Orthogonality for quantum Latin isometry squares: the search for perfection...

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PART 1 - INTRODUCTION

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First here are some definitions.

Basic definitions

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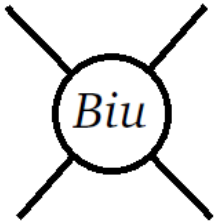
- 

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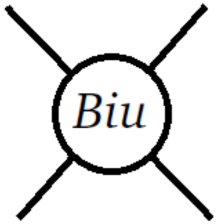
- 

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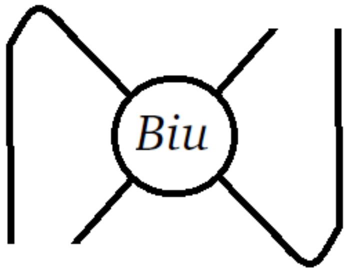
- 

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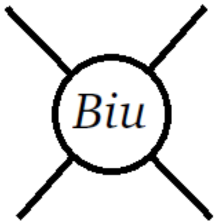
- 

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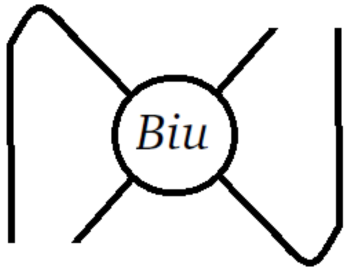
- 

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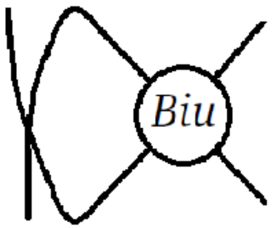
- 

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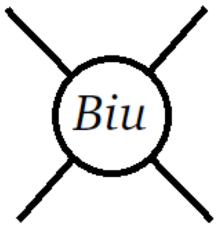
- 

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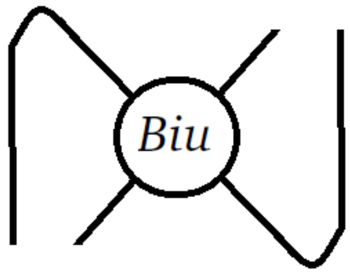
- 

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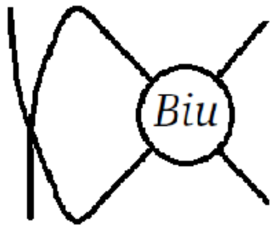
- 

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If the isometry is unitary then *Biu* is a *perfect tensor*.

PART 2 - BIUNITARIES

Quantum Latin squares

Definition. A *Latin square of order n* is an n-by-n grid of elements of $\{0, \ldots, n-1\}$, with every element appearing exactly once in each row and column.

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Here is a quantum Latin square arising from a Latin square:

$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$
$\lvert 1\rangle$	$\lvert 3\rangle$	$\lvert 0\rangle$	$\lvert 2\rangle$
$\lvert 2\rangle$	$\lvert 0\rangle$	$\lvert 3\rangle$	$\lvert 1\rangle$
$\lvert 3\rangle$	$\lvert 2\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

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Here is a quantum Latin square equivalent to one arising from a Latin square:

$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\|2\rangle$	$\|3\rangle$
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Here is a quantum Latin square *not* equivalent to one arising from a Latin square:

$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$
$\frac{1}{\sqrt{2}}(\lvert 1\rangle - \lvert 2\rangle)$	$\frac{1}{\sqrt{5}}(i\lvert 0\rangle + 2\lvert 3\rangle)$	$\frac{1}{\sqrt{5}}(2\lvert 0\rangle + i\lvert 3\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 2\rangle)$
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$\lvert 3\rangle$	$\lvert 2\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

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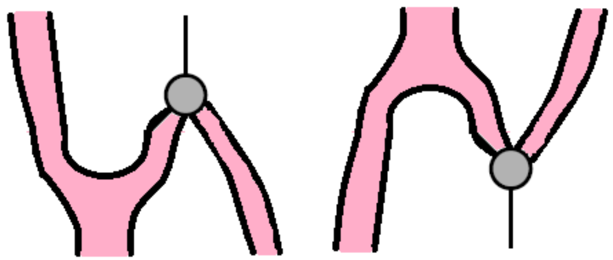
Definition. (MV, QPL 2015.)
A *quantum Latin square of order n* is a $(2, 1, n)$-tensor  such that the following are unitary:



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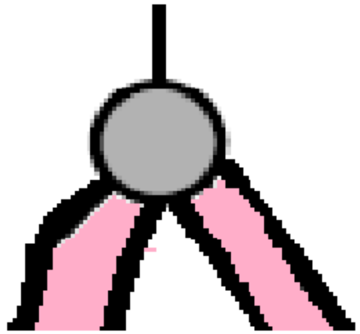

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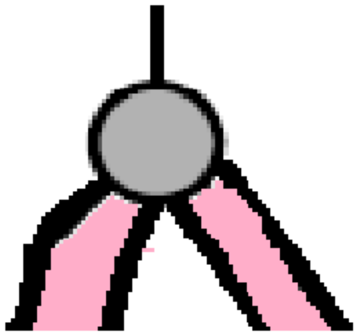

That's a 3-valent tensor, how can that be a biunitary?

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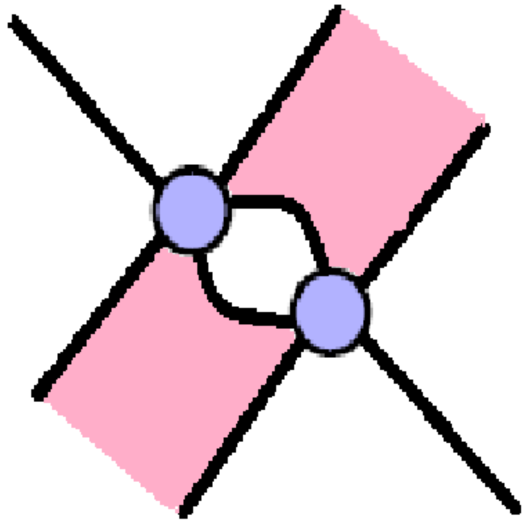

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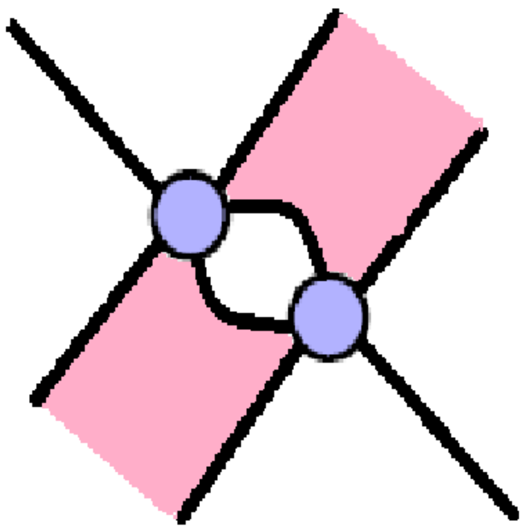

This is a grid on 1-dimensional projectors and is a biunitary.

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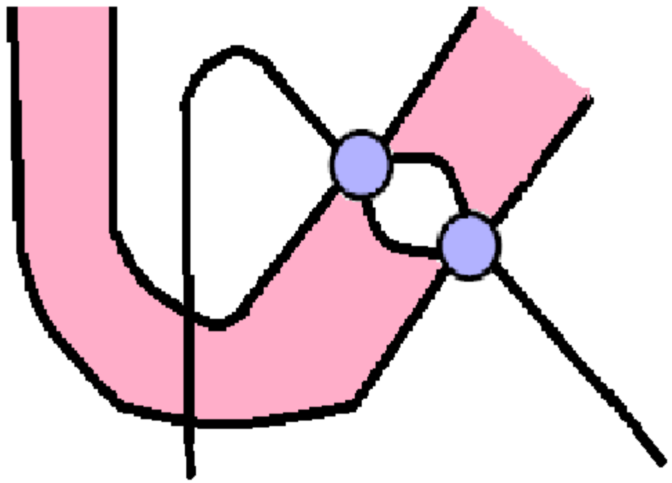

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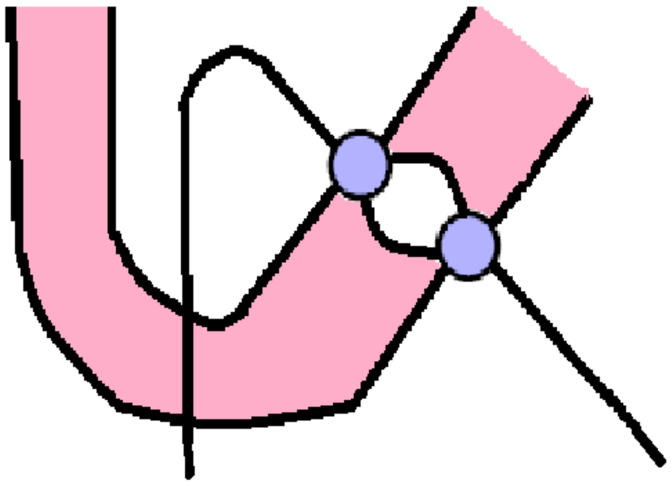

The answer is no, so the search for perfection goes on...

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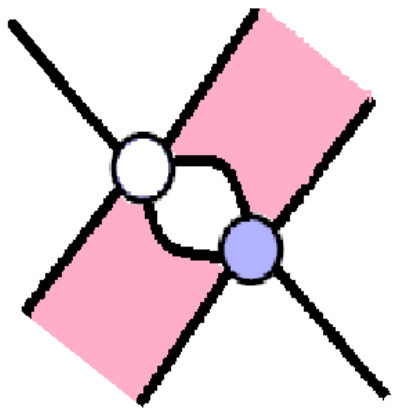

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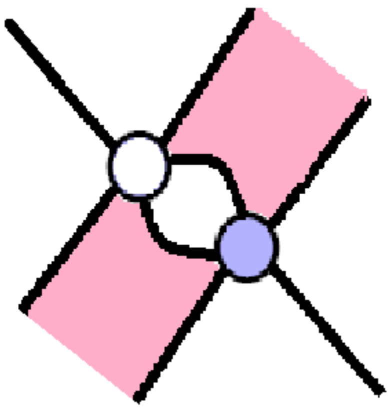

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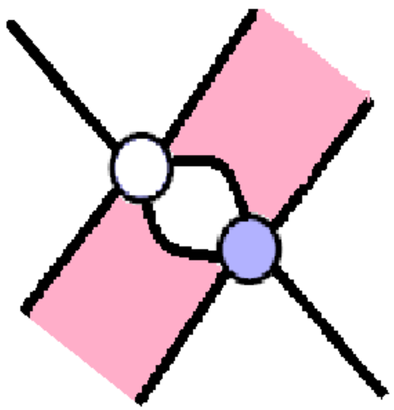

This is also biunitary. We will see later that a skew projective QLS can be perfect.

PART 3:- QUEST FOR PERFECTION

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Definition. A pair of quantum Latin squares P, Q of order n are *orthogonal* when the elements $|P_{i,j}\rangle \otimes |Q_{i,j}\rangle$ are orthonormal.

This is equivalent to a more complicated definition in GRMŻv1, and was independently described in GRMŻv2.

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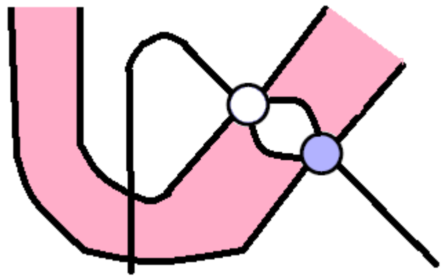

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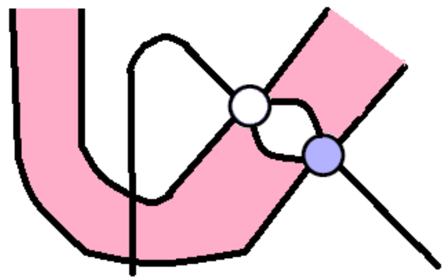

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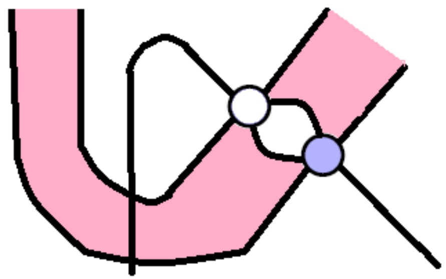

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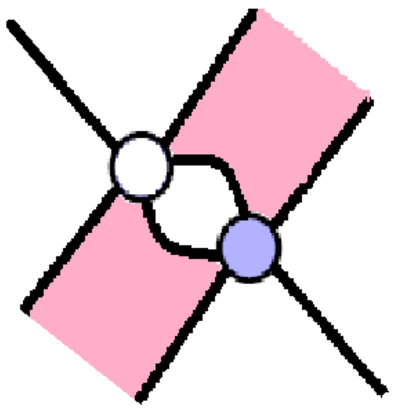

PART 4 - QUANTUM LATIN ISOMETRY SQUARES

Quantum Latin isometry squares

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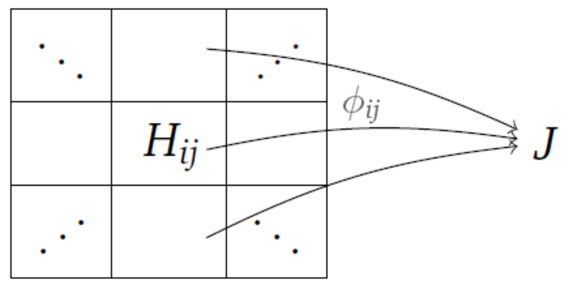
Definition. A *quantum Latin isometry square* (ϕ_{ij}, H_{ij}, J) of order n is an n-by-n grid of isometries $k_{ij} : \mathbb{C}^{a_{ij}} \to \mathbb{C}^d$, complete along each row and column.



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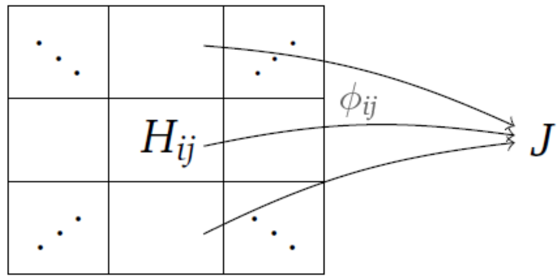

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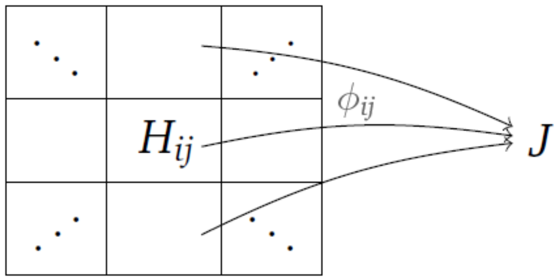

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Example. Write "$H\delta_{ij}$" for the family of Hilbert spaces which is H when $i = j$, and 0 otherwise. For any family of unitaries U_i on H, we get a quantum Latin isometry square $(U_i\delta_{ij}, H\delta_{ij}, H)$.

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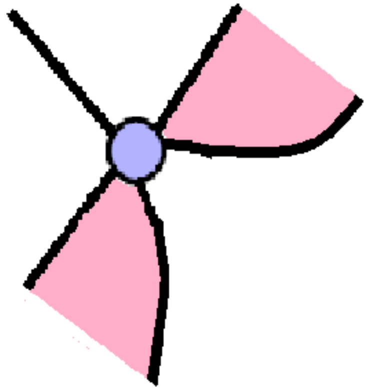

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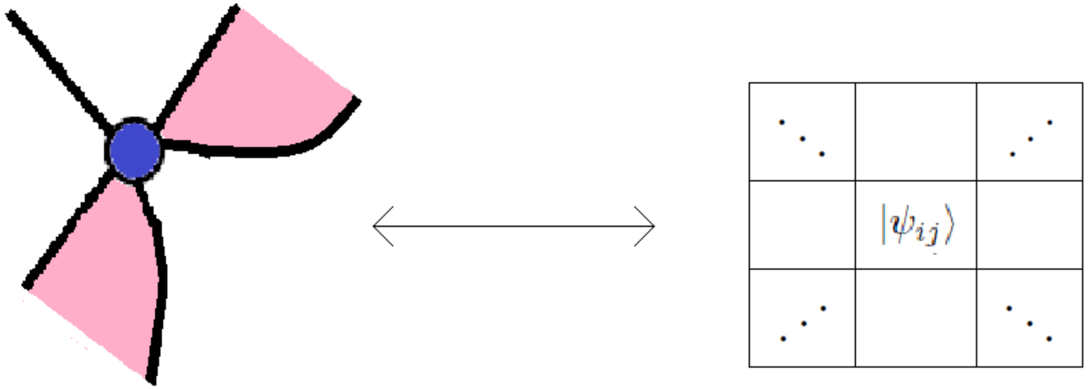

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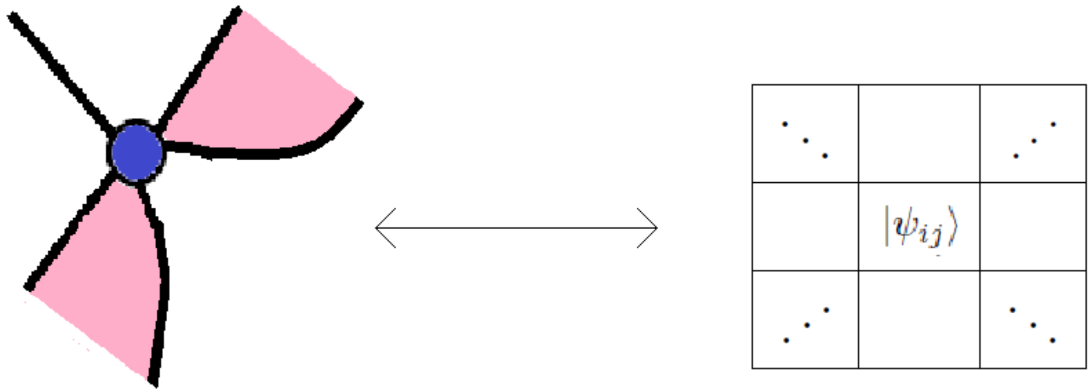

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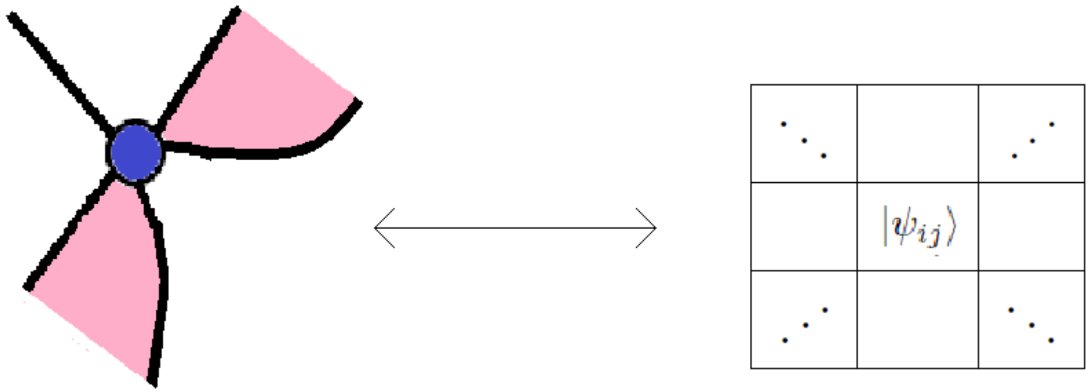

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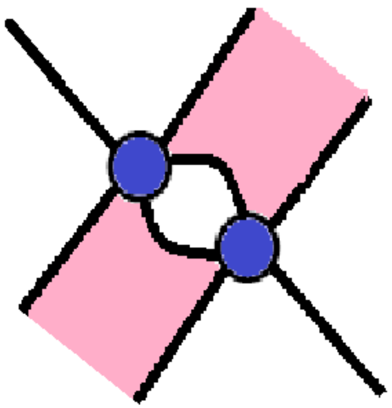

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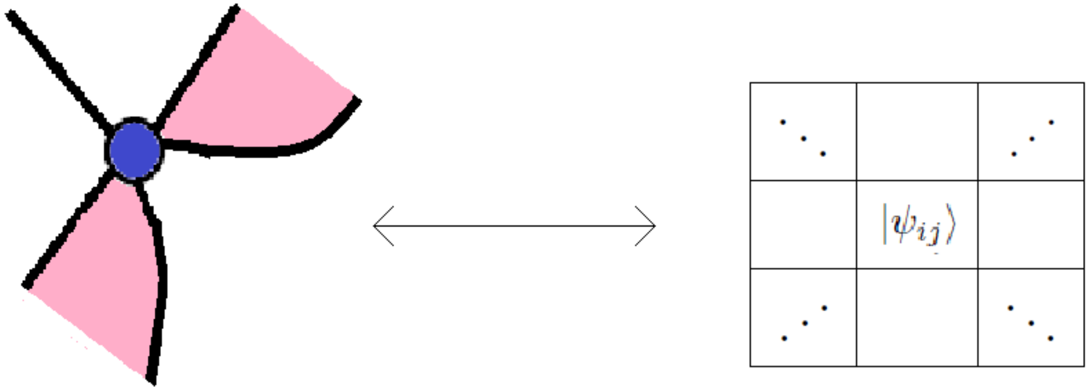

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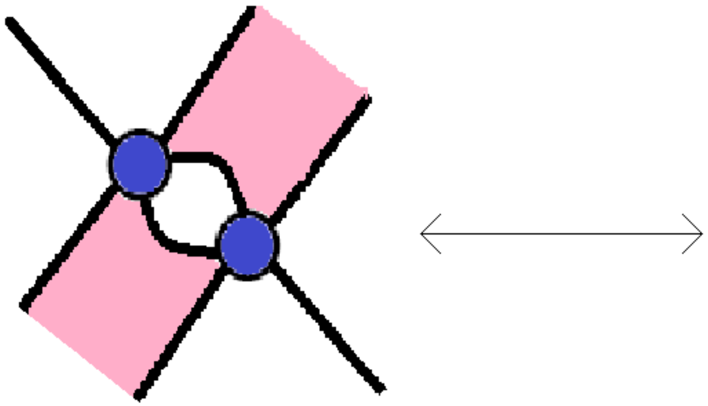

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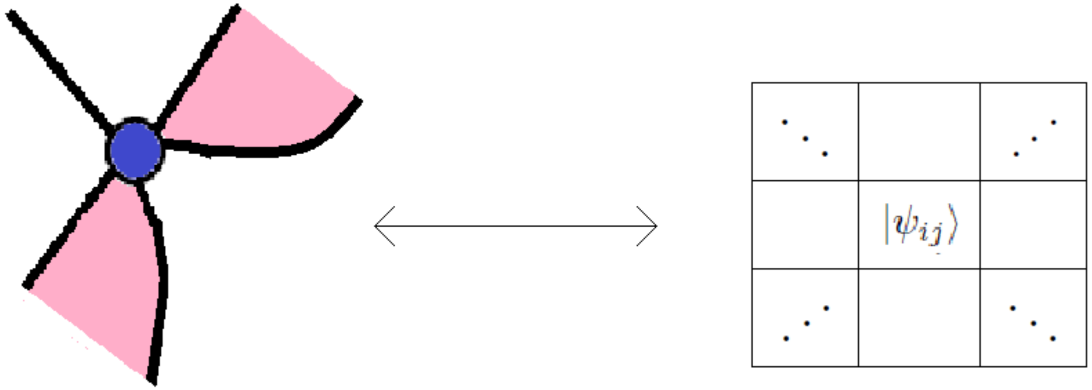

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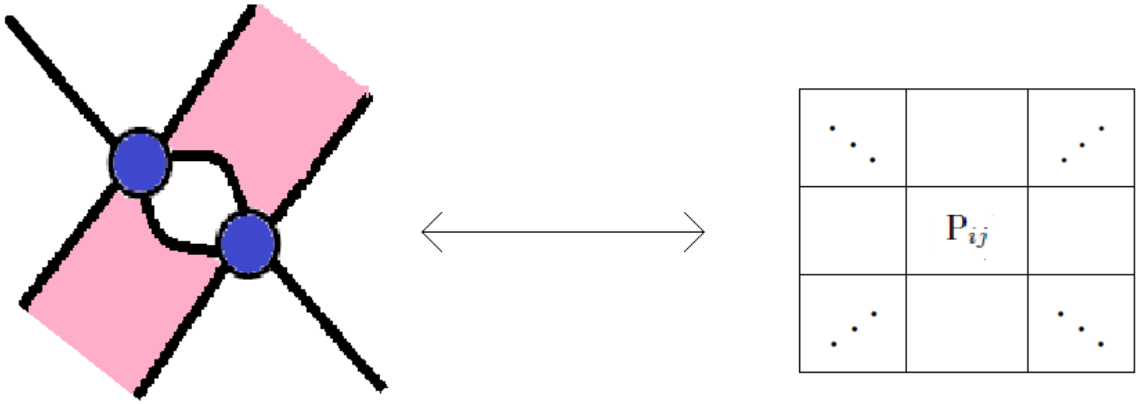

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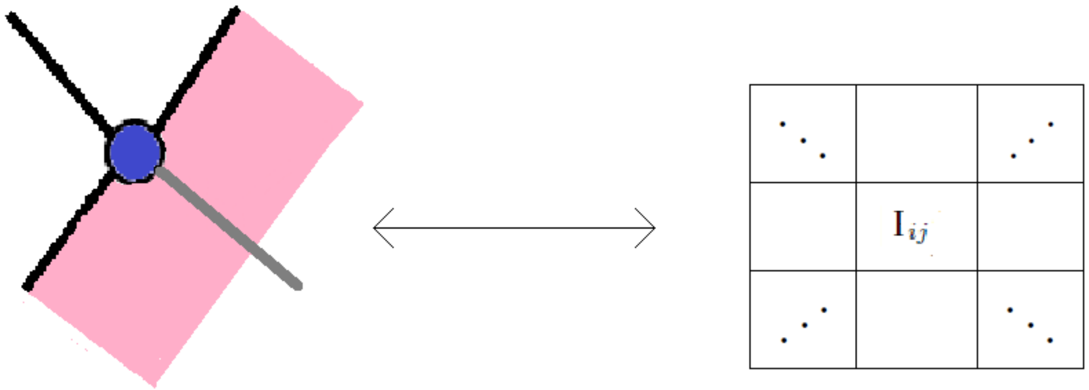

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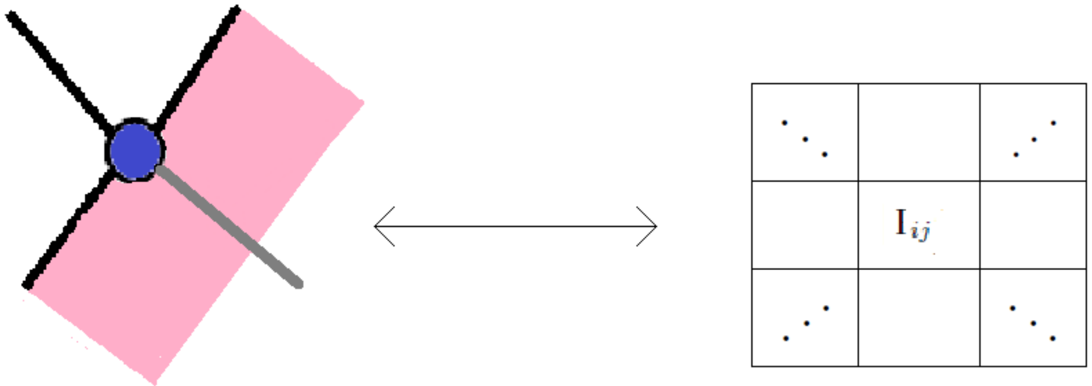
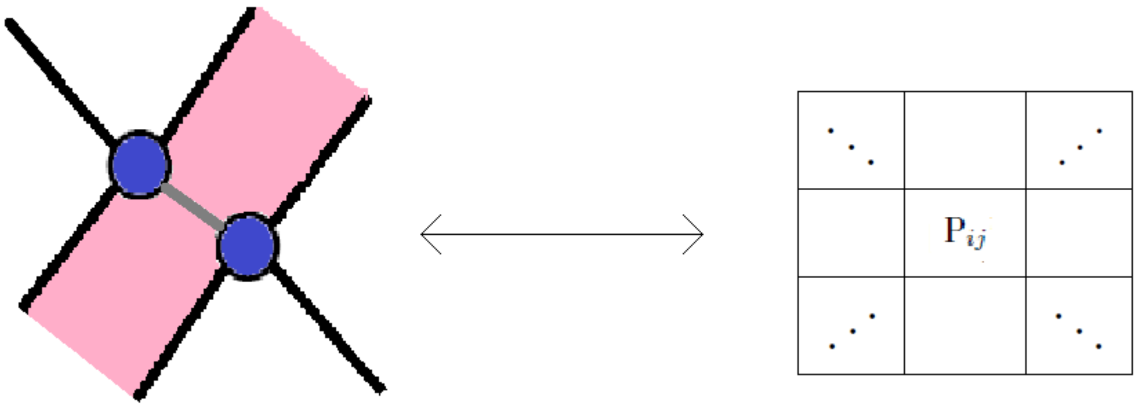


We can project out a QLiS to obtain a grid of projectors:



Graphical QLiSs

This is known as a projective permutation matrix (PPM), the projectors form POVMs on every row and column. PPMs have appeared in connection to non-local games for example in AMRSSV.

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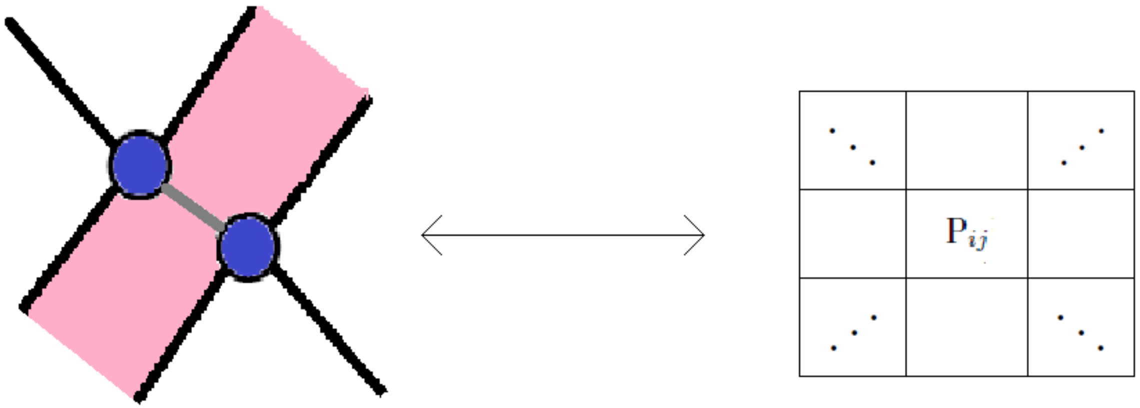

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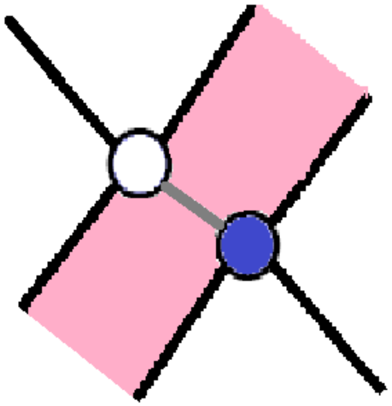

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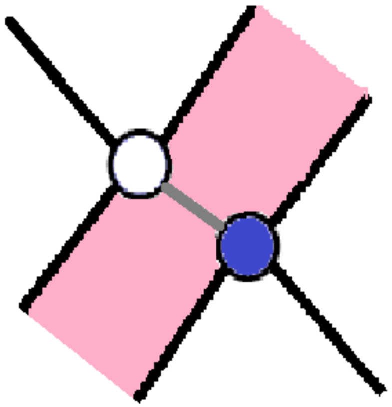

This a biunitary. It is known as a matrix of partial isometries and was shown to characterise quantum channels preserving pointer states in NB.

Orthogonal isometry squares

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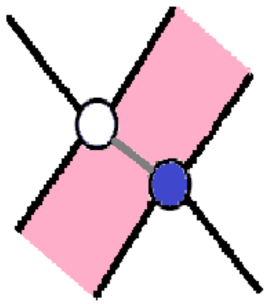

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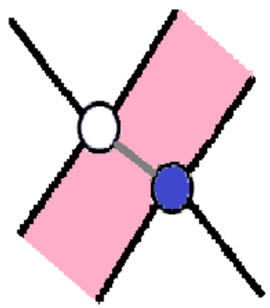

is an error detecting tensor.

Unitary error bases

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- the quantum Latin isometry squares $(U_i\delta_{ij}, H\delta_{ij}, n)$ and $(\mathrm{id}_H\delta_{ij}, H\delta_{ij}, n)$ are orthogonal;
- the family U_i forms a unitary error basis.