Categorical Quantum Dynamics

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Quantum Dynamics as Algebra

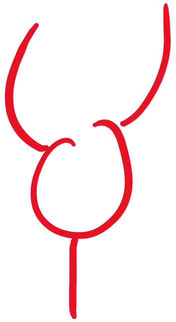
Dynamics = Time-translation symmetry

		time	energy
discrete periodic	$\longrightarrow$	$\mathbb{Z}_T$	$\mathbb{Z}_T$
discrete	$\longrightarrow$	$\mathbb{Z}$	$\mathbb{R}/T\mathbb{Z}$
continuous periodic	$\longrightarrow$	$\mathbb{R}/T\mathbb{Z}$	$\mathbb{Z}$
continuous	$\longrightarrow$	$\mathbb{R}$	$\mathbb{R}$

Quantising Groups

time-translation group

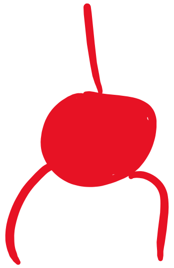
Work with the Hopf algebra $\underbrace{\mathbb{C}[G]}_{\text{call this } T\text{, for time}}$



$|t\rangle \otimes |t\rangle$

$\uparrow$

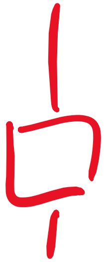
$|t\rangle$



$|t+s\rangle$

$\uparrow$

$|t\rangle \otimes |s\rangle$



$|-t\rangle$

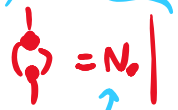
$\uparrow$

$|t\rangle$

antipode

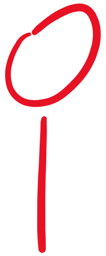
(special commutative †-Frobenius algebra)

(quasi-special symmetric †-Frobenius algebra)



$= N_0 \,|$

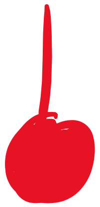
invertible



1

$\uparrow$

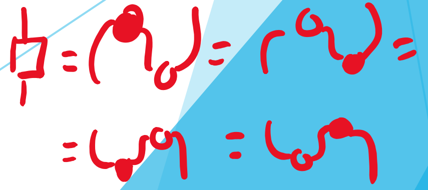
$|t\rangle$



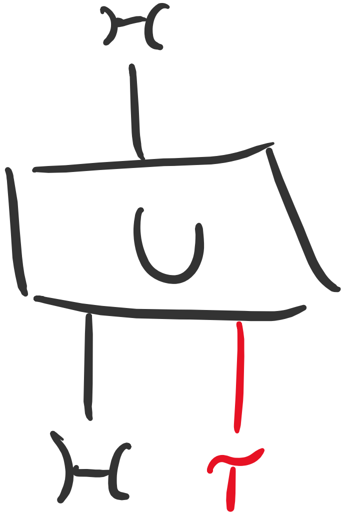
$|0\rangle$

$\uparrow$

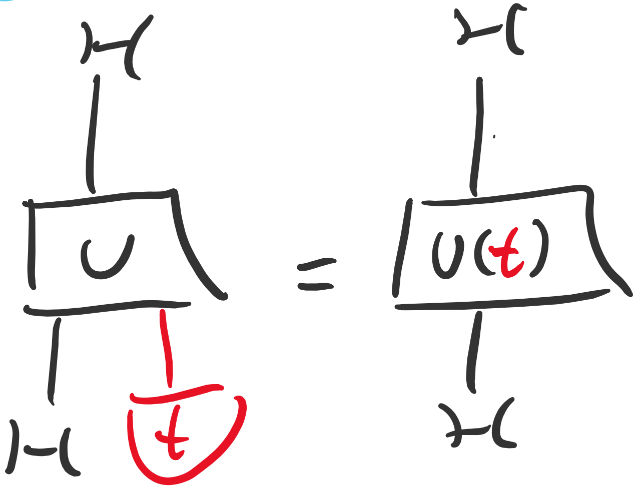
1

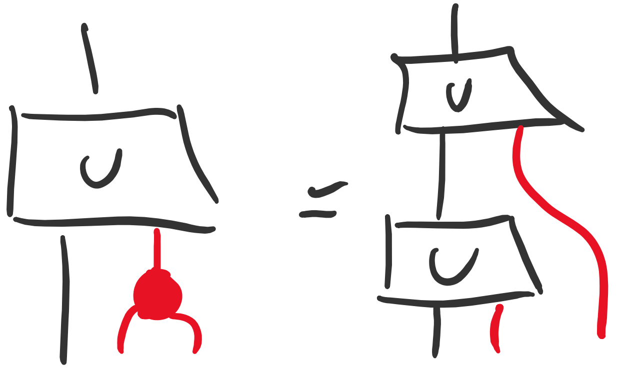


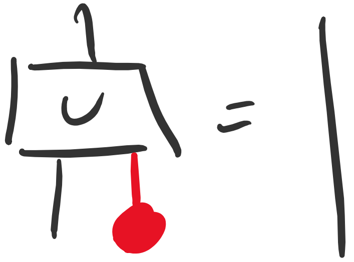
Quantising Algebras

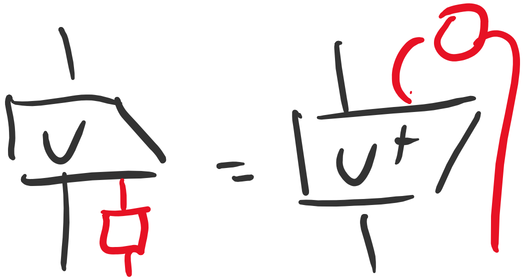


s.t.









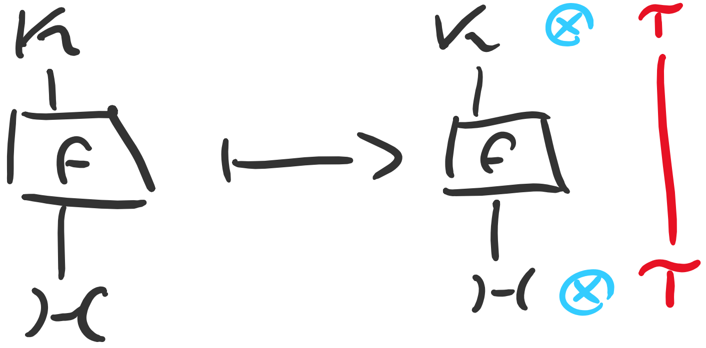
$U(t+s) = U(s)U(t)$ $\qquad U(0) = 1$ $\qquad U(-t) = U(t)^\dagger$

Algebra $\qquad$ unitary

Quantum Dynamical Systems

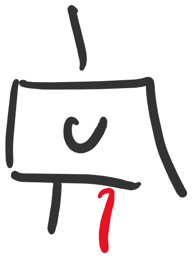
Monads:

(μ =  η = )

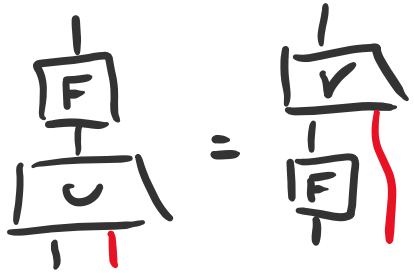


"add time on the side"

$\Downarrow$ En algebras (unitary)

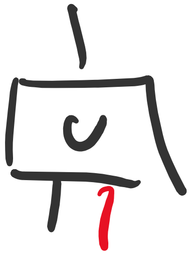
Objects: 

quantum dynamical systems

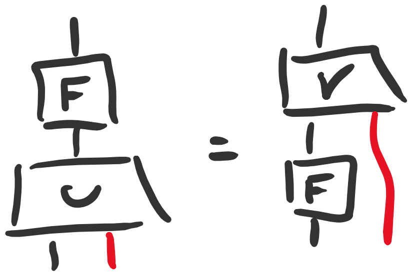
Morphisms: $U \longrightarrow V$ 

time-equivariant transformations

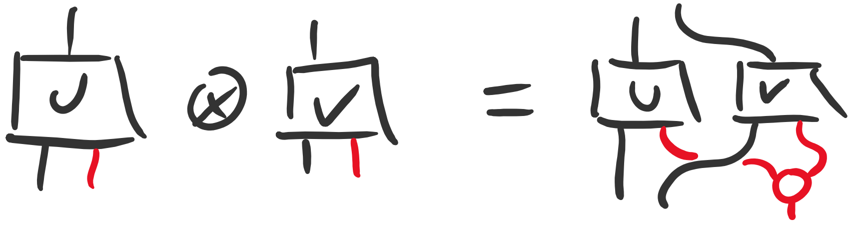
Quantum Dynamical Systems

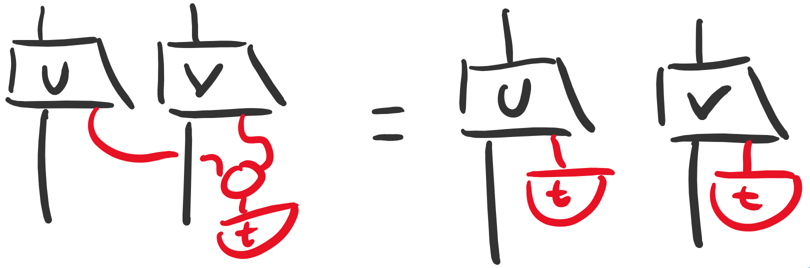
Objects: 

quantum dynamical systems

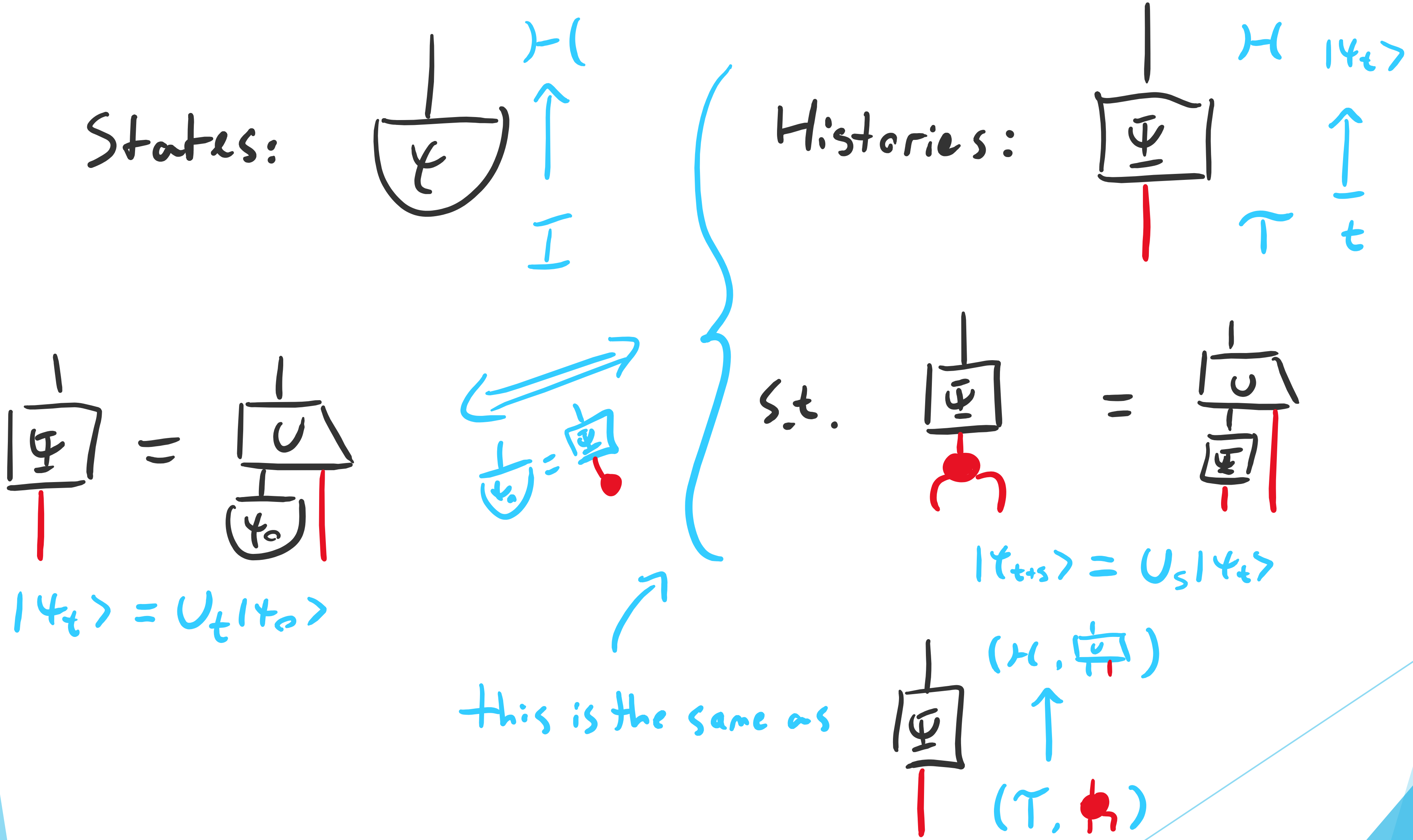
Morphisms: $U \longrightarrow V$ 

time-equivariant transformations

Composition: 

Synchronised quantum dynamical systems $\Rightarrow$ 

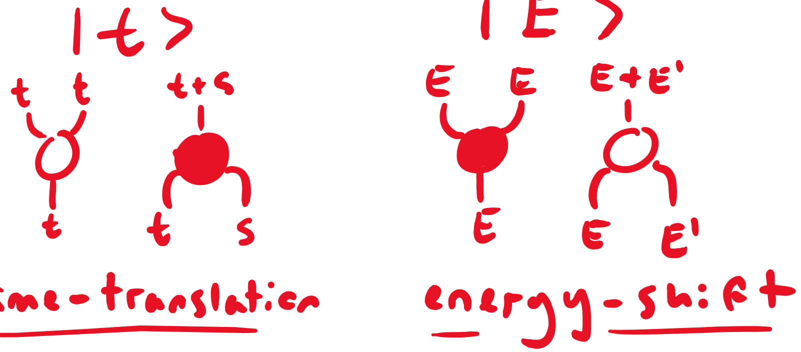
States and Histories



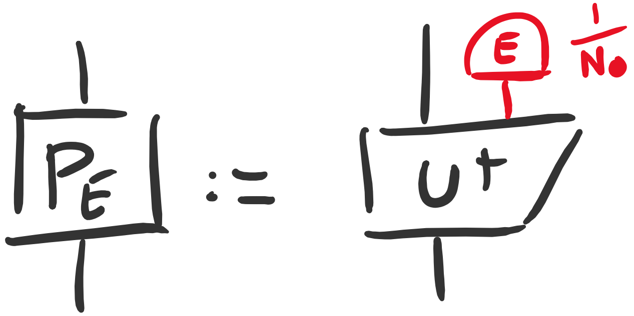
Hamiltonian as a Coalgebra

Time-Energy Duality:

$$\overset{\text{time}}{\mathbb{C}[G]} \cong \overset{\text{energy}}{\mathbb{C}[G^\wedge]}$$



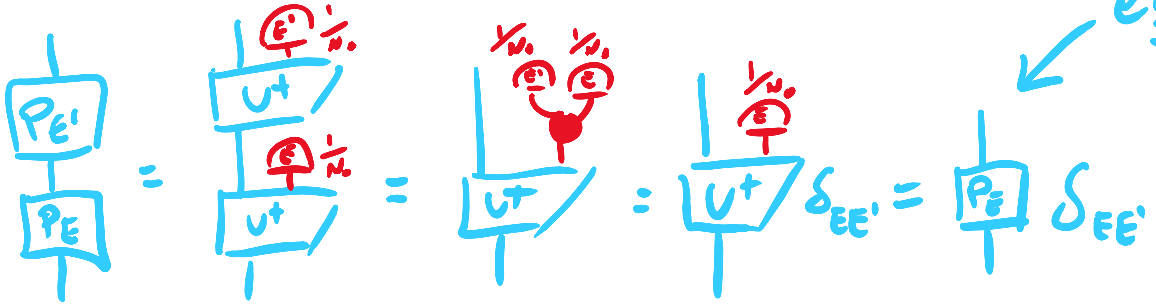
$$|E\rangle := \int e^{i\frac{Et}{\hbar}} |t\rangle dt$$



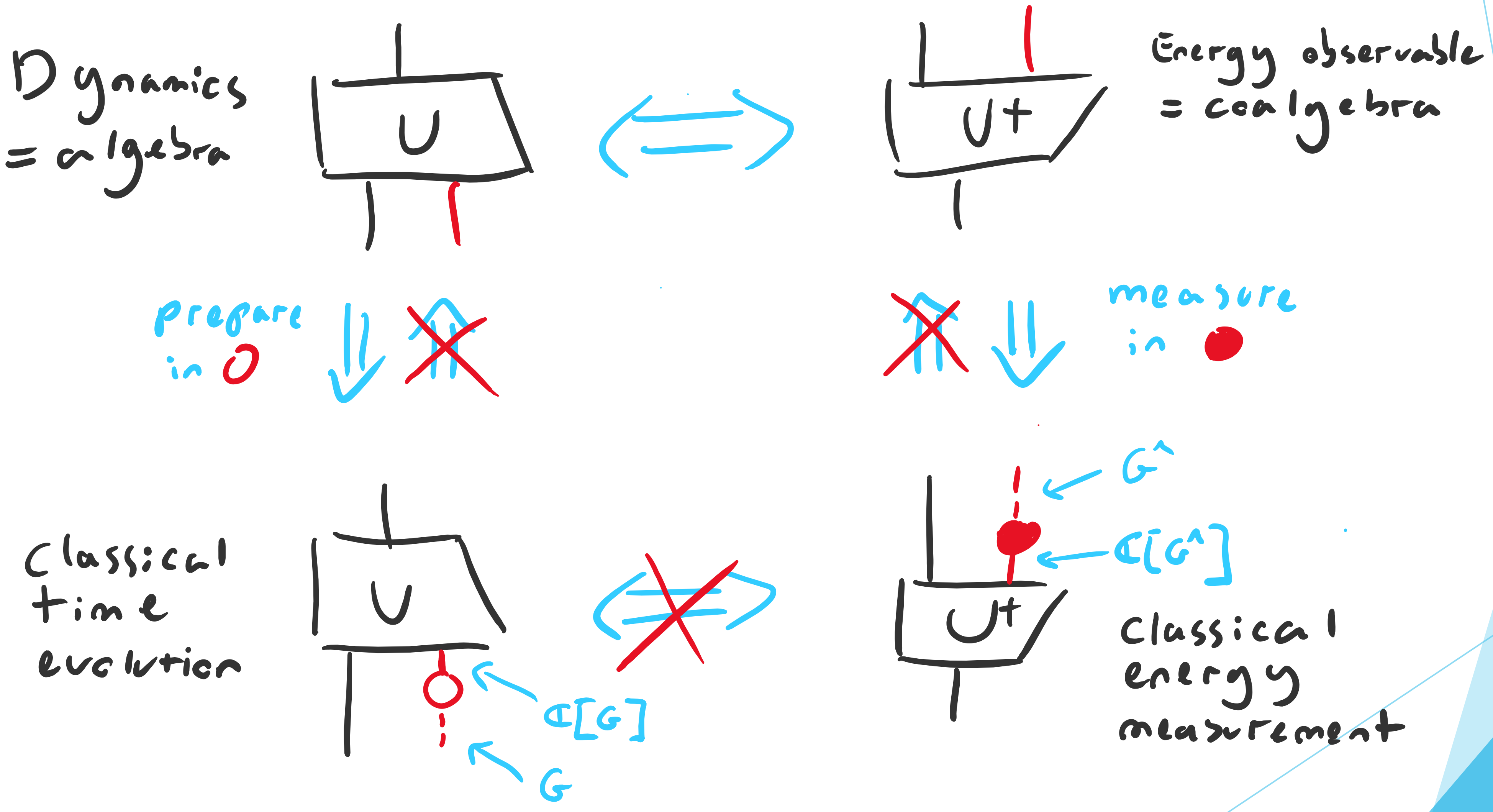
Projector over energy E eigenstate.

e.g. Orthoidempotence

$$P_{E'} P_E = P_E \delta_{EE'}$$



Hamiltonian as a Coalgebra



Schrödinger's Equation

$$H\,|\psi_E\rangle = E\,|\psi_E\rangle$$

$$U(t) = \exp\left[-i\frac{Ht}{\hbar}\right]$$

$$U(t)|\psi_E\rangle = e^{-i\frac{Et}{\hbar}}|\psi_E\rangle$$

$$i\hbar\frac{d}{dt}|\psi_t\rangle = H\,|\psi_t\rangle$$

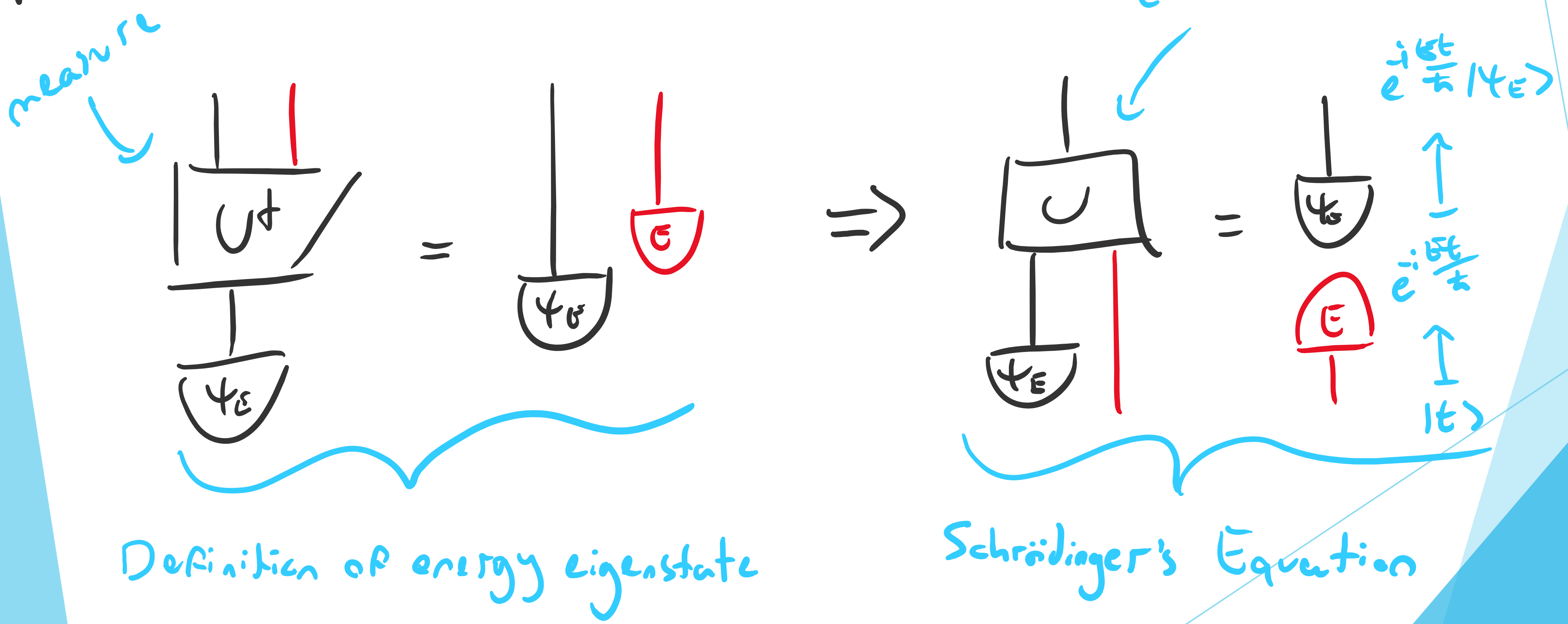
$$|\psi_{t+\delta t}\rangle = U(\delta t)|\psi_t\rangle$$

$$U(t)|\psi_E\rangle = e^{-i\frac{Et}{\hbar}}|\psi_E\rangle$$

Schrödinger's Equation

$$U(t)|\psi_E\rangle = e^{-i\frac{Et}{\hbar}}|\psi_E\rangle$$

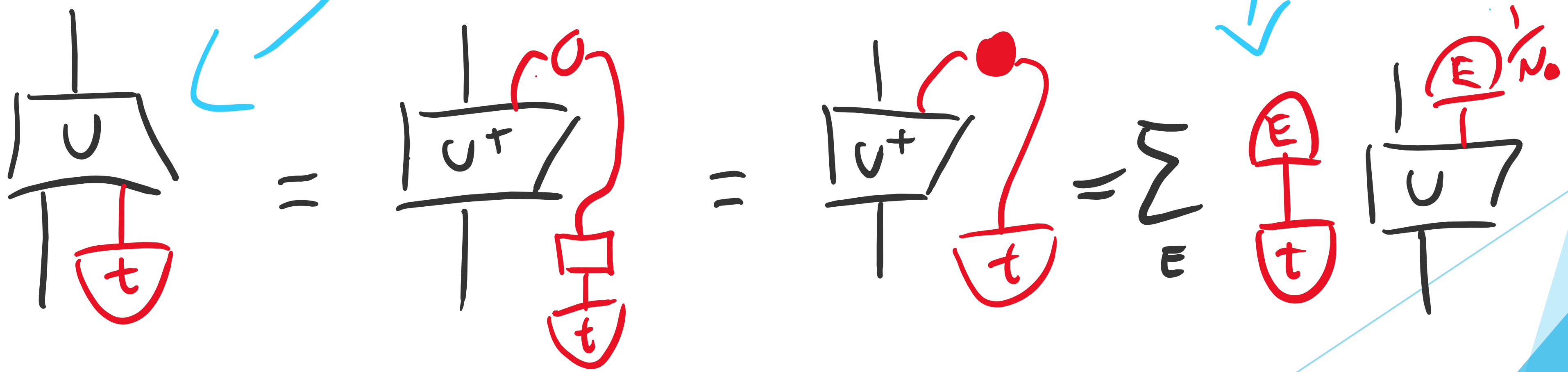
proof:



Stone's Theorem (with a really long name)

$$U(t) = \int e^{-i\frac{Et}{\hbar}} \, dP_\sigma$$

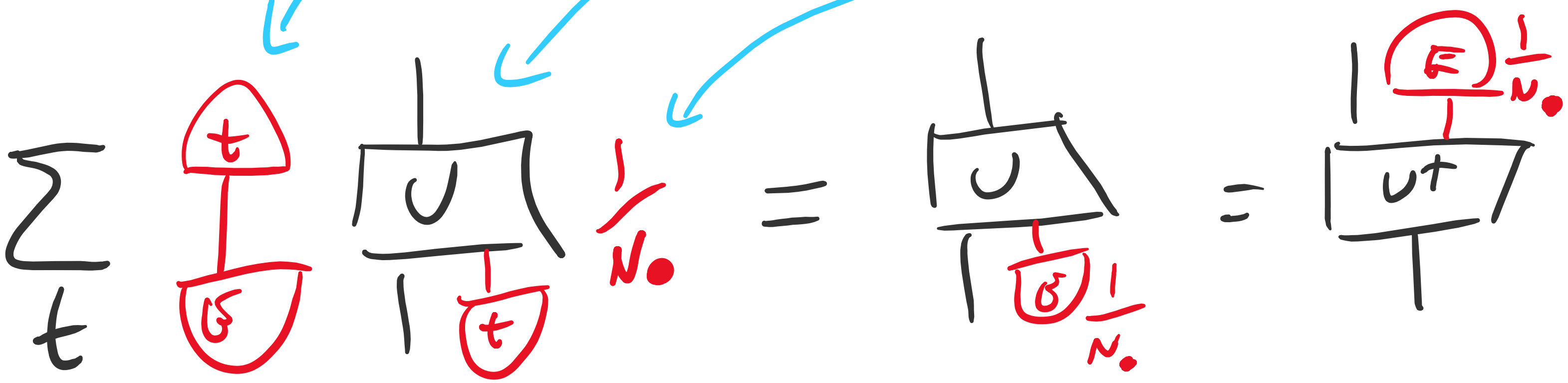
proof:



von Neumann's Ergodic Theorem

$$\lim_{T\to\infty}\int_0^T \underbrace{e^{i\frac{Et}{\hbar}}}\ \underbrace{U(t)}\ \underbrace{\frac{dt}{T}} = \underbrace{P_E}$$

Proof:



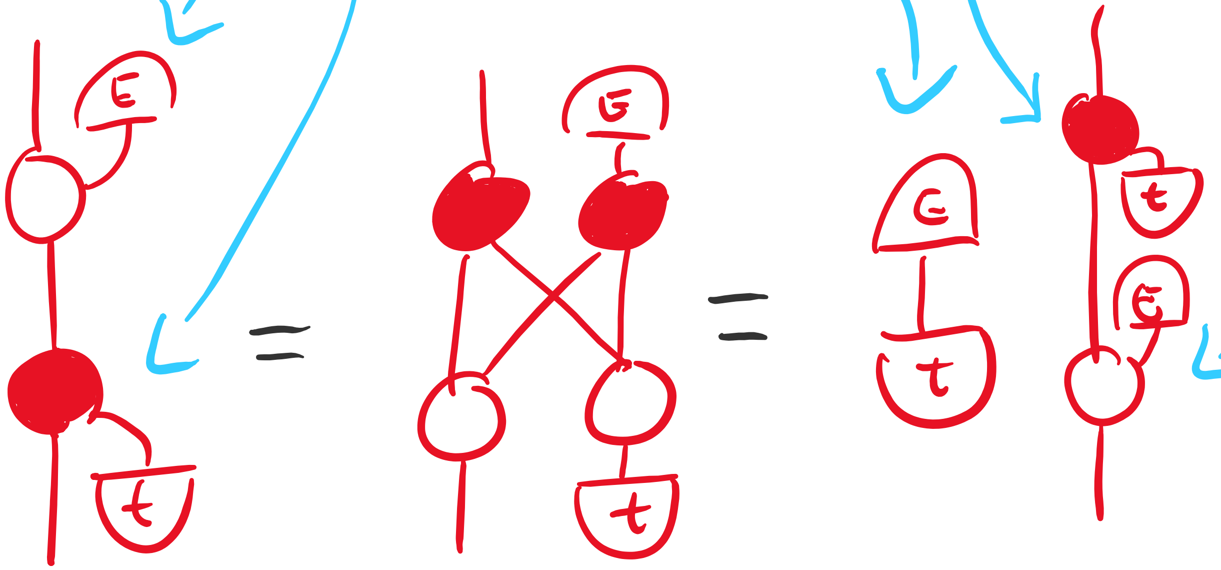
Weyl Canonical Commutation Relations

time-translation

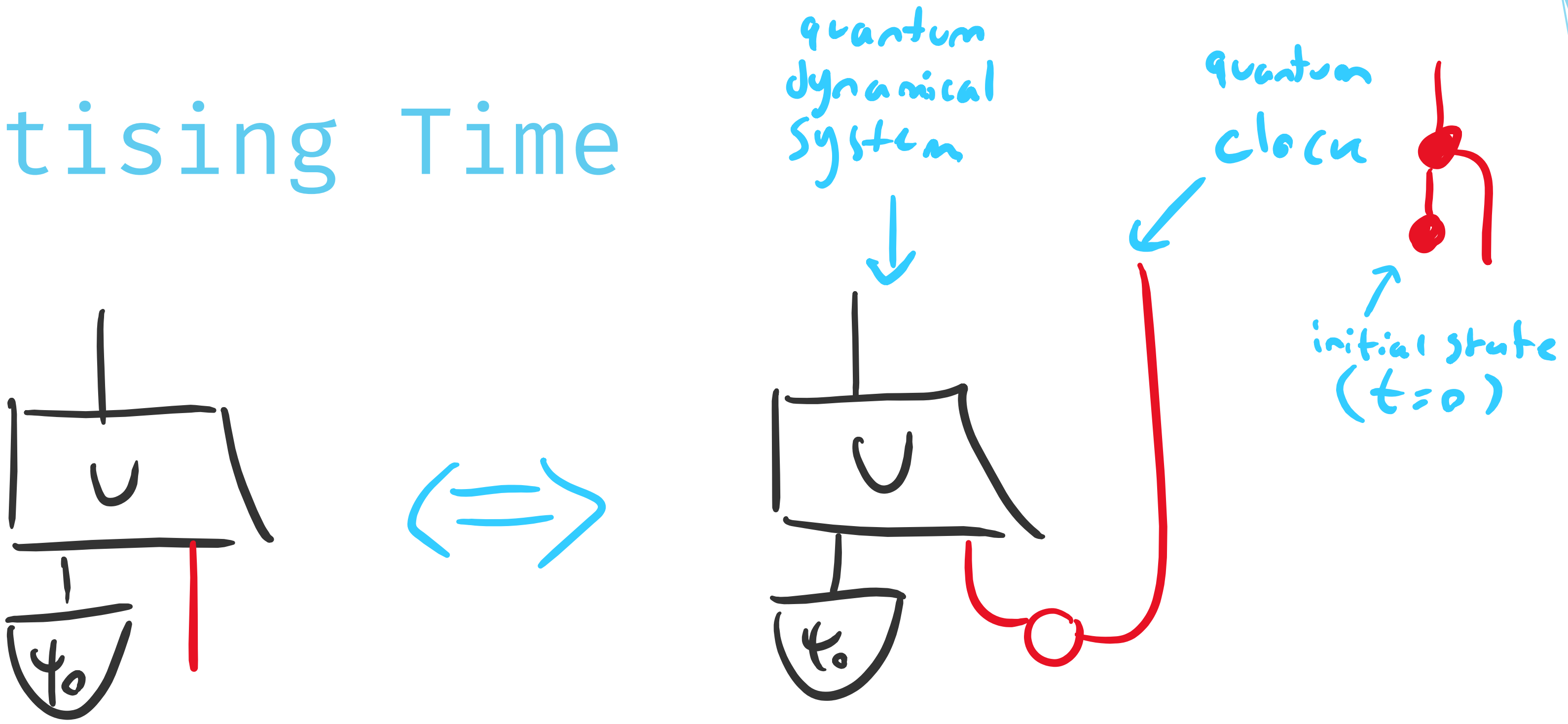
energy-shift

$$V_E U_t = e^{-i\frac{Et}{\hbar}} U_t V_E$$

proof:



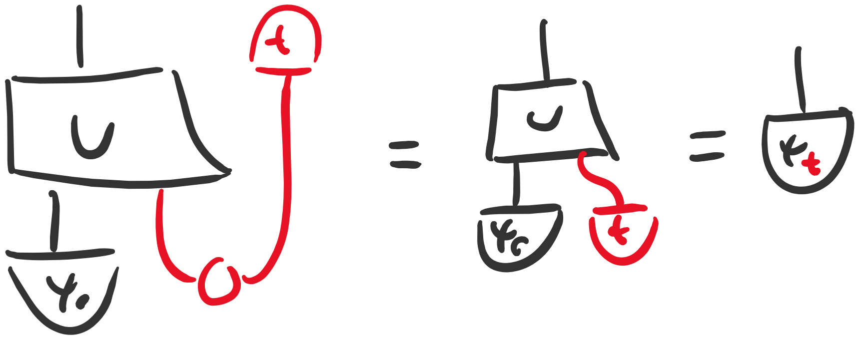
Quantising Time



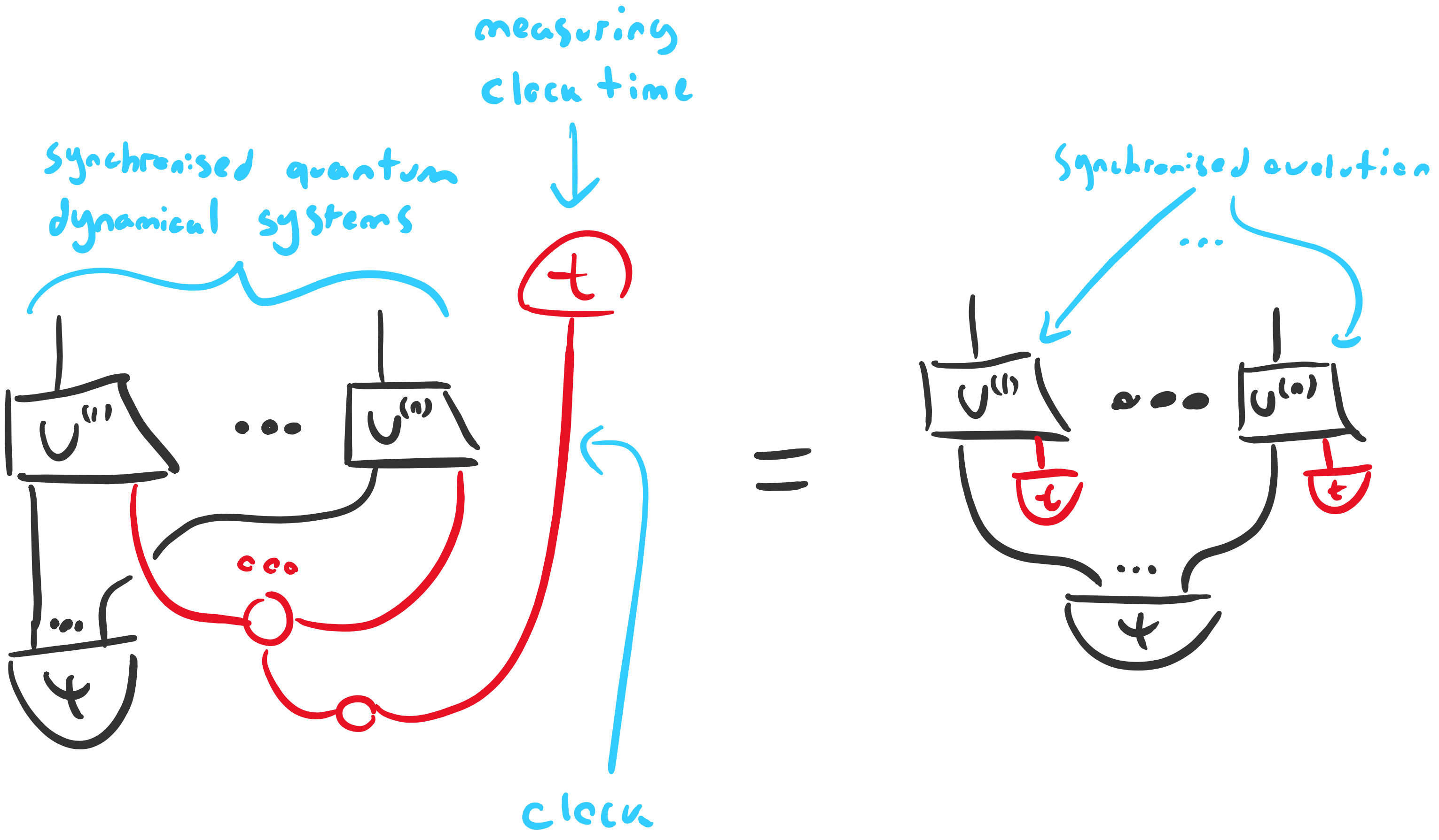
history

synchronised pair

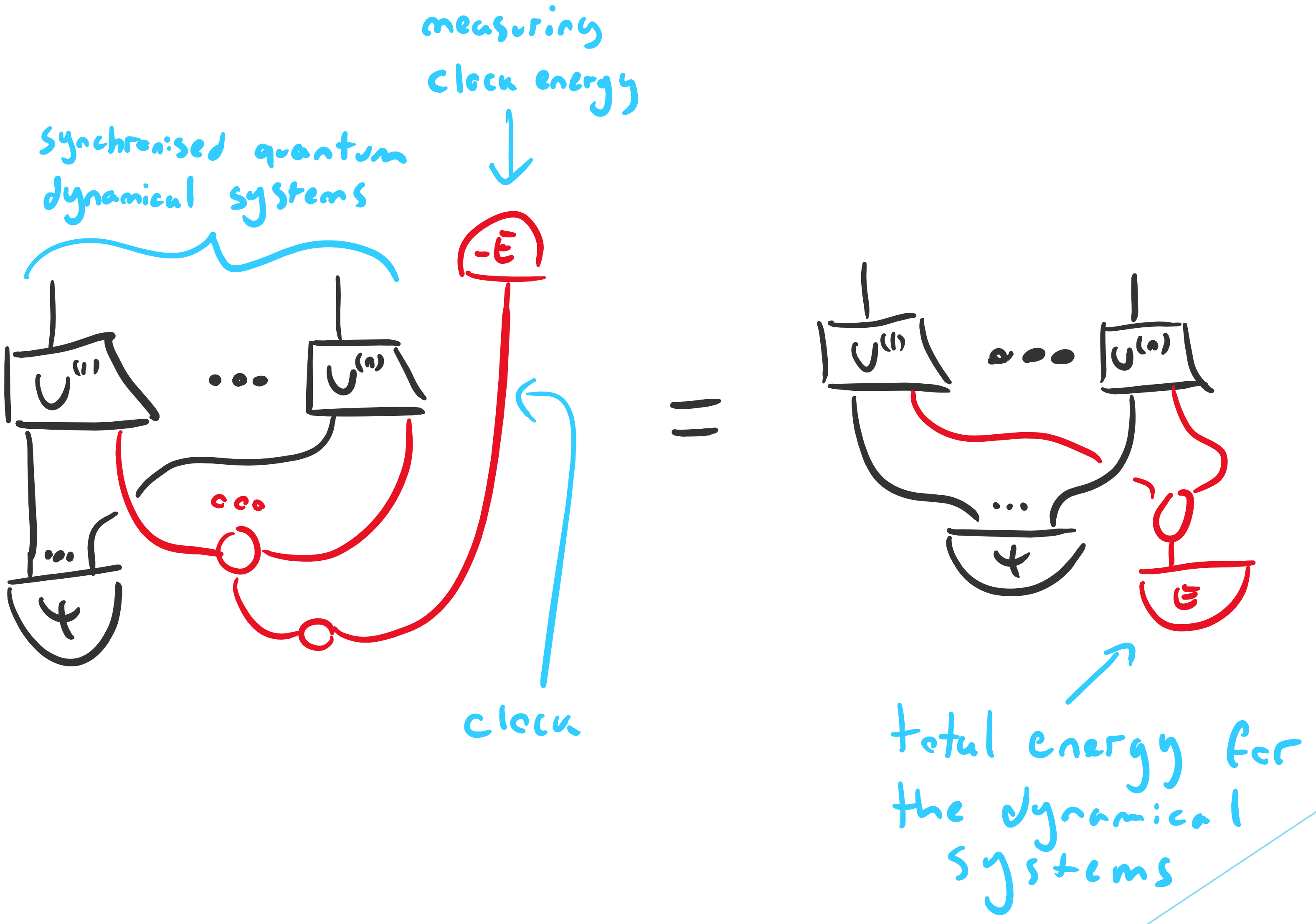
measuring clock time:



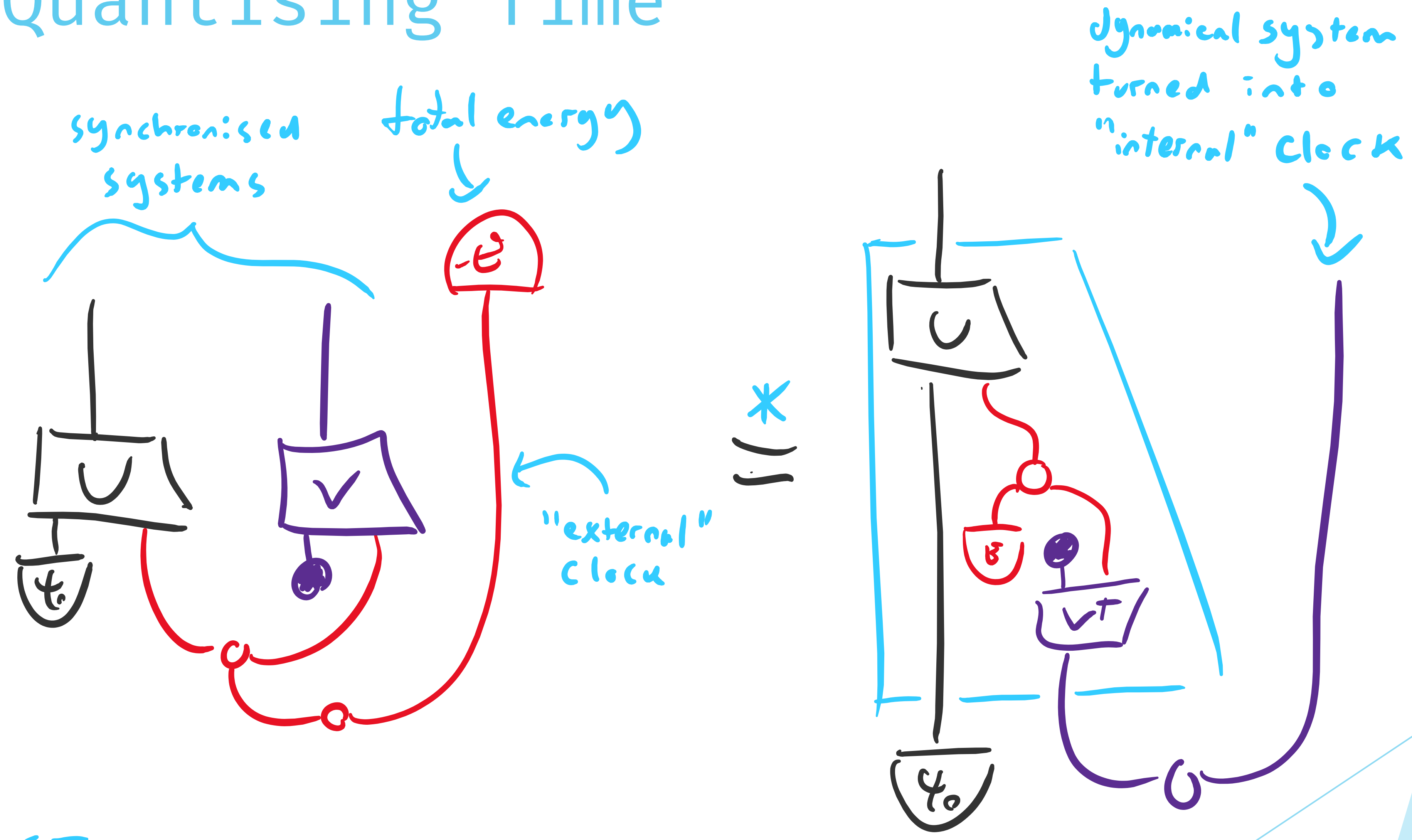
Quantising Time



Quantising Time

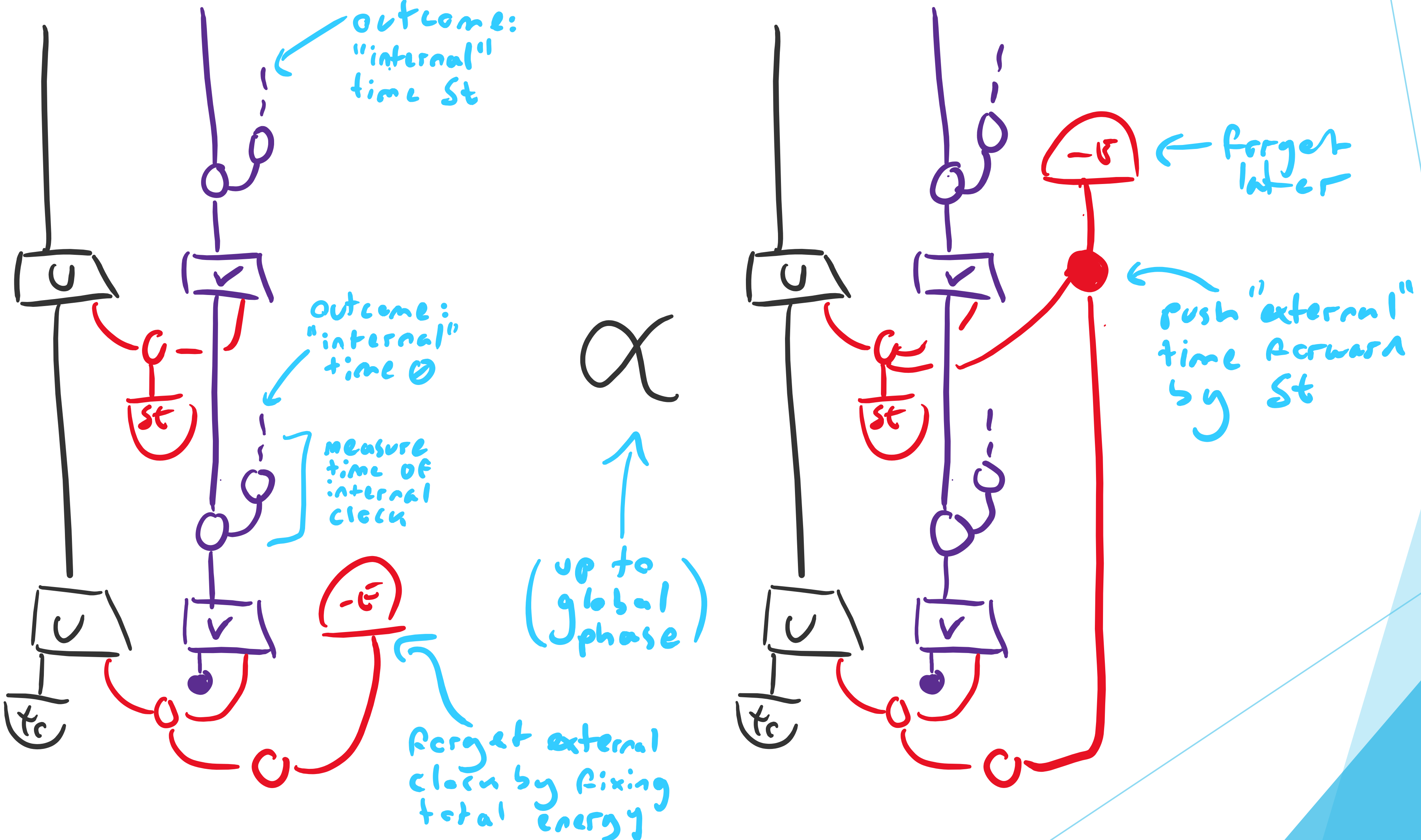


Quantising Time



* Terms and conditions apply.

As time goes by…



THANK YOU!

Any Questions?