Completely Positive Maps for Mixed Unitary Categories

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Linearly distributive categories

A **linearly distributive category (LDC)** has two monoidal structures $(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)$ and $(\oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$ linked by natural transformations called the linear distributors:

$$\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$
$$\partial_R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

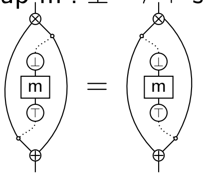
LDCs are equipped with a graphical calculus.

LDCs provide a categorical semantics for multiplicative linear logic.

Mix categories

A **mix category** is a LDC with a mix map $\mathsf{m} : \bot \to \top$ such that

$$\mathsf{mx}_{A,B} : A \otimes B \to A \oplus B :=$$



$$(1\oplus(u^L_\oplus)^{-1})(1\otimes(\mathsf{m}\oplus 1))\delta^L(u^R_\otimes\oplus 1) = ((u^R_\oplus)^{-1}\oplus 1)((1\oplus\mathsf{m})\otimes 1)\delta^R(1\oplus u^R_\otimes)$$

mx is called a **mixor**. The mixor is a natural transformation.

It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the mixor an isomorphism. However, it does make the above coherence automatic.

A **compact LDC** is an LDC in which every mixor is an isomorphism i.e., in a compact LDC $\otimes \simeq \oplus$.

†-isomix category

A **†-isomix category** is an isomix category equipped with a $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ functor and the following natural isomorphisms:

$$\begin{aligned}
\text{tensor laxors: } & \lambda_\oplus : A^\dagger \oplus B^\dagger \to (A \otimes B)^\dagger \\
& \lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger \\
\text{unit laxors: } & \lambda_\top : \top \to \bot^\dagger \\
& \lambda_\bot : \bot \to \top^\dagger \\
\text{involutor: } & \iota : A \to A^{\dagger\dagger}
\end{aligned}$$

such that certain coherence conditions hold.

Unitary category

A **unitary category** is a compact †-isomix category in which every object has an isomorphism

$$A \xrightarrow{\varphi_A} A^{\dagger}$$

called the **unitary structure map** for A, such that φ_A satisfies certain coherence conditions.

An isomorphism $A \xrightarrow{f} B \in \mathbb{U}$ is said to be a **unitary isomorphism** if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\dagger} \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^{\dagger}} \\ B & \xrightarrow{\varphi_B} & B^{\dagger} \end{array}$$

Mixed unitary category

A **mixed unitary category**, $M : \mathbb{U} \to \mathbb{C}$ consists of a

- $\dagger$-isomix category $\mathbb{C}$
- unitary category $\mathbb{U}$
- a strong $\dagger$- isomix functor $M : \mathbb{U} \to \mathbb{C}$
  - $(M, m_\otimes, m_\top)$ is strong monoidal on $\otimes$
  - $(M, n_\oplus, n_\bot)$ is strong comonoidal on $\oplus$
  - M is a linear functor
  - M preserves mix map: $n_\bot M(\mathsf{m}) m_\top = \mathsf{m}$
  - $(\rho, \rho^{-1}) : M((\_)^\dagger) \to M(\_)^\dagger$ is a linear natural isomorphism

An example of a MUC: $\mathbb{R} \subset \mathbb{C}$

Consider the discrete monoidal category $\mathbb{C}$:

Objects: $a + ib \in \mathbb{C}$

Maps: Identity maps only $c = c$

Tensor: multiplication
$(a + ib) \otimes (x + iy) := (ax - by) + i(ay + bx)$

Unit: 1

Dagger: $(a + ib)^{\dagger} := a - ib$

$\mathbb{C}$ is a compact LDC ($\otimes \simeq \oplus$) with a non-stationary dagger functor.

The subcategory $\mathbb{R}$ is a unitary category with the unitary structure map being the identity map.

$\mathbb{R} \subset \mathbb{C}$ is a mixed unitary category.

An example of a MUC: $\mathrm{Mat}(\mathbb{C}) \subset \mathrm{FMat}(\mathbb{C})$

A **finiteness space**, $(X, \mathcal{A}, \mathcal{B})$, consists of a set X and a subset $\mathcal{A}, \mathcal{B} \subseteq P(X)$ such that $\mathcal{B} = \mathcal{A}^\perp$, that is

$$\mathcal{B} = \{b | b \in P(X) \text{ with for all } a \in A, |a \cap b| < \infty\},$$

and $\mathcal{A} = \mathcal{B}^\perp$.

A **finiteness relation**, $(X, \mathcal{A}, \mathcal{B}) \xrightarrow{R} (Y, \mathcal{A}', \mathcal{B}')$ is relation $X \xrightarrow{R} Y$ such that

$$\forall A \in \mathcal{A}. AR \in \mathcal{A}' \quad \text{and} \quad \forall B' \in \mathcal{B}'. RB' \in \mathcal{B}$$

Finiteness spaces with finiteness relation form a $*$-autonomous category.

An example of a MUC: Mat($\mathbb{C}$) $\subset$ FMat($\mathbb{C}$)

FMat($\mathbb{C}$) is defined as follows:

Objects: Finiteness spaces $(X, \mathcal{A}, \mathcal{B})$

Maps: $(X, \mathcal{A}, \mathcal{B}) \xrightarrow{M} (Y, \mathcal{A}', \mathcal{B}')$ is a matrix $XxY \xrightarrow{M} \mathbb{C}$ such that

$$supp(M) := \{(x, y) | x \in X, y \in Y \text{ and } M(x, y) \neq 0\}$$

is a finiteness relation from $(X, \mathcal{A}, \mathcal{B})$ to $(Y, \mathcal{A}', \mathcal{B}')$.

Dagger: $(X, \mathcal{A}, \mathcal{B})^\dagger := (X, \mathcal{B}, \mathcal{A})$
$M^\dagger$ is the complex conjugate of M.

Mat($\mathbb{C}$) is a full subcategory of FMat($\mathbb{C}$) which is determined by the objects, $(X, P(X), P(X))$, where X is a finite set.

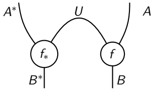
Mat($\mathbb{C}$) is a unitary category, indeed a well-known $\dagger$-compact closed category.

The inclusion Mat($\mathbb{C}$) $\subset$ FMat($\mathbb{C}$) is a mixed unitary category.

Goal

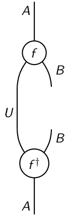
CP^{∞} construction

CPM construction



in †- compact closed categories

Selinger, 2007



in †-SMCs

Coecke and Heunen, 2011

?

in MUCs

Kraus maps

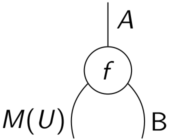
A Kraus map

$$(f, U) : A \longrightarrow B$$

in a mixed unitary category, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a map

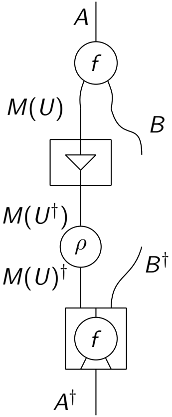
$$f : A \longrightarrow M(U) \oplus B \in \mathbb{C}$$

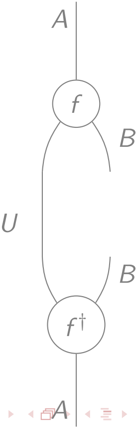
for some $U \in \mathbb{U}$ and $M(U)$ is called the ancillary system of f.



Combinator

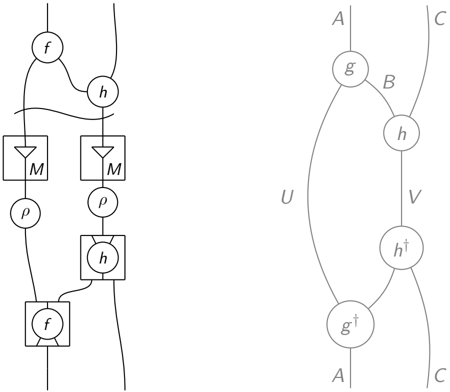
A Kraus map can be glued along the unitary structure map with the dagger of itself to get a combinator which takes positive maps to positive maps:





Test maps

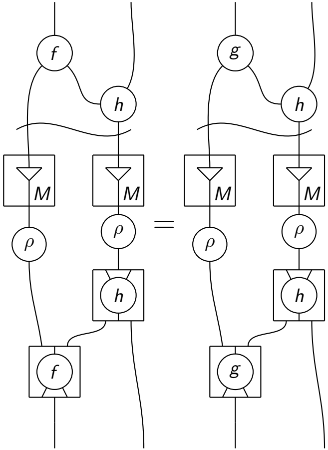
A combinator built from a Kraus map $(f, U) : A \longrightarrow B$ acts on **test maps**, $h : B \otimes C \longrightarrow M(V)$ as follows:

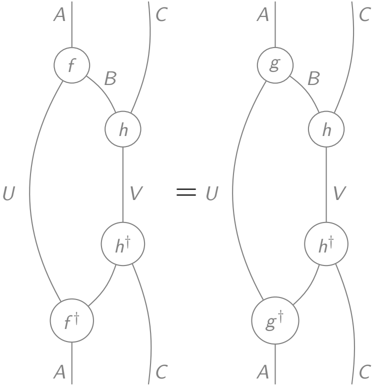


Test maps are glued along unitary structure map with its dagger to give a **positive map**.

Equivalence relation

$(f, U) \sim (g, V) : A \longrightarrow B$, if for all test maps maps $h : B \otimes C \to V$, the following holds:





Unitarily isomorphic $\Rightarrow$ equivalence

Lemma: For any two Kraus morphisms $(f, U), (g, V) : A \longrightarrow B$,

$$(f, U) \sim (g, V)$$

if

- $U \xrightarrow{\alpha} V$ is a unitary isomorphism, and,
- $f(M(\alpha) \oplus 1) = f'$

CP^∞ construction

$\mathrm{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ is given as follows:

Objects: Same as $\mathbb{C}$

Maps: A map

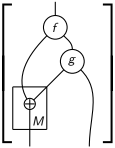
$$[(f, U)] : A \to B \in \mathrm{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$$

is an equivalence class of Kraus maps

$$(f, U) : A \to B \in \mathbb{C} \;/\; \sim$$

CP$^\infty$ construction cont...

Composition:

$$[(f, U)][(g, V)] :=$$ 

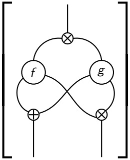
$$[A \xrightarrow{f} M(U) \oplus B \xrightarrow{1 \oplus g} M(U) \oplus (M(V) \oplus C) \xrightarrow{a_\oplus}$$

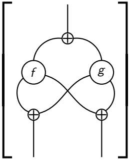
$$(M(U) \oplus M(V) \oplus C \xrightarrow{n_\oplus^{-1} \oplus 1} M(U \oplus V) \oplus C]$$

Identity: $1_A := [A \xrightarrow{(u_\oplus^L)^{-1}} \bot \oplus A \xrightarrow{n_\bot^{-1} \oplus 1} M(\bot) \oplus A]$

CP^∞ construction cont...

$\mathrm{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ has two tensor products:

$[(f, U)] \otimes [(g, V)] :=$ 

$[(f, U)] \oplus [(g, V)] :=$ 

Unit of $\otimes$ is $\top$ and the unit of $\oplus$ is $\bot$.

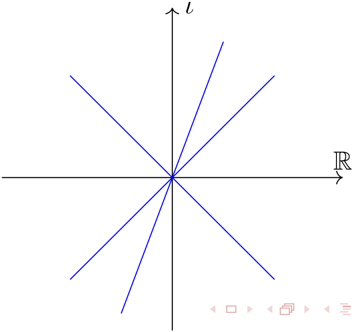
Lemma: $\mathrm{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ is an isomix category.

CP$^\infty(\mathbb{R} \subset \mathbb{C})$

Kraus maps in $\mathbb{C}$ are $(=, r) : c \longrightarrow c'$ such that $c = rc'$.

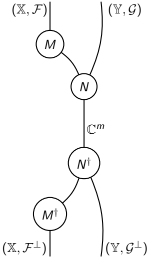
$c' \neq 0$ then there is at most one Kraus map $(=, r) : c \longrightarrow c'$, for all $c \in \mathbb{C}$ when $c = rc'$.

$c' = 0$ then $c = 0$ and for all $r' \in \mathbb{R}$, $(=, r) \sim (=, r') : c \longrightarrow c'$.



$CP^\infty(\mathrm{Mat}(\mathbb{C}) \subset \mathrm{FMat}(\mathbb{C}))$

A Kraus map $(M, \mathbb{C}) : A \longrightarrow B$ gives a **pure completely positive map**:



Choi's theorem for $\mathrm{FMat}(\mathbb{C})$**:**
Every map in $CP^\infty(\mathrm{Mat}(\mathbb{C}) \subset \mathrm{FMat}(\mathbb{C}))$ can be written as a sum of pure completely positive maps.

Environment structure

An **environment structure** for a mixed unitary category, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a pair

$$(F : \mathbb{C} \longrightarrow \mathbb{D}, \top\!\!\!\!\bot)$$

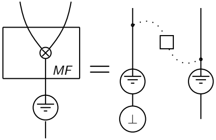
where,

- $\mathbb{D}$ is any isomix category,
- F is a strict Frobenius isomix functor, and
- $\top\!\!\!\!\bot : MF(U) \longrightarrow \bot$ is a family of maps indexed by the objects $U \in \mathbb{U}$ such that the following conditions hold:

Environment structure cont...

For all unitary objects $U, V \in \mathbb{U}$, the following diagrams commute:

$$\begin{array}{ccccc} MF(U) \otimes MF(V) & \xrightarrow{\text{mx}} & M(U) \oplus \mathbb{M}(V) & \xrightarrow{\text{⏚} \oplus \text{⏚}} & \bot \oplus \bot \\ \downarrow{\scriptstyle m_{\otimes}} & & & & \downarrow{\scriptstyle u^{L}_{\oplus}} \\ MF(U \otimes V) & & \xrightarrow{\text{⏚}} & & \bot \end{array}$$

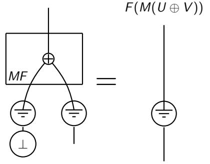
[Env.1a] 

Environment structure cont...

For all unitary objects $U, V \in \mathbb{U}$, the following diagrams commute:

$$
\begin{array}{ccccc}
MF(U \oplus V) & & \xrightarrow{\qquad \perp \qquad} & & \perp \\
\downarrow{\scriptstyle n_\oplus} & & & & \| \\
MF(U) \oplus MF(V) & \xrightarrow[\perp \oplus \perp]{} & \perp \oplus \perp & \xrightarrow[u_\oplus]{} & \perp
\end{array}
$$

[Env.1b]

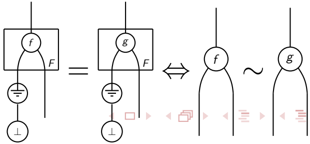


Environment structure cont...

$(f, U) \sim (g, V) \in \mathbb{C}$ if and only if the following equation holds:

$$
\begin{array}{ccccc}
 & & MF(A) & & \\
 & \swarrow^{MF(f)} & & \searrow^{MF(g)} & \\
MF(U \oplus B) & & & & MF(U \oplus B) \\
\downarrow F((n_{\oplus}^{M})^{-1}) & & & & \downarrow F((n_{\oplus}^{M})^{-1}) \\
MF(U) \oplus MF(B) & & & & MF(U) \oplus MF(B) \\
\downarrow \top\!\!\!\!\!\!\; \oplus 1 & & & & \downarrow \top\!\!\!\!\!\!\; \oplus 1 \\
\bot \oplus MF(B) & = & = & = & \bot \oplus MF(B)
\end{array}
$$

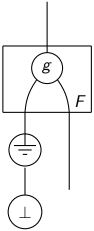
[Env.2]



The purification axiom

An environment structure $(F : {}_{\mathbb{U}}\mathbb{C}_M \longrightarrow \mathbb{D}, \top)$ satisfies **purification** axiom if

- F is injective on objects, and
- for all $f : A \longrightarrow B \in \mathbb{D}$, there exists a Kraus map $(g, U) : X \longrightarrow Y \in \mathbb{C}$ such that

[Env.3] $f =$ 

Equationally,

$$A \xrightarrow{f} B = F(A) \xrightarrow{F(g)} F(M(U) \oplus Y) \xrightarrow{n_{\oplus}} M(F(U)) \oplus F(Y) \xrightarrow{\top \oplus 1} \bot \oplus F(Y) \xrightarrow{u_{\oplus}} F(Y)$$

Examples

Lemma: For any mixed unitary category, $M : \mathbb{U} \to \mathbb{C}$, there exists an environment structure $(\mathbb{C} \xrightarrow{Q} \text{CP}^{\infty}(M : \mathbb{U} \to \mathbb{C}), \top)$ satisfying purification axiom where,

$$
\begin{aligned}
Q(A) &:= A \\
Q(f) &:= [(f(u_{\oplus}^{L})^{-1}(n_{\oplus})^{-1}, \bot)] \\
\top : F(M(U)) \to \bot &:= [((u_{\oplus}^{R})^{-1}, U)]
\end{aligned}
$$

Examples cont...

- Consider the MUC, $\mathbb{R} \subset \mathbb{C}$. Then,

$$(\mathbb{R} \xrightarrow{Q} \mathrm{CP}^\infty(\mathbb{R} \subset \mathbb{C}), \top_r : r \to 1)$$

is an environment structure where,

$$\top_r := (=, 1/r) : r \to 1$$

- Consider the MUC, $\mathrm{Mat}_\mathbb{C} \to \mathrm{FMat}(\mathbb{C})$. Then,

$$\mathrm{Mat}(\mathbb{C}) \xrightarrow{Q} \mathrm{CP}^\infty(\mathrm{Mat}(\mathbb{C}) \subset \mathrm{FMat}(\mathbb{C}))$$

is an environment structure where,

$$\top_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}; \rho \mapsto Tr(\rho)$$

Axiomatization of CP^∞ construction

Proposition: Let $\mathbb{D}$ be any isomix category. If $(F : \mathbb{C} \longrightarrow \mathbb{D}, \perp)$ is an environment structure for the mixed unitary category $M : \mathbb{U} \longrightarrow \mathbb{C}$ which satisfies the purification axiom, then

$$\mathbb{D} \simeq \mathrm{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C})$$

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Robin Cockett, Cole Comfort, and Priyaa Srinivasan.

Dagger linear logic for categorical quantum mechanics.
ArXiv e-prints, September 2018.

Coming soon on arXiv....
Completely positive maps for mixed unitary categories.