Building models from finite pieces

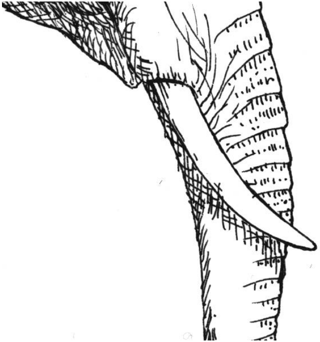
Morgan Rogers

rogers@lipn.univ-paris13.fr

Laboratoire d'Informatique de Paris Nord (LIPN)

Overview

Sketch 1



Geometric logic

In topos theory, we consider theories in the positive fragment of infinitary first-order logic, *geometric logic*. Formulas are constructed from typed variables, function and relation symbols from a signature Σ and the following connectives:

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Given an interpretation of the types, functions and relations from Σ in a topos $\mathcal{E}$, each formula defines a subobject. A *model* of $\mathbb{T}$ in $\mathcal{E}$ is an interpretation such that the axioms become inclusions of subobjects.

Classifying toposes

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We consider these theories because they capture representable structure in Grothendieck toposes.

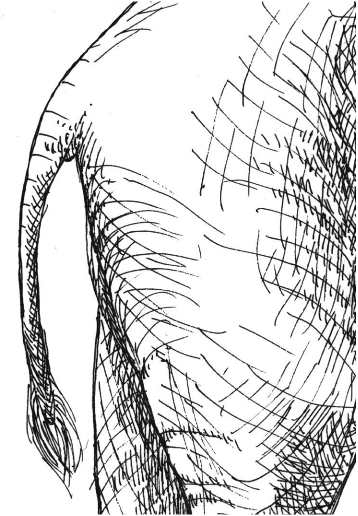
Fundamental theorem of classifying toposes

For a geometric theory $\mathbb{T}$ there is a topos $\mathbf{Set}[\mathbb{T}]$ (the **classifying topos of** $\mathbb{T}$) such that,

$$\mathbb{T}\text{-mod}(\mathcal{E}) \simeq \mathrm{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]),$$
$$M \mapsto \ulcorner M \urcorner$$

naturally in $\mathcal{E}$. Conversely, any topos $\mathcal{F}$ is equivalent to $\mathbf{Set}[\mathbb{T}]$ for some geometric theory $\mathbb{T}$.

Sketch 2



Toposes from monoids

Let L be a topological monoid. A *continuous action* of L on a set X is an action:

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Proposition [R]

The category of continuous actions of L on sets is a Grothendieck topos, $\mathrm{Cont}(L)$.

Note that this discards most 'connected' structure in L.
Interesting examples: ordinary (discrete) monoids; profinite completions $\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}}_p$; prodiscrete monoids; $\mathrm{End}(\mathbb{N})$ with the pointwise convergence topology.

Pointed toposes

A *point* of a topos $\mathcal{E}$ is a geometric morphism **Set** $\to \mathcal{E}$.

The forgetful functor Cont$(L) \to$ **Set** has a right adjoint and preserves finite limits, so determines a point, p.

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Theorem [R]

Pairs of the form $(\mathrm{Cont}(L), p)$ are coreflective in the 2-category of pointed toposes.

In particular, given a model L of a geometric theory $\mathbb{T}$ in $\mathbf{Set}$, the pointed topos $(\mathbf{Set}[\mathbb{T}], \ulcorner M \urcorner)$ coreflects to $\mathrm{End}(L)$ with the 'pointwise convergence' topology. (See previous SYCO talk!)

Monoids vs Theories

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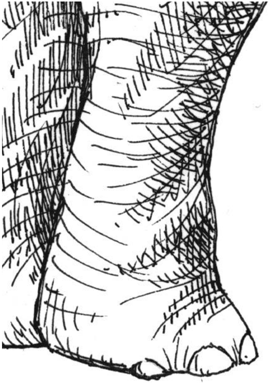
Note that p must correspond to a *special model* of any such theory $\mathbb{T}$, so another way of asking the question is:

Q. Which theories $\mathbb{T}$ have a special model?

Sketch 3



Sites of principal actions

A *site* consists of a small category $\mathcal{C}$ and a Grothendieck coverage J. This induces a reflective subcategory (subtopos):

$$\mathrm{Sh}(\mathcal{C}, J) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{PSh}(\mathcal{C})$$

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Example

For a topological monoid L, denote by $\mathcal{C}_s$ the category of (continuous) *principal actions* of L. This has a canonical Grothendieck coverage J_s generated by quotient maps $X \to X/\sim$. One can show that

$$\mathrm{Cont}(L) \simeq \mathrm{Sh}(\mathcal{C}_s, J_s).$$

Theories of presheaf type

A theory is said to be *of presheaf type* when $\mathbf{Set}[\mathbb{T}] \simeq \mathrm{PSh}(\mathcal{C})$. These theories are particularly convenient because we have:

$$\mathcal{K} := \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq \mathcal{C}^{\mathrm{op}} \quad \text{and} \quad \mathbb{T}\text{-mod}(\mathbf{Set}) = \mathrm{Ind}(\mathcal{K}),$$

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Meanwhile, a subtopos of $\mathbf{Set}[\mathbb{T}]$ classifies a ‘*quotient*’ of $\mathbb{T}$, meaning a theory obtained by adding axioms.

Corollary

If $\mathrm{Cont}(L)$ classifies a quotient of any theory $\mathbb{T}$ of presheaf type such that $\mathcal{K} := \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq \mathcal{C}_s^{\mathrm{op}}$.

Properties of principal actions

We can derive necessary conditions on the category of f.p. models by considering properties of $\mathcal{C}_s$. For instance $\mathcal{K}$ must satisfy:

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(JEP) The *joint embedding property* requires that for any pair of objects X, Y, there is an object Z and (strong) monomorphisms,

$$X \rightarrowtail Z \leftarrowtail Y.$$

(TP) The *transferability property* requires that any span with one leg a (strong) monomorphism can be completed to a square where the opposite side is also monic:

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Compare JEP and Amalgamation Property (AP) in Fraïssé theory.

The special model with respect to the OFS

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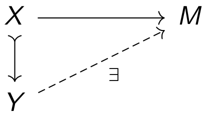
A *special model* M of $\mathbb{T}$, if it exists, satisfies:

(inj) M is *injective* with respect to $\mathcal{R}$, so any span in Ind($\mathcal{K}$) as follows (with left leg in $\mathcal{R}$) extends (non-uniquely) to a triangle:



(univ) M is *universal* for $\overline{\mathcal{R}}$, so any object X of $\mathcal{K}$ admits an $\overline{\mathcal{R}}$-morphism to M in Ind($\mathcal{K}$):

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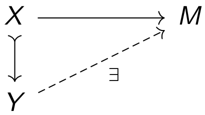

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These conditions are necessary *and sufficient*.

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Proof. We inductively construct a chain of objects U_i of $\mathcal{K}$ whose colimit has the desired properties.
Let $\pi : \omega \times \omega \times \omega \to \omega$ be a bijection such that $\pi(i, j, k) \geq k$.
Enumerate $\mathcal{L}$ as $\{l_i : X_i \to Y_i\}_{i<\omega}$.
Base case. Let $U_0 = A_0$. (We use the enumeration for objects too.)

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Induction step. Given U_k, we can enumerate spans of the following form,

$$\left\{ \begin{array}{ccc} X_i & \xrightarrow{f_{i,j,k}} & U_k \\ {\scriptstyle l_i}\downarrow & & \\ Y_i & & \end{array} \;\middle|\; j < \omega \right\}.$$

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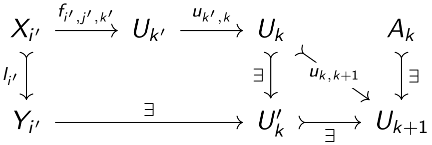
Let $(i', j', k') = \pi^{-1}(k)$. We use TP to define an intermediate object U'_k and then the JEP to define U_{k+1}:



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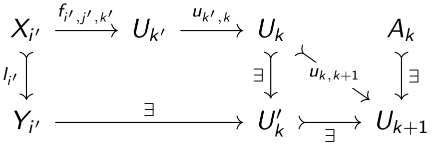

The colimit satisfies (dir) by construction. For (inj), any span must factorize through an enumerated one. Lastly, $A_k \rightarrowtail U_k \rightarrowtail M$ is in $\overline{\mathcal{R}}$, giving (univ).

Recap

Let $\mathbb{T}$ be a theory whose finitely presentable models form a countable category $\mathcal{K}$ satisfying OFS, JEP and TP, we obtain a special model M of an extension $\mathbb{T}'$ of $\mathbb{T}$ such that,

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Specifically, the extension $\mathbb{T}'$ requires each monomorphism $X \rightarrowtail Y$ in $\mathcal{K}$ to induce an epimorphism $\mathrm{Hom}(Y, M) \twoheadrightarrow \mathrm{Hom}(X, M)$.

We can add torsion axioms in $\mathbb{T}$ to force f.p. models to be finite models.

Classical cases

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- Dense linear orderings ($\mathbb{Q}$)
- Simple graphs (Rado graph)
- Fields of characteristic p ($\overline{\mathbb{F}}_p$)
- Groups (Hall's universal group)
- Boolean algebras (countable atomless Boolean algebra)

Caramello covered these cases.

Actions on Boolean algebras

Let $L = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a finitely generated monoid. Let $\mathbb{T}$ be the theory of finite Boolean algebras equipped with a right action of L. Explicitly, this is based on a signature Σ containing:

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These satisfy axioms making:

- $(B, 0, 1, \cup, \cap, \neg)$ a Boolean algebra,
- Each m^* commute with each of the Boolean algebra operations and the ordering,
- $m^* n^*(x) = (nm)^*(x)$ for $m, n \in L$ (it is a *right* action),

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If I have constructed this correctly, the finitely presentable models $\mathcal{K}$ are finite and we have a duality with finite principal *left* L-sets (by analogy with Stone duality).

A free example

The theorem applies: there exists a universal, injective Boolean algebra B equipped with a (locally finite, connected) right L-action into which all of the finite ones embed, and such that,

$$\mathbf{Set}[\mathbb{T}'] \simeq \mathsf{Cont}(\mathsf{End}(B)) \simeq \mathsf{Sh}(\mathcal{K}, J_{\mathrm{mono}^{\mathrm{op}}})$$

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Concrete example

Let $L = \mathcal{A}^*$ for some finite alphabet $\mathcal{A}$. A left L-set 'is' a set of states Q equipped with a transition function $\mathcal{A} \times Q \to Q$. [*Almost an automaton.*]

The theory of Boolean algebras with a (locally finite, connected) right action of L is classified by the topos of transition functions in which every state generates a finite subset. [*Almost finite automata; the corresponding languages are regular!*]

Why so complicated?

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Algebraic theories can't work: if $\mathcal{K}$ has products, (TP) implies (AP), so the subcategory of monos is enough for classical Fraïssé theory!

Fin

Thank you! Questions?