SYCO 3

Differentiating proofs for programs

Marie Kerjean

Inria Bretagne

1. Linear Logic

2. Smooth classical models

3. LPDEs

4. Higher-Order

Linear logic

Usual Implication

Linear Logic

$$A \Rightarrow B = \; !A \multimap B$$
$$\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$$

A proof is linear when it uses only once its hypothesis A.

Linear logic

Usual implication

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Linear implication

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Exponential

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A focus on linearity

- Higher-Order is about *Seely's isomoprhism.*

$$!A \hat{\otimes} !B \simeq !(A \times B)$$

- Classicality is about a linear involutive negation :

$$A^{\perp\perp} \simeq A \qquad\qquad A \simeq A''$$

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

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$$!A \multimap \perp = A \Rightarrow \perp$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

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- Seely's isomorphism corresponds to the *Kernel theorem*:

$$\mathcal{C}^\infty(E, \mathbb{R})' \tilde{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

Just a glimpse at Differential Linear Logic

$$A, B := A \otimes B | 1 | A \parr B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A$$

Exponential rules of DiLL$_0$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

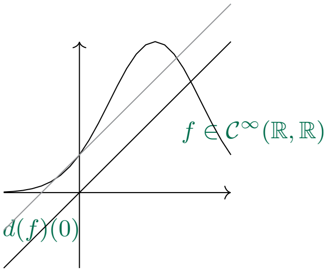
Differential Linear Logic

$$\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?A^\perp}\, d$$

A linear proof is in particular non-linear.

$$\frac{\vdash \Delta, A}{\vdash \Delta, !A}\, \bar{d}$$

From a non-linear proof we can extract a linear proof



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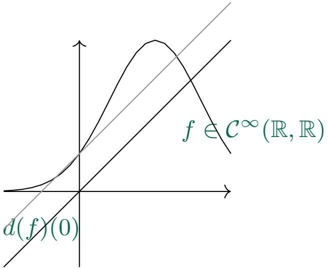

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Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : !A}{\vdash \Gamma, !A}\,\bar{d} \qquad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp}\,d}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\Gamma, \Delta}\,\text{cut}$$

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$$\rightsquigarrow \qquad \frac{\vdash \Gamma, v : A \qquad \vdash \Delta, \ell : A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(x) = \ell(x) : \mathbb{R} = \perp}\ \text{cut}$$

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

The computational content of differentiation

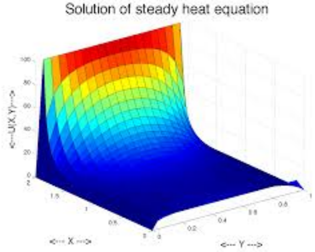
Historically, resource sensitive syntax and discrete semantics

- Quantitative semantics : $f = \sum_n f_n$
- Probabilistic Programming and Taylor formulas : $M = \sum_n M_n$

[Ehrhard, Pagani, Tasson, Vaux, Manzonetto ...]

Differentiation in Computer Science can have a different flavour :

- Numerical Analysis and functional analysis
- Ordinary and Partial Differential Equations



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Can we match the requirement of models of LL with the intuitions of physics ?

(YES, we can.)

Smooth and classical models of Differential Linear Logic

What's the good category in which we interpret formulas ?

Smoothness and Duality

Smoothness

Spaces : E is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

The two requirements works as opposite forces .

- ✓ A cartesian closed category with smooth functions.

  $\rightsquigarrow$ Completeness, and a dual topology fine enough.

- ✓ Interpreting $(E^\perp)^\perp \simeq E$ without an orthogonality:

  $\rightsquigarrow$ Reflexivity : $E \simeq E''$, and a dual topology coarse enough.

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What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

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The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
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MLL in TopVect

It's a mess.

Duality is not an orthogonality in general :

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically *not* preserved by $\otimes$.
- It is in the canonical case not an orthogonality : E'_β is not reflexive.

Monoidal closedness does not extends easily to the topological case :

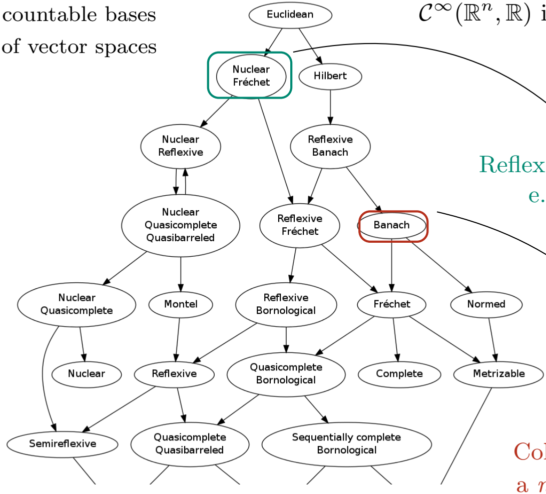
- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathcal{L}_\mathcal{B}(E \otimes_\mathcal{B} F, G) \simeq \mathcal{L}_\mathcal{B}(E, \mathcal{L}_\mathcal{B}(F, G))$
  $\Leftrightarrow$ "Grothendieck problème des topologies".

Topological models of DiLL

[Ehr02] [Ehr05] [DE08]

countable bases

of vector spaces



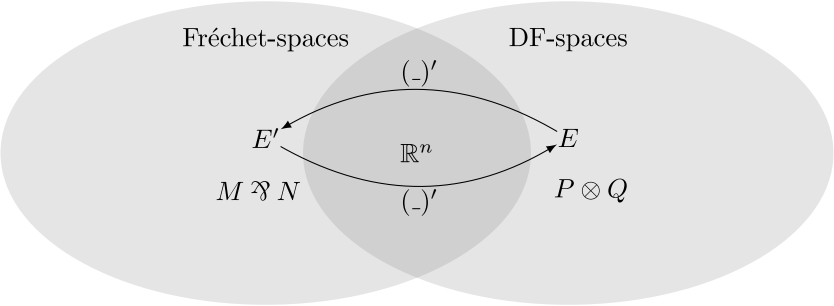
$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

Reflexive anc complete :
e.g. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$

Coherent Banach spaces [Gir99]
a *norm* is too restrictive

Fréchet and DF spaces

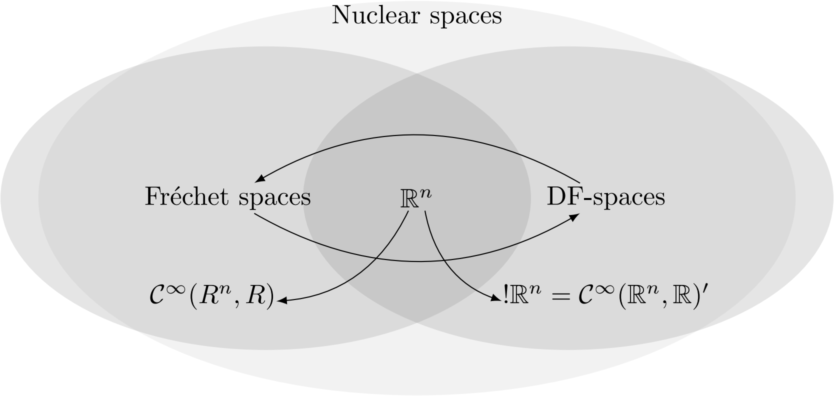
- Fréchet : metrizable complete spaces.
- (DF)-spaces : such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.



These spaces are in general not reflexive.

A polarized model of Smooth differential Linear Logic

Typical Nuclear Fréchet spaces are spaces of [smooth, holomorphic, rapidly decreasing ...] functions.



And more : $\uparrow$ is the completion $\rightsquigarrow$ Chiralities *[Mellies]*.

What we can get from semantics

- Higher-Order : how do we construct $\mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$.
- Partial Differentiation Equations : Distribution theory allows to generalize the interaction between *linearity and non-linearity* to the interaction between the*s*olutions and the parameters to a differential equation.

$\rightsquigarrow$ *interactions between theorical computer science and applied mathematics.*

A Logical account
for Linear Partial Differential Equations

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

$$\begin{aligned} f \in \mathcal{C}^\infty(A, \mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D_0(f)(x) \\ & \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g \\ \phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, D_0(\phi) = \psi \\ \phi \in !_D A \quad & \textit{iff } \exists \psi \in !A, D(\phi) = \psi \end{aligned}$$

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$$\bar{d} : \begin{cases} E'' \to \mathcal{C}^\infty(E, \mathbb{R})', \\ \phi = ev_x \mapsto \phi \circ D_0 = (f \mapsto ev_x(D_0(f)) \end{cases} \qquad d : \begin{cases} !E \to E'' \\ \psi \mapsto \psi_{|E'} \end{cases}$$

As $\mathcal{L}(E, \mathbb{R}) = D_0(\mathcal{C}^\infty(E, \mathbb{R}))$:

$$\bar{d} : \begin{cases} (D_0(\mathcal{C}^\infty(E, \mathbb{R})))' \to \mathcal{C}^\infty(E, \mathbb{R})', \\ \phi \mapsto \phi \circ D_0 \end{cases} \quad d : \begin{cases} \mathcal{C}^\infty(E, \mathbb{R})' \to (D_0(\mathcal{C}^\infty(E, \mathbb{R}))' \\ \psi \mapsto \psi_{|D_0(\mathcal{C}^\infty(E, \mathbb{R})} \end{cases}$$

Dereliction and co-dereliction, again.

$$\bar{d} : \begin{cases} (D_0(\mathcal{C}^\infty(E,\mathbb{R})))' \to \mathcal{C}^\infty(E,\mathbb{R})', \\ \phi \mapsto \phi \circ D_0 \end{cases} \quad d : \begin{cases} \mathcal{C}^\infty(E,\mathbb{R})' \to (D_0(\mathcal{C}^\infty(E,\mathbb{R}))' \\ \psi \mapsto \psi_{|D_0(\mathcal{C}^\infty(E,\mathbb{R})} \end{cases}$$

$$\bar{d}_D : \begin{cases} (D(\mathcal{C}^\infty(E,\mathbb{R}))' \to \mathcal{C}^\infty(E,\mathbb{R})' \\ \phi \mapsto \phi \circ D \end{cases} \quad d_D : \begin{cases} \mathcal{C}^\infty(E,\mathbb{R})' \to (D(\mathcal{C}^\infty(E,\mathbb{R}))' \\ \psi \mapsto \underline{\psi_{|D(\mathcal{C}^\infty(E,\mathbb{R}))}} \end{cases}$$

Another exponential is possible

$$!_D E := D^{-1}((\mathcal{C}^\infty(E,\mathbb{R})') \subset (\mathcal{C}_c^\infty(E,\mathbb{R}))'$$

The exponential is the space of solutions to a differential equation.

- $!_{D_0} E := E'' \simeq E$.
- $!_{Id} E := !E = \mathcal{C}^\infty(E,\mathbb{R})'$.

Linear Partial Differential Equations with constant coefficient

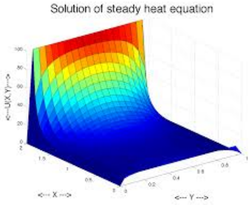
Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})'$ such that :

$$D(E_D) = \delta_0$$

and thus : output $D(E_D * \phi) = \phi$ input

D-DiLL

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

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$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \int D : ?_D A}\, w_D \qquad \frac{\vdash \Gamma, f : ?A, g : ?_D A}{\vdash \Gamma, f.g : ?_D A}\, c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\, d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\, \bar{w}_D \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\, \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\, \bar{d}_D$$

A deterministic cut-elimination.

A Logical Account for LPDEs, K. LICS 2018.

How to compute with higher-order distributions ?

joint work with JS Lemay.

Finite dimensional vector spaces into E

For every linear continuous injective function $f : \mathbb{R}^n \multimap E$:

$$\mathcal{E}'_f(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)'$$

Higher-order distributions

$$\mathcal{E}'(E) := \varinjlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}'_f(\mathbb{R}^n)$$

directed under the inclusion maps defined as

$$S_{f,g} : \mathcal{E}'_g(\mathbb{R}^n) \to \mathcal{E}'_f(\mathbb{R}^m), \phi \mapsto (h \mapsto \phi(h \circ \iota_{n,m}))$$

when $f = g \circ \iota_{n,m}$.

functorial only on injective linear maps : no promotion.

All about reflexivity

When E is reflexive, so is $\mathcal{E}'(E)$.

Duality works well :

$$\mathcal{E}'(E) \simeq (\varprojlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}_f(\mathbb{R}^n))'$$

but we still are in a polarized model.

A strong monoidal functor on isomorphisms

$$! : \begin{cases} \textsc{Refl}_{iso} \to \textsc{Refl}_{iso} \\ E \mapsto \mathcal{E}'(E) \\ \ell : E \multimap F \mapsto !\ell \in \mathcal{E}(F') \end{cases}$$

where $!\ell(\mathbf{f}_f) = \mathbf{f}_{\ell \circ f : \mathbb{R}^n \multimap F}$.

Higher-order dereliction and co-dereliction

$$d_E : \begin{cases} !(E) \to E'' \simeq E \\ \quad \phi \mapsto (\ell \in E' \mapsto \phi((\ell \circ f)_{f:\mathbb{R}^n \multimap E} \in \mathcal{E}(E)) \end{cases}$$

$$\bar{d}_E : \begin{cases} E \to !E \simeq (\mathcal{E}(E))' \\ x \mapsto (\mathbf{f}_f \in \mathcal{C}^\infty_f(\mathbb{R}^n, \mathbb{R}))_{f:\mathbb{R}^n \multimap E'} \mapsto D_0 \mathbf{f}_f(f^{-1}(x)) \\ \quad \text{where } f \text{ is injective such that } x \in Im(f)\ . \end{cases}$$

Computing in higher-dimension

$$!E = \overline{< \delta_x, x \in E >}$$

By Frölicher, as used by Blute, Ehrhard and Tasson.

That's a *discretisation scheme* :
let's embed numerical schemes into cut-elimination, through compositionality.

Conclusion

Polarization

From mathematics to proof-theory and back.

Differential Equations

A coalgebraic structure on D

Weakening

$w :!_D E \to \mathbb{R}$ comes from $t : E \to \{0\}$.

If $E = \mathbb{R}^n$, define $\mathbb{R}^{n'}$ another copy of E. Then

$$\begin{aligned} D(\mathcal{C}^\infty(E,\mathbb{R})) &\to D(\mathcal{C}^\infty(E \times E,\mathbb{R})) \\ &= D(\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{n'},\mathbb{R})) \\ &= D(\mathcal{C}^\infty(E,\mathbb{R}) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R})) \\ &= D(\mathcal{C}^\infty(E,\mathbb{R})) \parr \mathcal{C}^\infty(\mathbb{R}^{n'},\mathbb{R}) \end{aligned}$$

Contraction

We thus have $c :!_D E \to !E \otimes !_D E$.

What's typable with D-DiLL

Consider D a Smooth Linear Partial Differential Operator : D : $\mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$. D acts on $E \times E$:

$$\hat{D} = (D \otimes Id_F)\mathcal{C}^\infty(E \times E, \mathbb{R}) \to \mathcal{C}^\infty(E \times E, \mathbb{R})$$

Then Green's function is the operator $K_{x,y} :!E\ to!E$ such that :

$$K_{x,y} \circ (\hat{D})' = \delta_{x-y}$$

$$\dfrac{\dfrac{\vdash \Gamma, ?_D E^\perp, ?E^\perp}{\vdash ?_D E^\perp}\, c_D \qquad \dfrac{\vdash \Delta, ?_D E \qquad \dfrac{\vdash}{\vdash !_D E}\, \bar{w}_D}{\vdash ?_D \Delta, !_D E}\, c_D}{\vdash \Gamma, \Delta}\, cut$$

A closer look to Kernels

A answer to a well-known issue :

- Any $k \in (\mathrm{L}_p(\mu \otimes \eta))'$ gives rise to a compact operator $T_k : \mathrm{L}_p(\mu) \to \mathrm{L}_{p^*}(\eta) \simeq (\mathrm{L}_p(\eta))' : T_k(f)(g) = k(f.g)$.
- This is not a surjection : if $p = p^* = 2$, for $T_k = Id$ one should have $k = \delta_{x-y}$, which is *not a function*.
- The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions spaces, generalizing L_p :

Kernel theorems

$$\begin{aligned} \mathcal{L}(\mathcal{C}^\infty(E,\mathbb{R})', \mathcal{C}^\infty(F,\mathbb{R})'') &\simeq \mathcal{C}^\infty(E,\mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F,\mathbb{R})' \\ &\simeq \mathcal{C}^\infty(E \times F,\mathbb{R})' \\ T_k &\mapsto K_{x,y} \end{aligned}$$

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Nuclearity

A closer look to Kernels

A answer to a well-known issue :

- Any $k \in (\mathrm{L}_p(\mu \otimes \eta))'$ gives rise to a compact operator $T_k : \mathrm{L}_p(\mu) \to \mathrm{L}_{p^*}(\eta) \simeq (\mathrm{L}_p(\eta))' : T_k(f)(g) = k(f.g)$.
- This is not a surjection : if $p = p^* = 2$, for $T_k = Id$ one should have $k = \delta_{x-y}$, which is *not a function*.
- The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions spaces, generalizing L_p :

Kernel theorems

$$\begin{aligned} \mathcal{C}^\infty(E,\mathbb{R})'\hat{\otimes}\mathcal{C}^\infty(F,\mathbb{R})' &\simeq \mathcal{L}(\mathcal{C}^\infty(E,\mathbb{R})', \mathcal{C}^\infty(F,\mathbb{R})'') \\ &\simeq \mathcal{C}^\infty(E \times F, \mathbb{R})' \end{aligned}$$

Density