{(0)}[x]).$$

CS3: Many-valued semantics for modal logic [Fitting]

- Truth-value space: A finite (or complete, or perfect) Heyting algebra $\mathbb{A}$.
- Formulas of $L_{\mathbb{A}}$:

$$\varphi := \boldsymbol{t} \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi$$

- Models $\mathfrak{M} = (W, R, V)$, where
  - $W \neq \emptyset$
  - $R : W \times W \rightarrow \mathbb{A}$
  - $V : (Prop \times W) \rightarrow \mathbb{A}$.

Semantics

- $V(\boldsymbol{t}, w) = t \in A$
- $V(\Diamond p, w) = \bigvee^{\mathbb{A}}\{Rwu \wedge^{\mathbb{A}} V(p, u) \mid u \in W\}$
- $V(\Box p, w) = \bigwedge^{\mathbb{A}}\{Rwu \rightarrow^{\mathbb{A}} V(p, u) \mid u \in W\}$

Correspondence theory for MV-modal logic

This is work of **Britz, Conradie, and Morton**.
Let $\mathbb{A}$ be a perfect Heyting algebra and $a \in A$.

Theorem

Every inductive formula has a effectively computable local frame a-correspondent of the class of $\mathbb{A}$-frames.

Corollary

Every Sahlqvist formula has an effectively computable local frame a-correspondent of the class of $\mathbb{A}$-frames.

A preservation result

This is work of **Britz, Conradie, and Morton**.

Restricted Sahlqvist formulas

A restricted Sahlqvist implication is an implication $\varphi \to \psi$ in which

1. φ is built from boxed atoms ($\Box^n p$) by applying $\wedge$, $\vee$ and $\Diamond$.
2. ψ is positive.
3. For each $p \in Prop$ in ψ, p does not occur in any subformula α such that $\alpha \to \gamma$ is a subformula of φ.

A restricted Sahlqvist formula is built from restricted Sahlqvist implications by applying $\wedge$ and $\Box$.

Theorem

Let φ be a restricted Sahlqvist formula and let α be its classical local frame correspondent. Then:

$$\mathfrak{F}, w \Vdash_a \varphi \to \psi \quad \text{iff} \quad \mathfrak{F} \models_a \alpha[x := w]$$

Example: a-validity of reflexivity $p \to \Diamond p$

	$\forall p[a \leq p \to \Diamond p]$	
iff	$\forall p[p \wedge a \leq \Diamond p]$	
iff	$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \wedge a \ \& \ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	
iff	$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq a \ \& \ \mathbf{i} \leq p \ \& \ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	splitting
iff	$\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq a \ \& \ \Diamond \mathbf{i} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	Ackermann
iff	$\forall \mathbf{i}[\mathbf{i} \leq a \Rightarrow \mathbf{i} \leq \Diamond \mathbf{i}]$	inv. Ackermann
i.e.	$\Delta \subseteq R$ relativized to a.	

Preliminary conclusions

- Notation, notation, notation.
- From parametricity to modularity and back.
- Both syntactic and semantic parameters.
- Preservation of syntactic shape but not of meaning; preservation of meaning but not of syntactic shape.
- Is there more than an optical illusion?