A comonadic view of simulation and quantum resources

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Overview of the talk

- Crash course on contextuality
- What are we trying to formalize?
- Free operations on empirical models. Free transformations.
- Simulations.
- Equivalence of the viewpoints
- No-cloning
- Further topics

Measurement scenarios

A measurement scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X a finite set of measurements
- Σ is a simplicial complex on X, whose faces are called the *measurement contexts*.
- $O = (O_x)_{x \in X}$ specifies for each measurement $x \in X$ a finite non-empty set of possible outcomes O_x;
- Note: X and each O_x finite.

Events and distributions

Let $\langle X, \Sigma, O \rangle$ be a scenario. For any $U \subseteq X$, we write

$$\mathcal{E}_O(U) := \prod_{x \in U} O_x$$

for the set of assignments of outcomes to each measurement in the set U. When U is a valid context, this is be the set of possible joint outcomes for the measurements U

For any set Y, let $D(Y)$ denote the set of finitely supported probability distributions over Y

Empirical models

- An empirical model $e\colon \langle X, \Sigma, O \rangle$ is a family $(e_\sigma)_{\sigma \in \Sigma}$ where e_σ is a distribution over the available joint outcomes, i.e.

$$e_\sigma \in \mathsf{D} \circ \mathcal{E}_O(\sigma) = \mathsf{D}\left(\prod_{x \in \sigma} O_x\right)$$

- We assume (generalized) no-signalling, i.e. that marginal distributions are well-defined: for any $\sigma, \tau \in \Sigma$ with $\tau \subseteq \sigma$, it holds that

$$e_\tau = e_\sigma|_\tau = \mathsf{D} \circ \mathcal{E}(\tau \subseteq \sigma)(e_\sigma) \;;$$

concretely, for any $t \in \mathcal{E}(\tau)$,

$$e_\tau(t) = \sum_{s \in \mathcal{E}(\sigma), s|_\tau = t} e_\sigma(s) \;.$$

Contextuality

- Contextuality: Is there a joint distribution d on $\mathcal{E}_O(X)$ such that $d|_\sigma = e_\sigma$ for each $\sigma \in \Sigma$?

- Strong contextuality: Is there a joint outcome $s \in \mathcal{E}_O(X)$ consistent with e?

- Non-contextual fraction $NCF(e) \in [0, 1]$: what fraction of e is non-contextual? $CF(e) = 1 - NCF(e)$

Examples

Bell:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(x_0, y_0)	$1/2$	0	0	$1/2$
(x_0, y_1)	$3/8$	$1/8$	$1/8$	$3/8$
(x_1, y_0)	$3/8$	$1/8$	$1/8$	$3/8$
(x_1, y_1)	$1/8$	$3/8$	$3/8$	$1/8$

Examples

PR box:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(x_0, y_0)	$1/2$	0	0	$1/2$
(x_0, y_1)	$1/2$	0	0	$1/2$
(x_1, y_0)	$1/2$	0	0	$1/2$
(x_1, y_1)	0	$1/2$	$1/2$	0

Towards morphisms

- A bunch of mathematical objects has been defined, but what are the morphisms?
- Given $e\colon \langle X, \Sigma, O\rangle$ and $d\colon \langle Y, \Theta, P\rangle$, a morphism $d \to e$ is a way of transforming d to e using free operations.
- Alternatively: a morphism $d \to e$ is a way of *simulating* e using d.

Examples from the literature

- Any two-outcome bipartite box can be simulated with PR boxes (Barrett-Pironio).
- An explicit two-outcome three-partite box that cannot be simulated with PR boxes (Barrett-Pironio).
- No finite set of bipartite boxes can simulate all of them (Dupuis et al).

Free operations

We have

- Zero model z: the unique empirical model on the empty measurement scenario

$$\langle \emptyset, \Delta_0 = \{\emptyset\}, () \rangle \ .$$

- Singleton model u: the unique empirical model on the one-outcome one measurement scenario

$$\langle \mathbf{1} = \{\star\}, \Delta_1 = \{\emptyset, \mathbf{1}\}, (O_\star = \mathbf{1}) \rangle \ .$$

- Probabilistic mixing: Given empirical models e and d in $\langle X, \Sigma, O \rangle$ and $\lambda \in [0, 1]$, the model $e +_\lambda d : \langle X, \Sigma, O \rangle$ is given by the mixture $\lambda e + (1 - \lambda) d$

Free operations

- Tensor: Let $e : \langle X, \Sigma, O \rangle$ and $d : \langle Y, \Theta, P \rangle$ be empirical models. Then

$$e \otimes d : \langle X \sqcup Y, \Sigma * \Theta, (O_x)_{x \in X} \cup (P_y)_{y \in Y} \rangle$$

  represents running e and d independently and in parallel. Here $\Sigma * \Theta := \{\sigma \cup \theta | \sigma \in \Sigma, \theta \in \Theta\}$.

- Coarse-graining: given $e : \langle X, \Sigma, O \rangle$ and a family of functions $h = (h_x \colon O_x \longrightarrow O'_x)_{x \in X}$, get a coarse-grained model

$$e/h : \langle X, \Sigma, O' \rangle$$

- Measurement translation: given $e : \langle X, \Sigma, O \rangle$ and a simplicial map $f \colon \Sigma' \longrightarrow \Sigma$, the model $f^* e : \langle X', \Sigma', O \rangle$ is defined by pulling e back along the map f.

Free operations

Given a simplicial complex Σ and a face $\sigma \in \Sigma$, the link of σ in Σ is the subcomplex of Σ whose faces are

$$\mathsf{lk}_\sigma \Sigma := \{\tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\} \ .$$

- Conditioning on a measurement: Give $e : \langle X, \Sigma, O \rangle$, $x \in X$ and a family of measurements $(y_o)_{o \in O_x}$ with $y_o \in \mathsf{Vert}(\mathsf{lk}_x \Sigma)$. Consider a new measurement $x?(y_o)_{o \in O_x}$, abbreviated $x?y$. Get

  $$e[x?y] : \langle X \cup \{x?y\}, \Sigma[x?y], O[x?y \mapsto O_{x?y}] \rangle$$

  that results from adding $x?y$ to e.

Summary of operations

The operations generate terms

$$\begin{aligned}\text{Terms} \ni t := & a \in \text{Var} \mid \text{z} \mid \text{u} \mid f^* t \mid t/h \\ & \mid t +_\lambda t \mid t \otimes t \mid t[x?y]\end{aligned}$$

typed by measurement scenarios.

Morphisms as free transformations

Proposition

A term without variables always represents a noncontextual empirical model. Conversely, every noncontextual empirical model can be represented by a term without variables.

Can d be transformed to e?

Formally: is there a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] = e$?

Morphisms as simulations

- Think of a measurement scenario as a concrete experimental setup, where for each measurement there is a grad student responsible for it.
- The grad student responsible for measuring $x \in X$, should have instructions which measurement $\pi(x) \in Y$ to use instead.
- Given a result for those measurements, should be able to determine the outcome to output.
- The outcome statistics should be identical to those of e.

Dependencies on multiple measurements and stochastic processing added as a comonadic effect.

Deterministic morphisms

Definition

Let $\mathbf{X} = \langle X, \Sigma, O \rangle$ and $\mathbf{Y} = \langle Y, \Theta, P \rangle$ be measurement scenarios.
A *deterministic morphism* $\langle \pi, h \rangle \colon \mathbf{Y} \longrightarrow \mathbf{X}$ consists of:

- a simplicial map $\pi \colon \Sigma \longrightarrow \Theta$;
- a natural transformation $h \colon \mathcal{E}_P \circ \pi \longrightarrow \mathcal{E}_O$;
  equivalently, a family of maps $h_x \colon P_{\pi(x)} \longrightarrow O_x$ for each $x \in X$.

Let $e \colon \mathbf{X}$ and $d \colon \mathbf{Y}$ be empirical models. A *deterministic simulation* $\langle \pi, h \rangle \colon d \longrightarrow e$ is a deterministic morphism $\langle \pi, h \rangle \colon \mathbf{Y} \longrightarrow \mathbf{X}$ that takes d to e.

Example simulation

If $h = (h_x \colon O_x \longrightarrow O'_x)_{x \in X}$, can coarse-grain e to get e/h.

There is a deterministic simulation $e \to e/h$:

If you need to measure $x \in X$ for e/h, just measure $x \in X$ in the experiment e and apply h to the outcome.

Beyond deterministic maps

- Deterministic morphisms aren't enough: a deterministic model can't simulate (deterministically) a coinflip

- Need classical (shared) correlations

- Moreover, to simulate $x \in X$ one might want to run a whole *measurement protocol* on $\langle Y, \Theta, P \rangle$.

Measurement protocols

Definition

Let $\mathbf{X} = \langle X, \Sigma, O \rangle$ be a measurement scenario. We define recursively the *measurement protocol completion* $MP(X)$ of X by

$$\mathrm{MP}(\mathbf{X}) ::= \emptyset \mid (x, f)$$

where $x \in X$ and $f \colon O_x \to \mathrm{MP}(\mathrm{lk}_x \Sigma)$.
$\mathrm{MP}(\mathbf{X})$ can be given the structure of a measurement scenario,and if $e \colon \langle X, \Sigma, O \rangle$, can extend it to $\mathrm{MP}(e) \colon \mathrm{MP}(\mathbf{X})$

General simulations

Definition

Given empirical models e and d, a *simulation* of e by d is a deterministic simulation $\mathsf{MP}(d \otimes c) \to e$ for some noncontextual model c.
We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Theorem

MP *defines a comonoidal comonad on the category of empirical models.*

Roughly: comultiplication $\mathsf{MP}(\mathbf{X}) \to \mathsf{MP}^2(\mathbf{X})$ by "flattening", unit $\mathsf{MP}(\mathbf{X}) \to \mathbf{X}$, and $\mathsf{MP}(\mathbf{X} \otimes \mathbf{Y}) \to \mathsf{MP}(\mathbf{X}) \otimes \mathsf{MP}(\mathbf{Y})$

The viewpoints agree

Theorem

Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] \simeq e$.

Proof.

(Sketch) If $d \rightsquigarrow e$, then e can be obtained from $\mathrm{MP}(d \otimes x)$ by a combination of a coarse-graining and a measurement translation.
There is a term representing x and MP can be built by repeated controlled measurements.
For the other direction, build a simulation $d \rightarrow t[d/a]$ inductively on the structure of t. □

No-cloning

Theorem (No-cloning)

$e \rightsquigarrow e \otimes e$ if and only if e is noncontextual.

Further questions

- Study the preorder induced by $d \rightsquigarrow e$.
- What can you simulate with arbitrarily many copies of d?
- The same for possibilistic empirical models. Connections to CSPs.
- Changing the free class of "free" models allows for more general simulations. What can be said about e.g. quantum simulations? Does the no-cloning result generalize?
- Comparison with other approaches to contextuality.

Further questions 2

- Multipartite non-locality
- Graded structure on the comonad?
- MBQC?
- Generating all empirical models?
- Bell inequalities?

Summary

- Intraconversions of contextual resources formalized in terms of
  - free operations
  - simulations
- These viewpoints agree and capture known examples
- A no-cloning result
- Several avenues for further work