

Categorical Quantum Dynamics

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Quantum Dynamics as Algebra

Dynamics = Time-translation
Symmetry

		<u>time</u>	<u>energy</u>
discrete periodic	→	\mathbb{Z}_T	\mathbb{Z}_T
discrete	→	\mathbb{Z}	$\mathbb{R}/T\mathbb{Z}$
continuous periodic	→	$\mathbb{R}/T\mathbb{Z}$	\mathbb{Z}
continuous	→	\mathbb{R}	\mathbb{R}

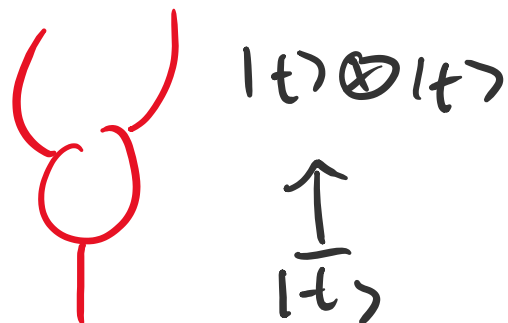
Quantising Groups

time-translation
group \rightarrow

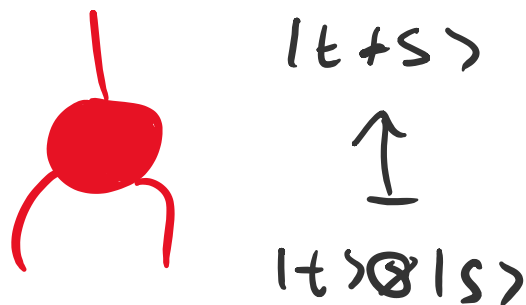
Work with the Hopf algebra

$\mathbb{C}[G]$

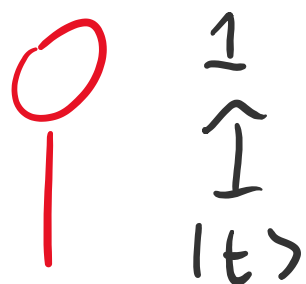
call this \mathcal{T} , for time



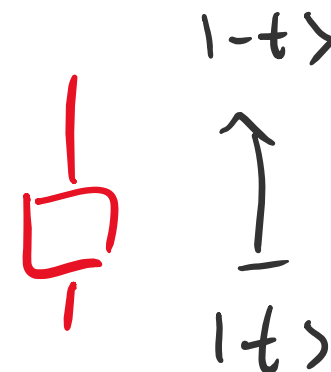
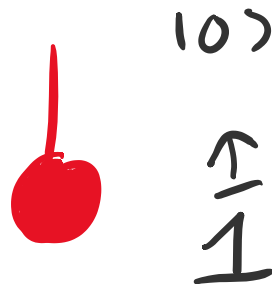
(special commutative
t-Frobenius algebra)



(quasi-special symmetric
t-Frobenius algebra)



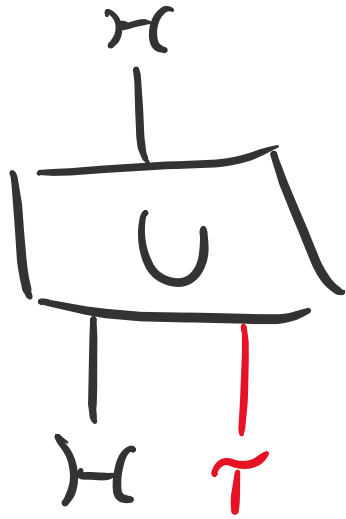
$\bullet = N_0$
 \uparrow
invertible



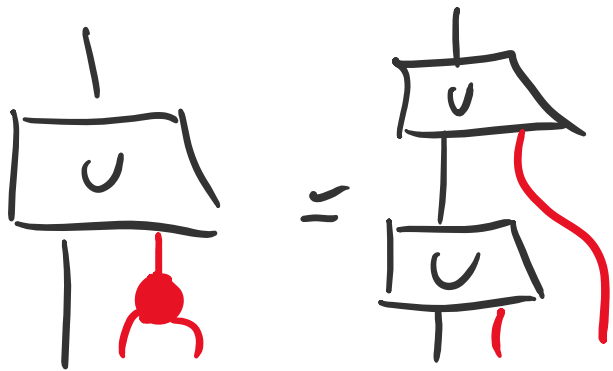
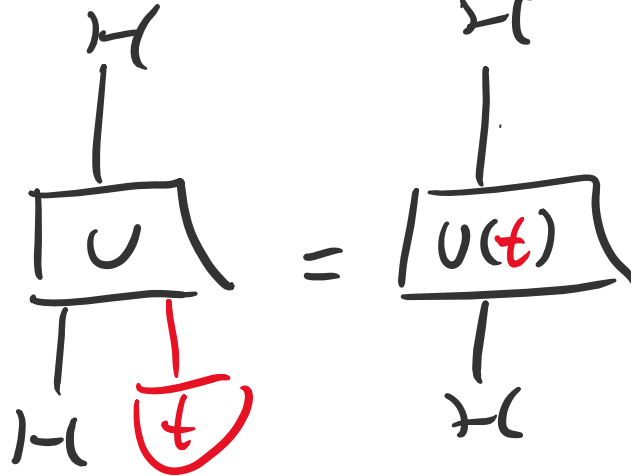
antipode

$$\begin{aligned} \square &= (m \circ \eta) = (r \circ \eta) = \\ &= \eta \circ m = \eta \circ r \end{aligned}$$

Quantising Algebras

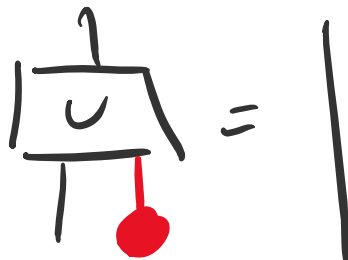


s.t.

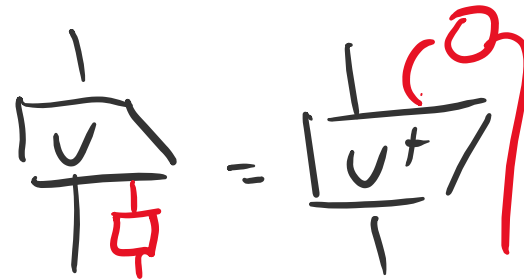


$$U(t+s) = U(s)U(t)$$

Algebra



$$U(0) = 1$$



$$U(-t) = U(t)^\dagger$$

unitary

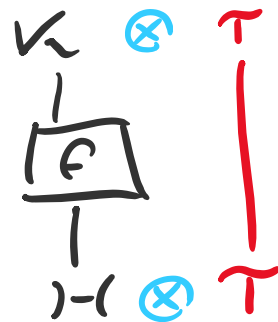
Quantum Dynamical Systems

Nodes:

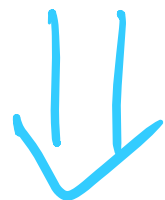
($\mu = 1$  $\eta = 1$ )



\mapsto



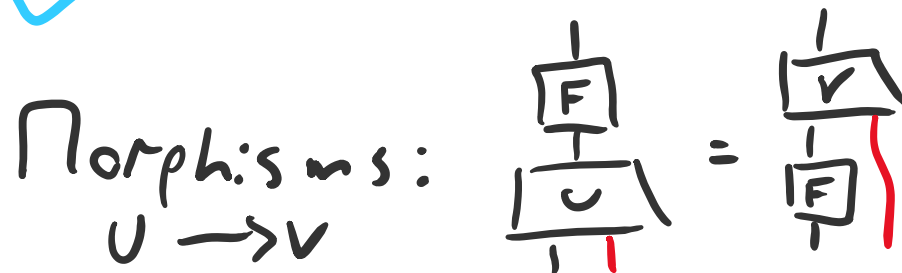
"add time
on the side"



EN algebras (unitary)




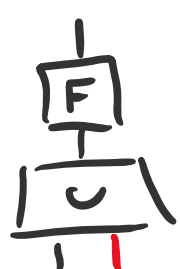

quantum dynamical
systems

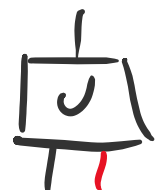

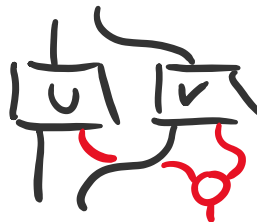


time-equivariant transformations

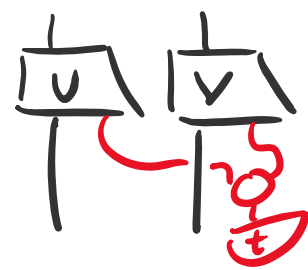
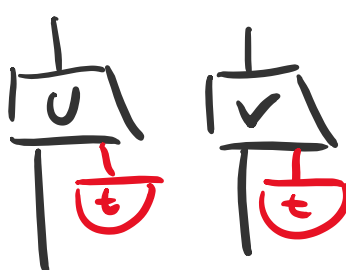
Quantum Dynamical Systems

Objects: 
quantum dynamical
systems

Morphisms: $U \rightarrow V$  = 
time-equivariant transformations

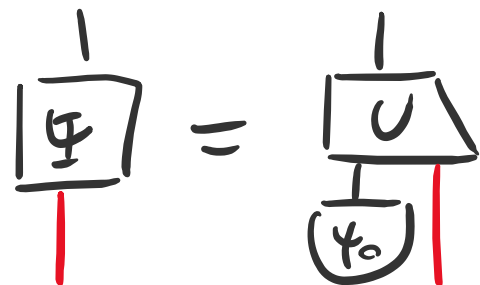
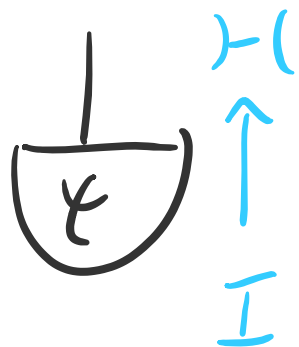
Composition:  \otimes  = 

Synchronised

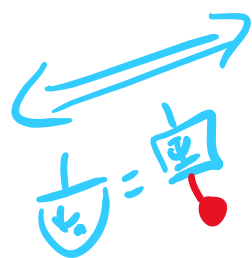
quantum dynamical systems \Rightarrow  = 

States and Histories

States:

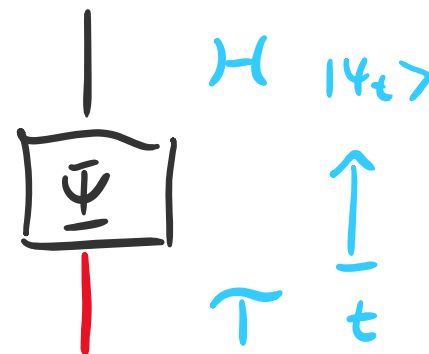


$$|\psi_t\rangle = U_t |\psi_0\rangle$$

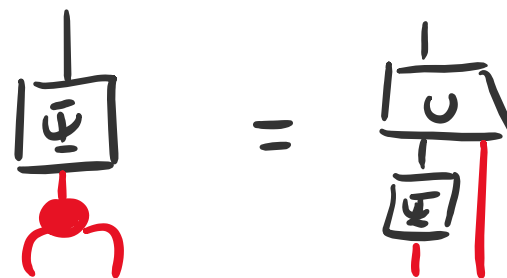


this is the same as

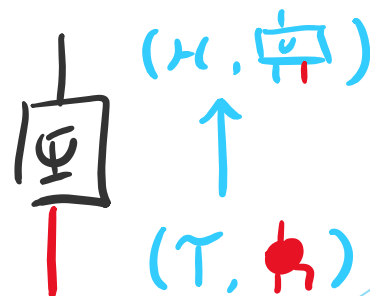
Histories:



s.t.



$$|\psi_{t+s}\rangle = U_s |\psi_t\rangle$$



Hamiltonian as a Coalgebra

Time-Energy Duality:

$$\boxed{P_E} := \boxed{U^\dagger} \quad \text{with } E \text{ in red}$$

Projector over energy E eigenstate.

Diagram illustrating the simplification of the product of two projectors:

$$P_{E'} P_E = \begin{array}{|c|} \hline P_{E'} \\ \hline P_E \\ \hline \end{array} = \begin{array}{|c|} \hline U^\dagger \\ \hline U \\ \hline \end{array} = \begin{array}{|c|} \hline U^\dagger \\ \hline U \\ \hline \end{array} = \begin{array}{|c|} \hline U^\dagger \\ \hline U \\ \hline \end{array} \delta_{EE'} = \begin{array}{|c|} \hline P_E \\ \hline \end{array} \delta_{EE'}$$

The diagram shows the simplification of the product of two projectors, $P_{E'}$ and P_E , using the unitary U and its adjoint U^\dagger . The final result is $P_E \delta_{EE'}$.

eg. Orthoidempotence
 $P_E' P_E = P_E S_{EE'}$

$$P_{E'} P_E = P_E S_{EE'}$$

time

energy

$$\mathbb{C}[G] \cong \mathbb{C}[G^*]$$

$|t\rangle$

$|E\rangle$

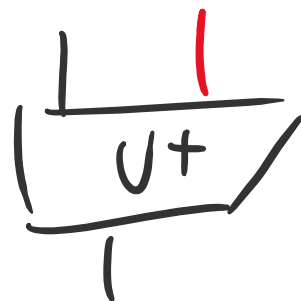
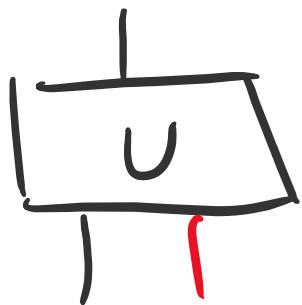
time-translation

energy-shift

$$|E\rangle := \int e^{iEt/\hbar} |t\rangle dt$$

Hamiltonian as a Coalgebra

Dynamics
= algebra



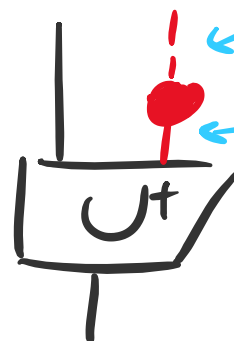
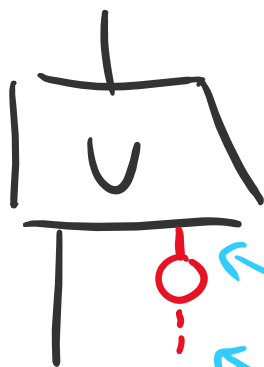
Energy observable
= coalgebra

prepare
in \circ



measure
in \bullet

classical
time
evolution



\hat{G}
 $\mathbb{E}[\hat{G}]$

classical
energy
measurement

$\mathbb{E}[G]$
 G

Schrödinger's Equation

$$H |\psi_E\rangle = E |\psi_E\rangle$$

$$U(t) = \exp\left[-i\frac{Ht}{\hbar}\right]$$

$$U(E) |\psi_E\rangle = e^{-i\frac{Et}{\hbar}} |\psi_E\rangle$$

$$i\hbar \frac{d}{dt} |\psi_t\rangle = H |\psi_t\rangle$$

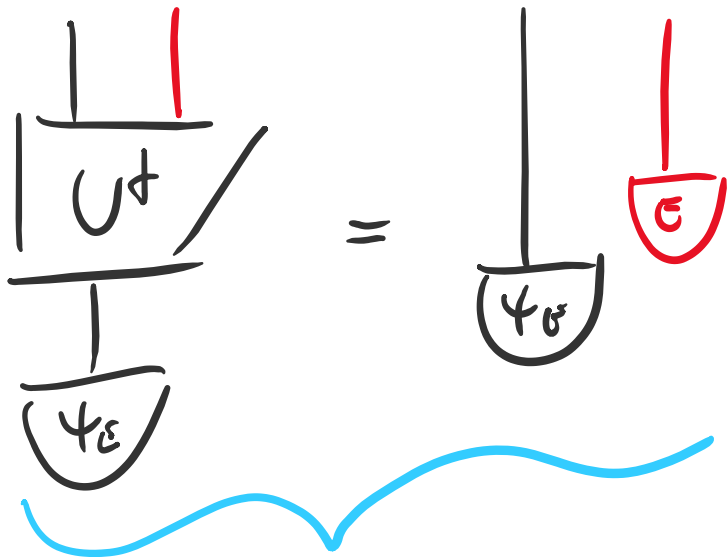
$$|\psi_{t+\delta t}\rangle = U(\delta t) |\psi_t\rangle$$

$$U(t) |\psi_E\rangle = e^{-i\frac{Et}{\hbar}} |\psi_E\rangle$$

Schrödinger's Equation

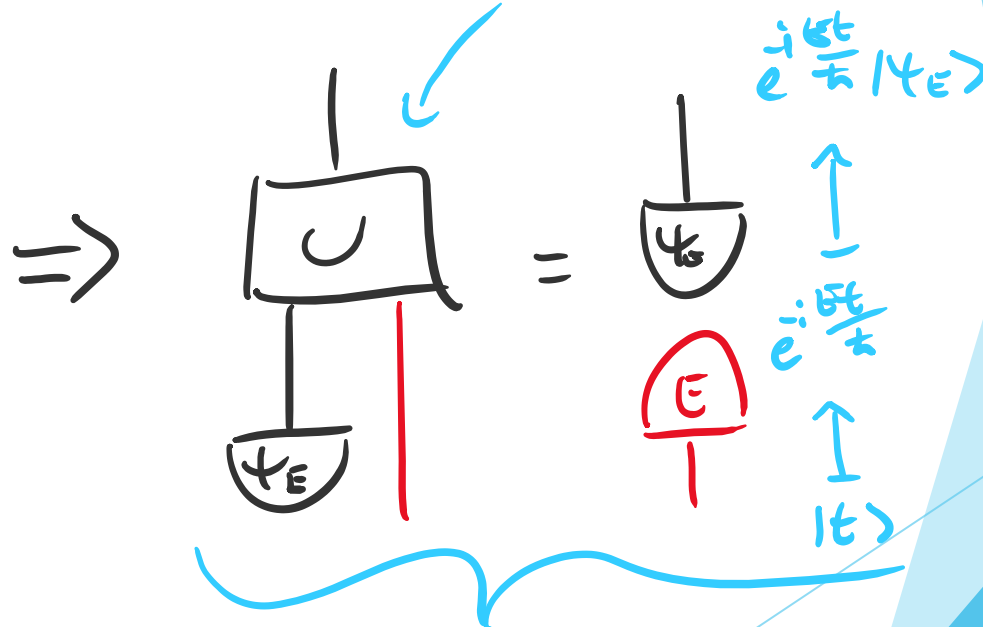
$$U(t) |\psi_E\rangle = e^{-i\frac{E}{\hbar}t} |\psi_E\rangle$$

proof:
measure



Definition of energy eigenstate

evolve

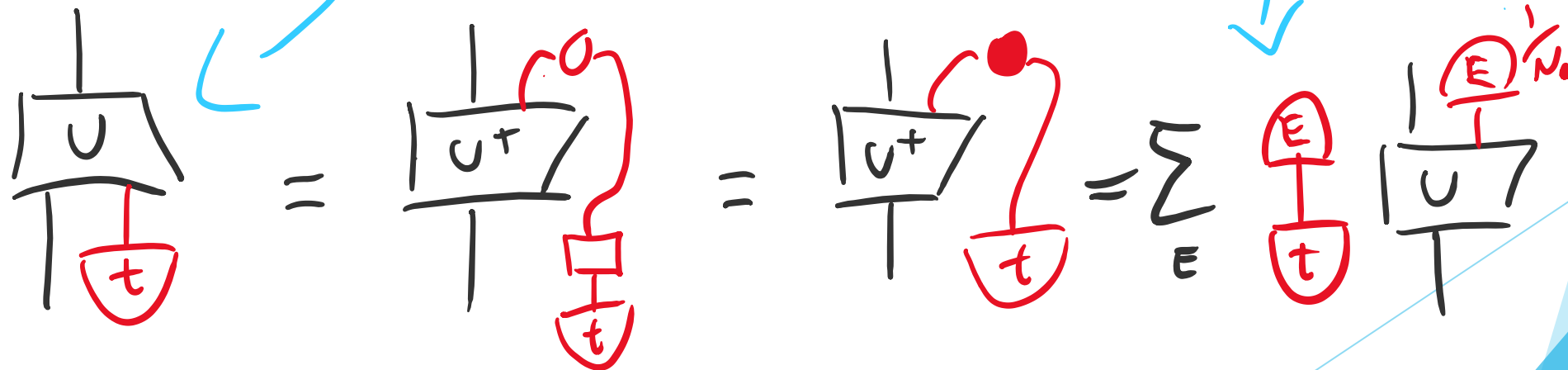


Schrödinger's Equation

Stone's Theorem (with a really long name)

$$U(t) = \int e^{-i \frac{E t}{\hbar}} d\rho_E$$

Proof:



von Neumann's Ergodic Theorem

$$\lim_{T \rightarrow \infty} \int_0^T e^{i \frac{\omega t}{T}} U(t) \frac{dt}{T} = P_E$$

Proof:

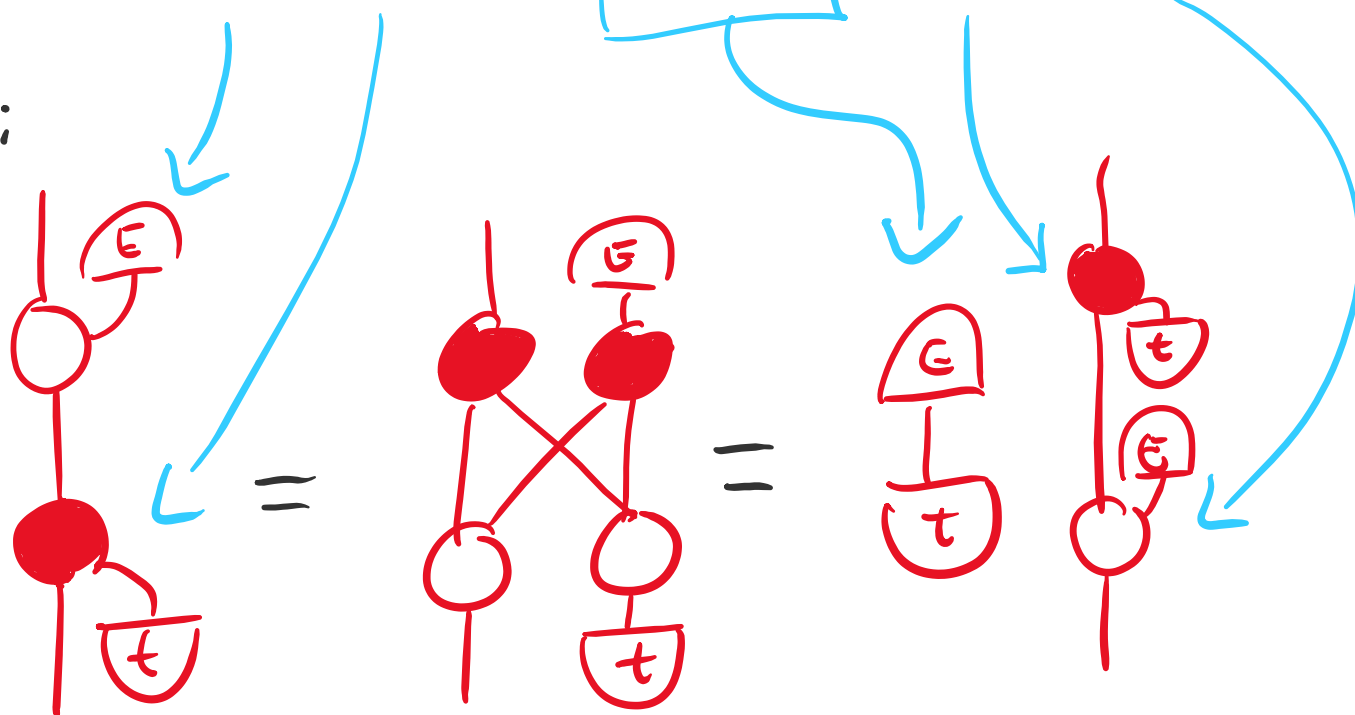
$$\sum_t \frac{\begin{array}{|c|} \hline t \\ \hline \end{array}}{\begin{array}{|c|} \hline \omega \\ \hline \end{array}} \frac{\begin{array}{|c|} \hline U \\ \hline \end{array}}{\begin{array}{|c|} \hline t \\ \hline \end{array}} \frac{1}{N_0} = \frac{\begin{array}{|c|} \hline U \\ \hline \end{array}}{\begin{array}{|c|} \hline \frac{1}{\omega} \\ \hline \end{array}} \frac{1}{N_0} = \frac{\begin{array}{|c|} \hline E \\ \hline \end{array}}{\begin{array}{|c|} \hline U \\ \hline \end{array}} \frac{1}{N_0}$$

Weyl Canonical Commutation Relations

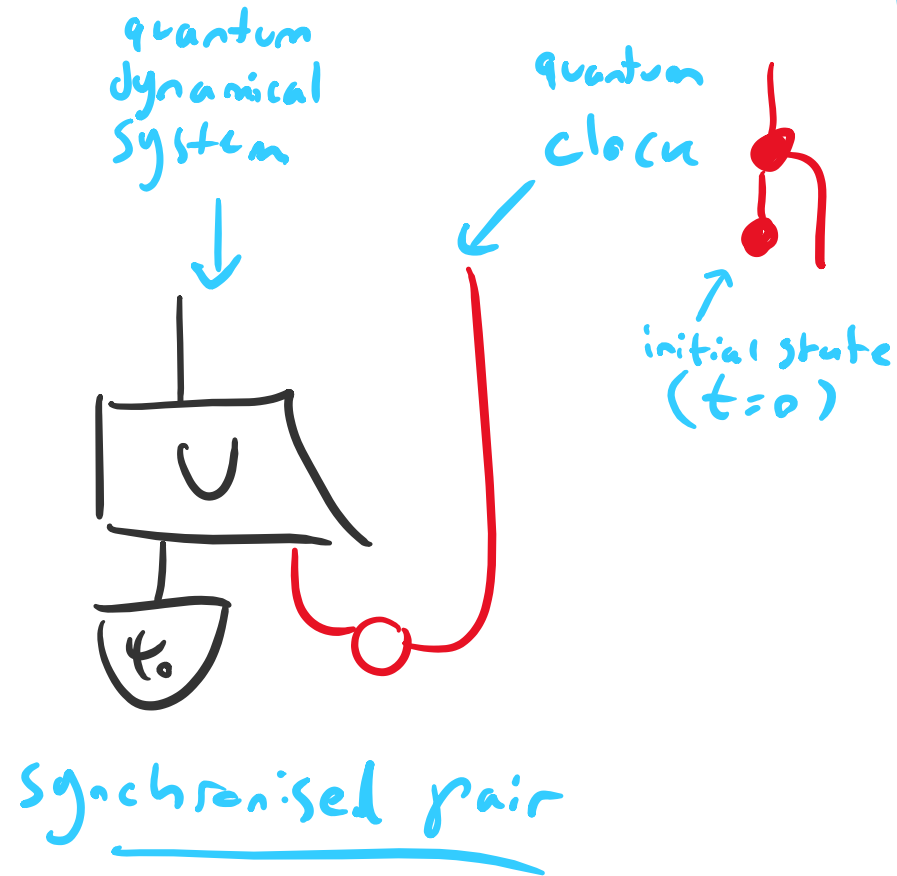
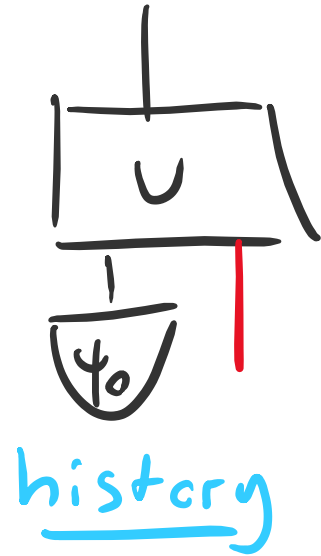
$$V_E U_t = e^{-i \frac{Et}{\hbar}} U_t V_E$$

time-translation
energy-shift

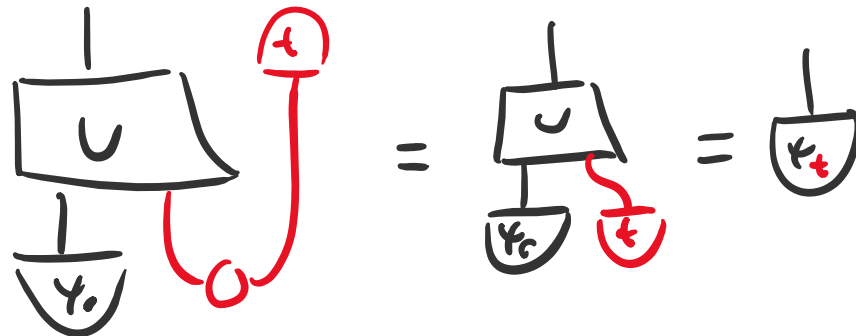
proof:



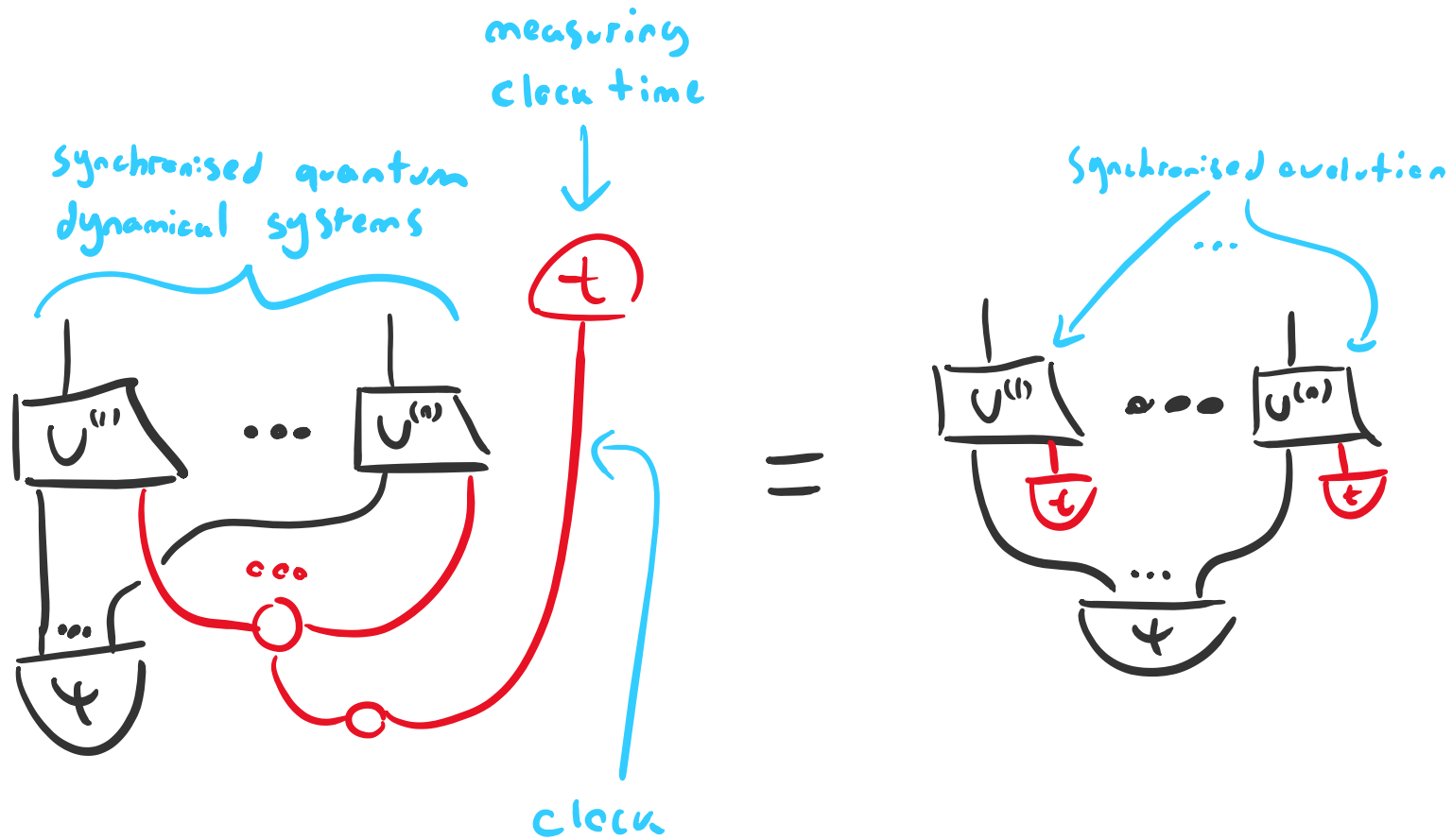
Quantising Time



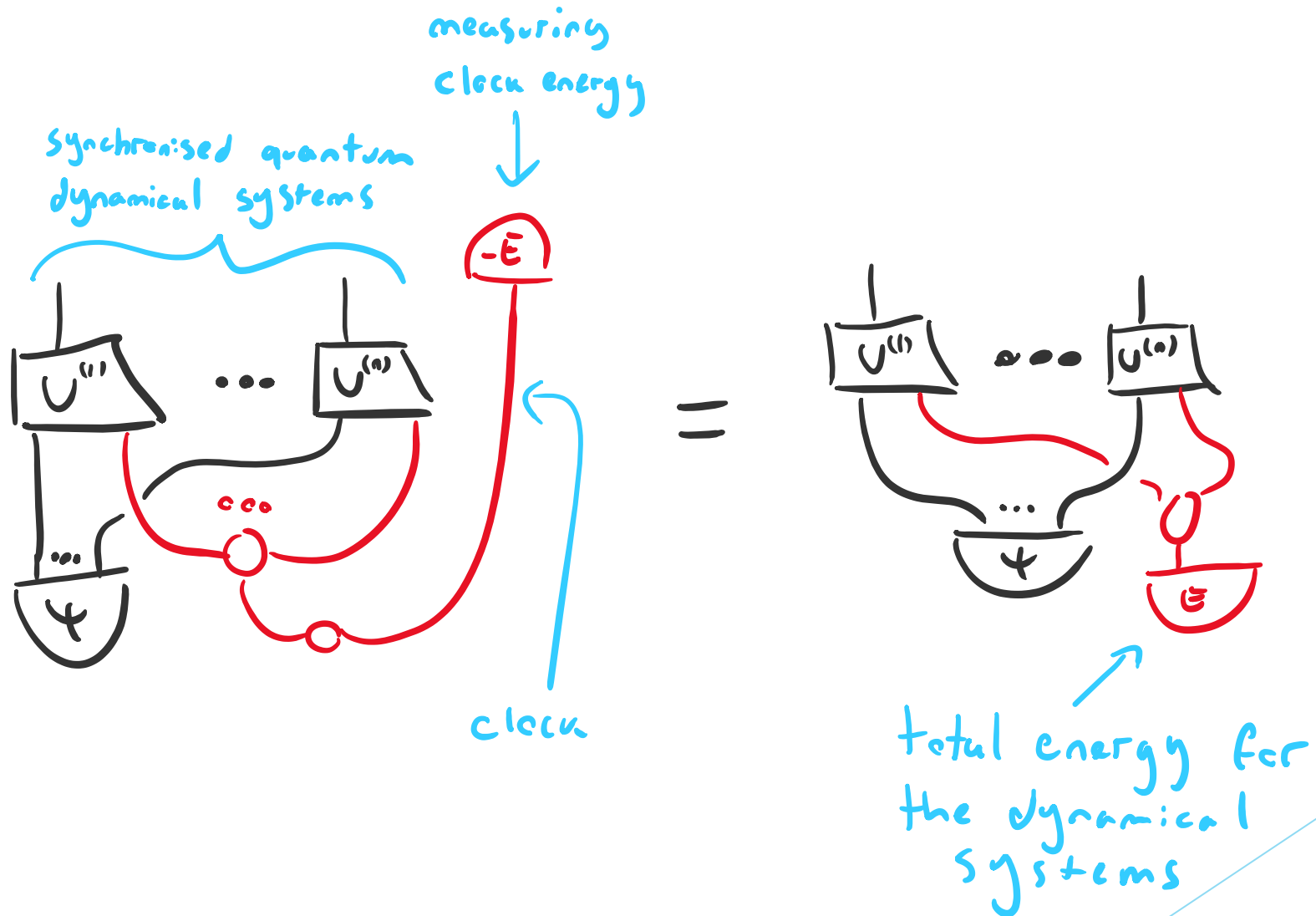
measuring
clock time :



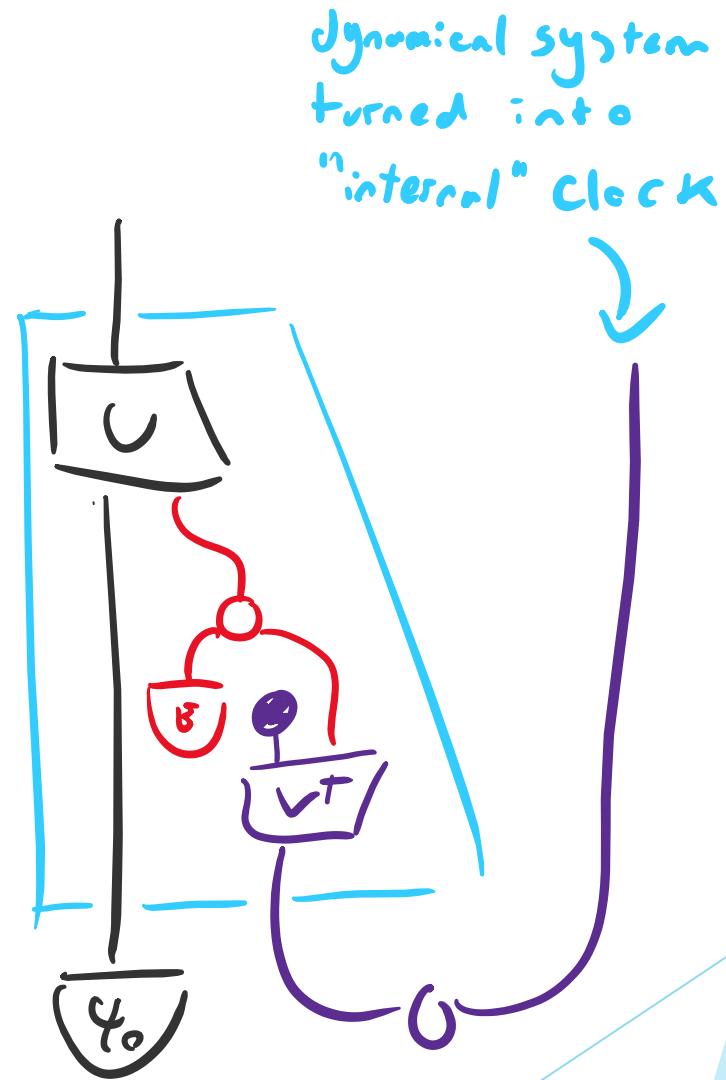
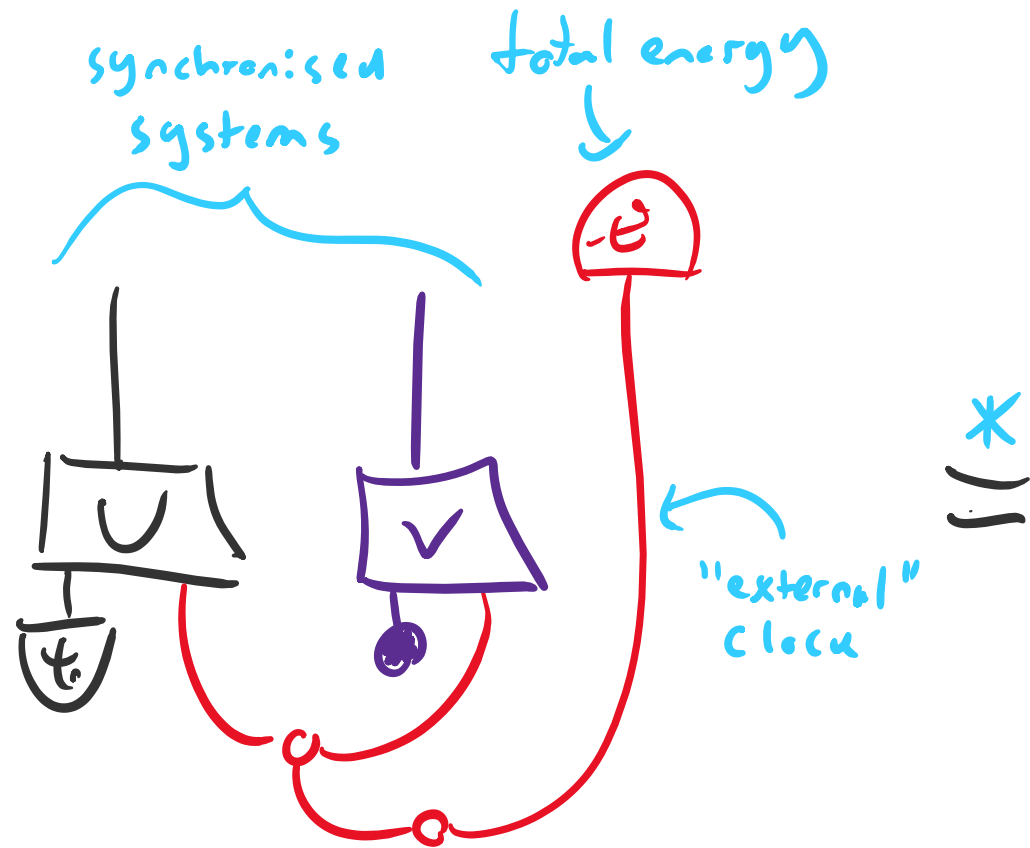
Quantising Time



Quantising Time

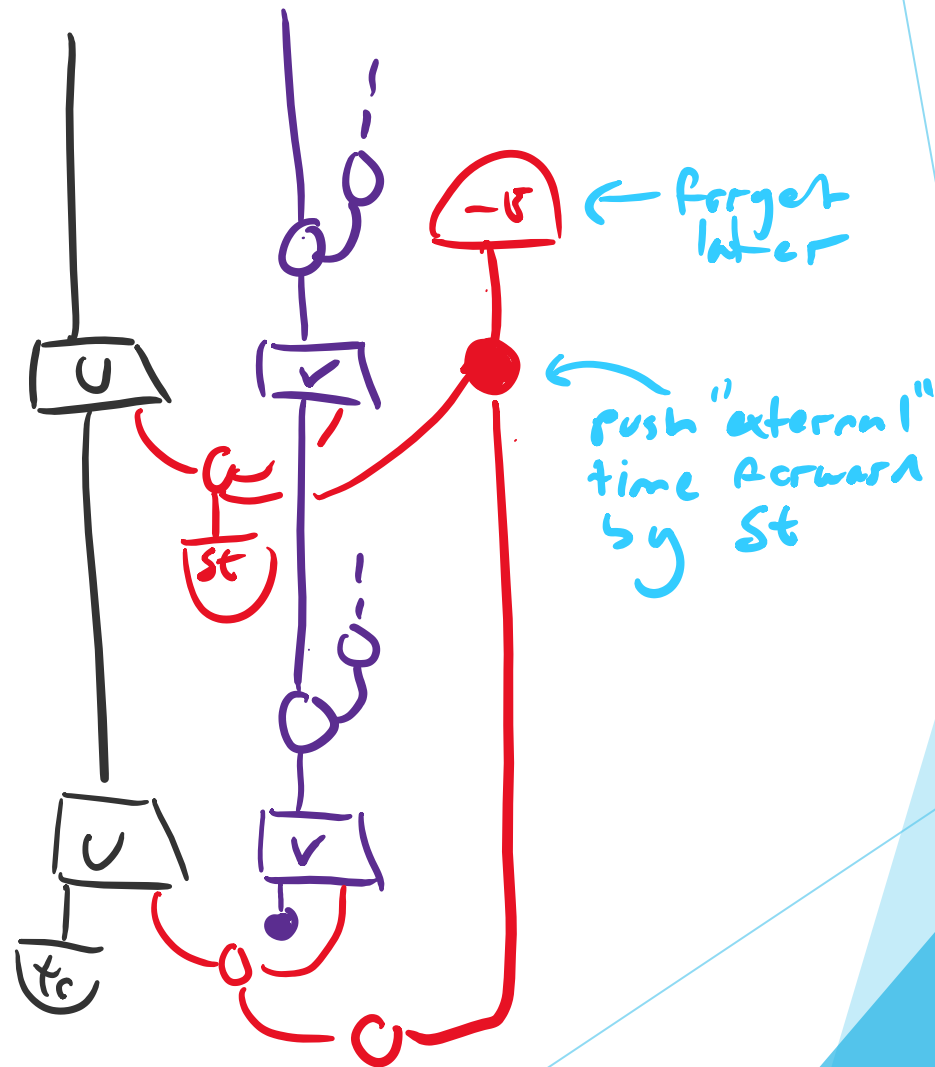
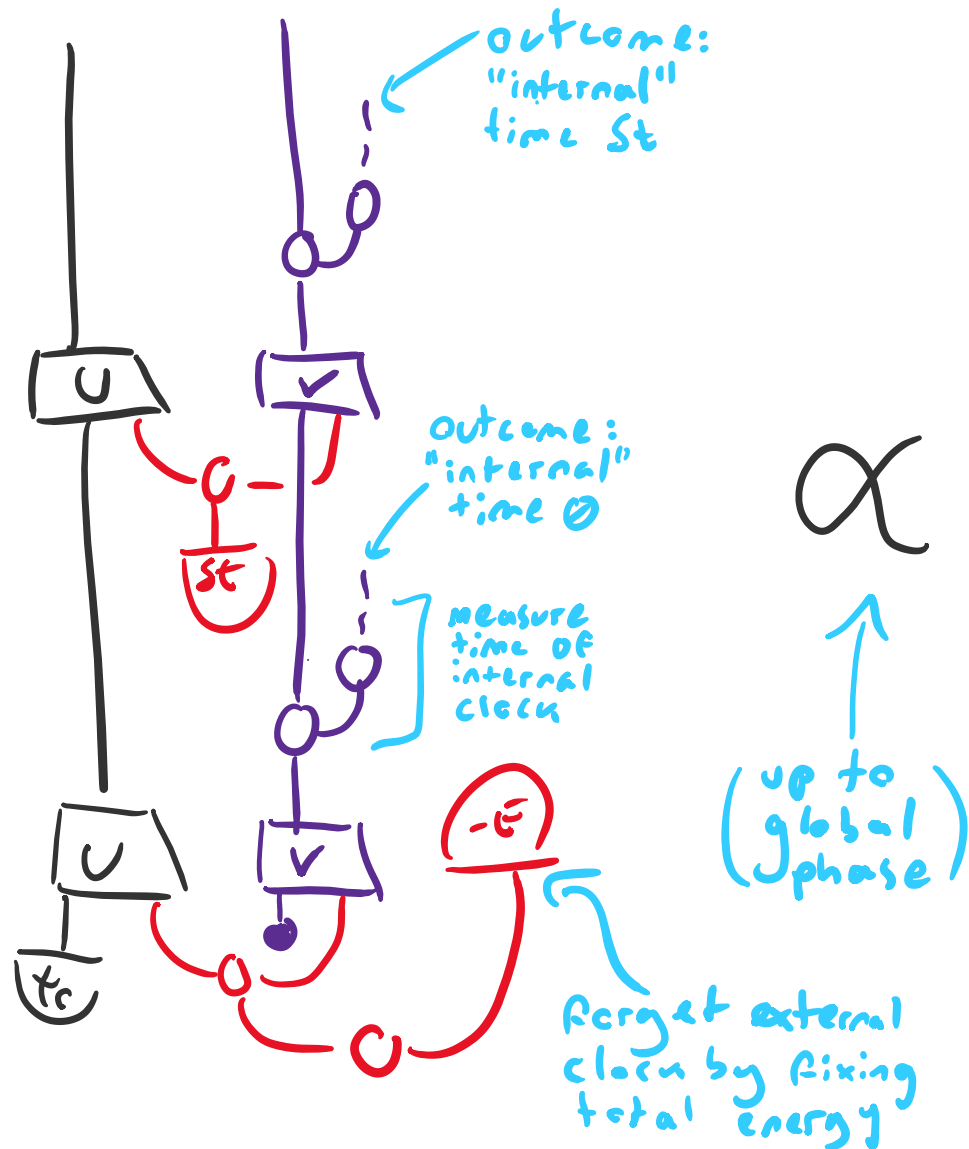


Quantising Time



* Terms and conditions apply.

As time goes by...



The background features abstract, overlapping geometric shapes in various shades of blue, ranging from light sky blue to deep navy blue. These shapes are primarily located on the right side of the frame, creating a modern, dynamic feel. The rest of the background is a solid, very light blue-grey.

THANK YOU!

Any Questions?