SYCO 13

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PARTIAL MARKOV CATEGORIES

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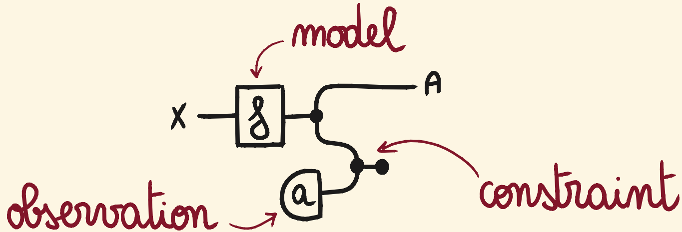
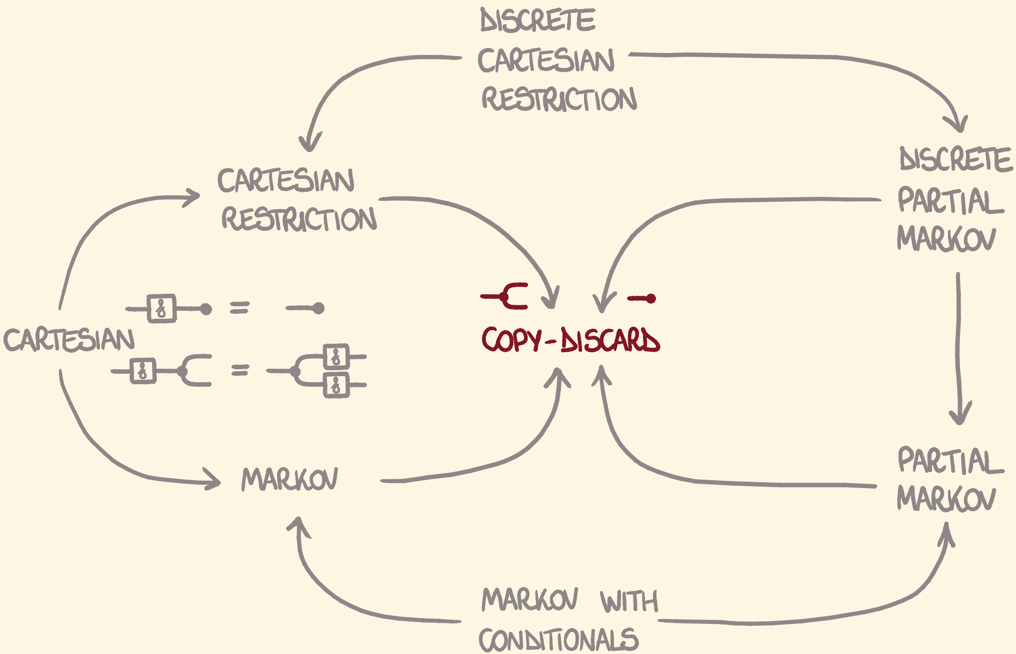
MOTIVATION

- Find the algebraic structure to express belief updates.
- Markov categories express probabilistic processes.



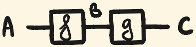
Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.

OUTLINE


DISCRETE CARTESIAN RESTRICTION
CARTESIAN RESTRICTION
DISCRETE PARTIAL MARKOV
CARTESIAN
COPY-DISCARD
MARKOV
PARTIAL MARKOV
MARKOV WITH CONDITIONALS

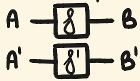
STRING DIAGRAMS FOR COPY-DISCARD CATEGORIES

composition



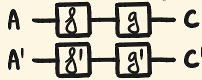
$f ; g : A \to C$

monoidal product



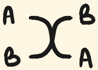
$f \otimes f' : A \otimes A' \to B \otimes B'$

interchange



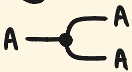
$(f \otimes f') ; (g \otimes g') = (f ; g) \otimes (f' ; g')$

symmetries



$A \otimes B \to B \otimes A$

copy

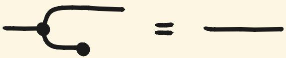


$A \to A \otimes A$

discard

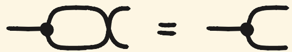
$A \to I$

counitality



coassociativity

cocommutativity



[Corradini, Gadducci (1999)]

EXAMPLES

- Kleisli of monoidal monads on cartesian categories

  cartesian categories = $\mathrm{Kl}(\mathrm{id})$ $\quad$ $\mathrm{Kl}\mathcal{D}$ $\quad$ $\mathrm{Kl}\mathcal{G}$

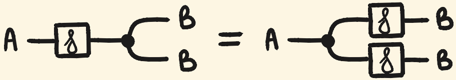
  $\mathrm{Rel} = \mathrm{Kl}\mathcal{P}$ $\quad$ $\mathrm{Par} = \mathrm{Kl}(\bullet + 1)$ $\quad$ $\mathrm{Kl}\mathcal{D}_{\leq}$ $\quad$ $\mathrm{Kl}\mathcal{G}_{\leq}$

- not only Kleisli categories

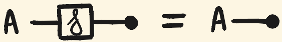
  Gauss $\quad$ hypergraph categories $\quad$ Mat_{R}

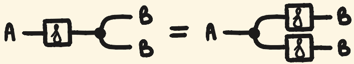
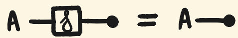
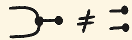
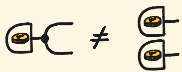
DETERMINISTIC & TOTAL MAPS

Deterministic maps can be copied.



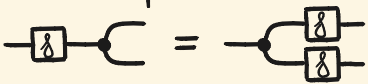
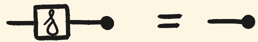
Total maps can be discarded.



EXAMPLES		
Set	✓	✓
Par	✓	✗ 
$\mathrm{Kl}(\mathcal{D})$	✗ 	✓
Rel	✗	✗
$\mathrm{Kl}(\mathcal{D}_{\leq 1})$	✗	✗

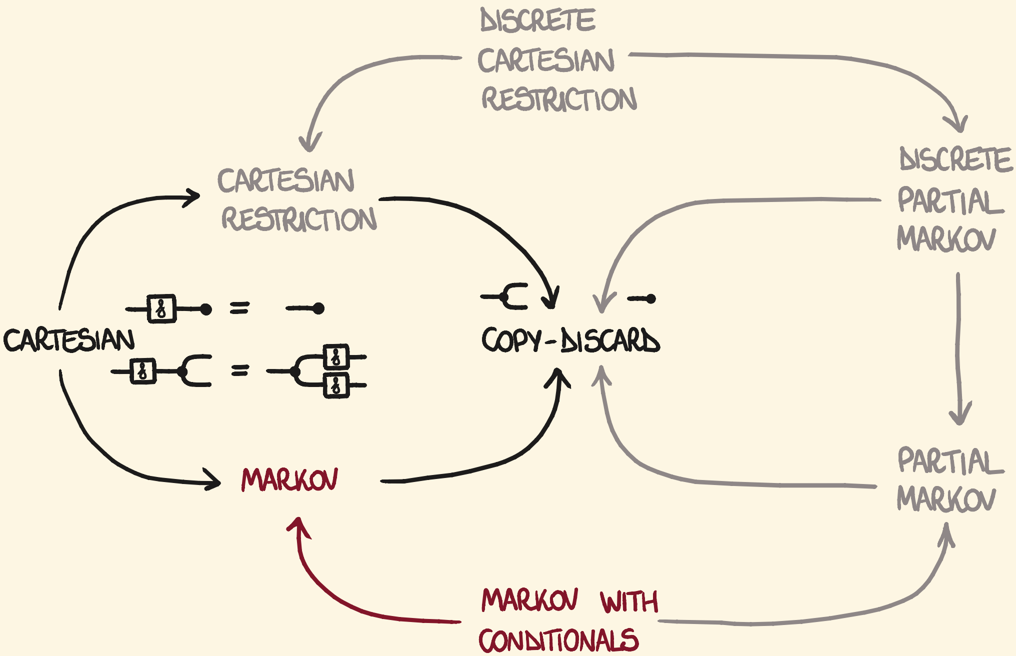
FOX'S THEOREM

A copy-discard category is cartesian if and only if all morphisms are deterministic and total,

 and  for all f.

[Fox 1976]

OUTLINE



DISCRETE CARTESIAN RESTRICTION
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COPY-DISCARD
MARKOV
PARTIAL MARKOV
MARKOV WITH CONDITIONALS

PROBABILISTIC PROCESSES

Markov categories express probabilistic processes, for example

- throwing a coin

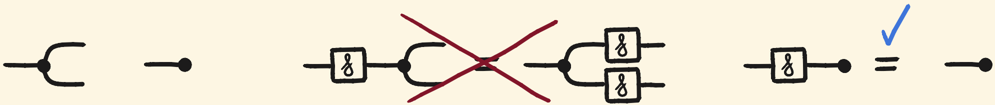
- tomorrow's weather given today's clouds

- developing cancer given smoking habits

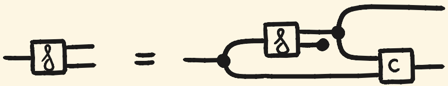
MARKOV CATEGORIES & CONDITIONALS

A Markov category with conditionals is a copy-discard category with conditionals where all morphisms are total.

COPY - DISCARD STRUCTURE



CONDITIONALS



[Fritz (2020); Cho, Jacobs (2019)]

FINITARY DISTRIBUTIONS

a *finitary distribution* $\sigma \in \mathcal{D}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is 1, $\sum_{a \in A} \sigma(a) = 1$.

$\rightsquigarrow$ $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$ is a monad

a morphism $X \text{—} \boxed{f} \text{—} A$ in $\mathsf{Kl}\mathcal{D}$ is a function $X \to \mathcal{D}(A)$

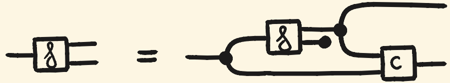
$f(a|x)$ = "probability of a given x"

composition is

$$X \text{—} \boxed{f} \text{—} \boxed{g} \text{—} B \; (b|x) := \sum_{a \in A} f(a|x) \cdot g(b|a)$$

CONDITIONALS

$\mathrm{Kl}\mathcal{D}$ has conditionals.



$$(a|x) = \sum_{b \in B} f(a, b|x)$$

$$c(b|a,x) := \begin{cases} \dfrac{f(a,b|x)}{\sum_{b'} f(a,b'|x)} & \text{if } \sum_{b'} f(a,b'|x) \neq 0 \\ \sigma(b) & \text{if } \sum_{b'} f(a,b'|x) = 0 \end{cases}$$

any distribution on B

NON-DISCRETE EXAMPLES

- GIRY MONAD. $\mathcal{G}: \mathsf{Meas} \to \mathsf{Meas}$. $\mathsf{Kl}\mathcal{G}$ is Markov

  $\mathcal{G}_B: \mathsf{Borel} \to \mathsf{Borel}$ $\quad$ $\mathsf{Kl}\mathcal{G}_B$ is Markov with conditionals

  $f: (X, \sigma) \longrightarrow (A, \tau)$ is $f: \tau \times X \longrightarrow [0,1]$

  $f(S|x) =$ "probability of event S given input x"

- RADON MONAD. $\mathcal{R}: \mathsf{CHaus} \to \mathsf{CHaus}$. $\mathsf{Kl}\mathcal{R}$ is Markov

- $\mathcal{G}_Q: \mathsf{QBS} \to \mathsf{QBS}$ $\quad$ $\mathsf{Kl}\mathcal{G}_Q$ is Markov

- Gauss is Markov with conditionals

  $f: n \to m$ is $(M_f \in \mathbb{R}^{m \times n}, C_f \in \mathbb{R}^{m \times m}, s_f \in \mathbb{R}^m)$, C_f positive semidefinite

  $Y = M_f \cdot X + z \quad z \sim N(s_f, C_f)$

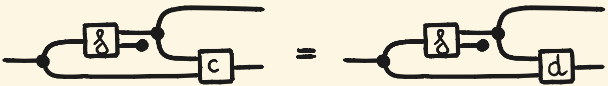
NON-UNIQUENESS OF CONDITIONALS

ex $\mathrm{Kl}\mathcal{D}$. $\quad c(b|a,x) := \begin{cases} \dfrac{f(a,b|x)}{\sum_{b'} f(a,b'|x)} & \text{if } \sum_{b'} f(a,b'|x) \neq 0 \\ \sigma(b) & \text{if } \sum_{b'} f(a,b'|x) = 0 \end{cases}$

PROPOSITION [Fritz (2020)]

In a Markov category $\mathcal{C}$ with conditionals, if conditionals are unique then $\mathcal{C}$ is a preorder.

⇝ conditionals of f are  - almost surely unique

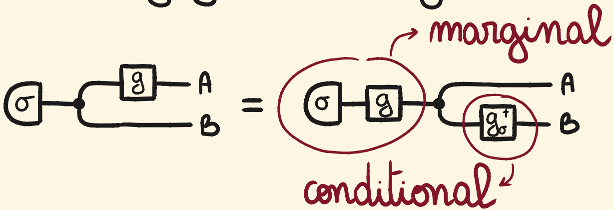


BAYES INVERSION

The Bayes inversion of a channel $g : B \to A$ with respect to a distribution $\sigma : I \to B$ is classically defined as

$$g^\dagger_\sigma(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b' \in B} g(a|b')\,\sigma(b')}$$

In a Markov category, it is a $g^\dagger_\sigma : A \to B$ such that



Bayes inversions are instances of conditionals.

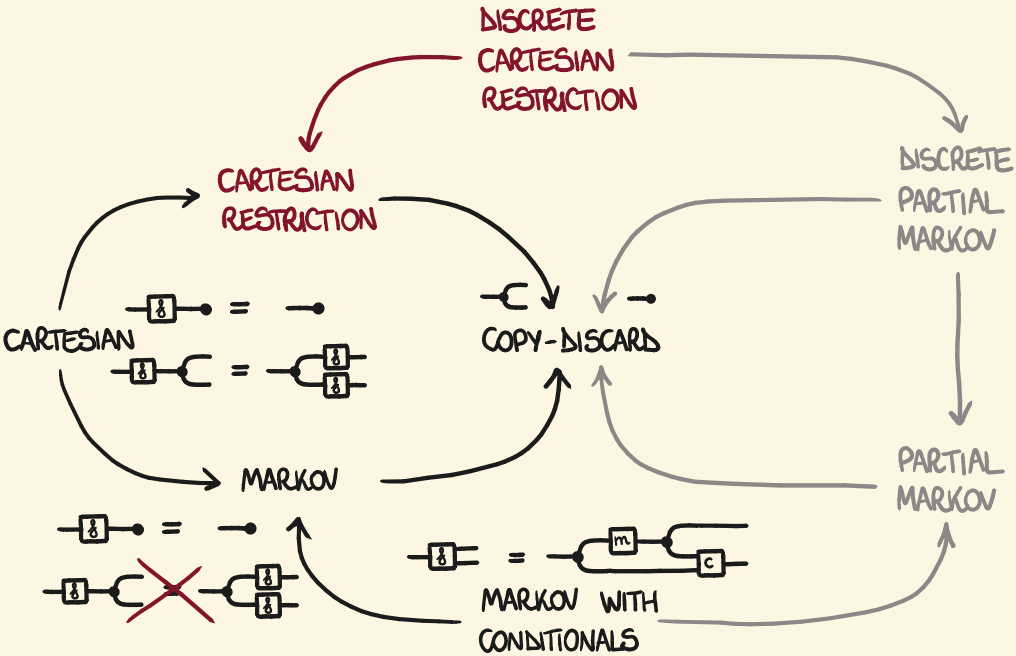
[Cho, Jacobs (2019)]

MARKOV CATEGORIES: A VERY ACTIVE FIELD

- Markov categories and conditionals
  [Cho, Jacobs (2019); Fritz (2020)]
- Theorems from probability theory
  [Fritz, Gonda, Perrone (2021); Fritz, Rischel (2020); Perrone, Van Belle (2024)]
- Probabilistic programming constructions
  [Stein, Staton (2023); Moss, Perrone (2022); Heunen, Kammar, Staton, Yang (2017)]
- Inference methods
  [Jacobs, Stein (2023); Jacobs, Zanasi, Kissinger (2021)]

⋮

OUTLINE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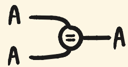
MARKOV WITH
CONDITIONALS

PARTIAL PROCESSES

Cartesian restriction categories express partial computations, for example

- computing $\frac{1}{x}$

- checking equality

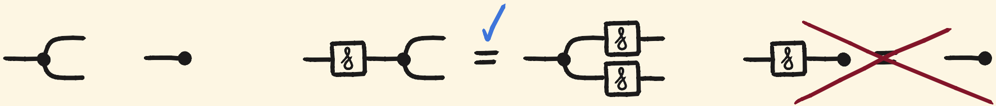


- non-terminating computations

CARTESIAN RESTRICTION CATEGORIES

A cartesian restriction category is a copy-discard category where all morphisms are deterministic.

COPY - DISCARD STRUCTURE



[Cockett, Lack (2003, 2007), Nester (2024)]

PARTIAL FUNCTIONS

$(\mathsf{Par}, \times, \{*\})$ is a cartesian restriction category

- objects are sets $A, B, C, \ldots$
- morphisms are partial functions $f : A \rightharpoonup B$, $g : B \rightharpoonup C, \ldots$
  i.e. functions $f : A \to B+1$, $g : B \to C+1, \ldots$
- composition is

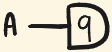
$$f ; g\,(a) := \begin{cases} g(f(a)) & \text{if } f(a) \neq \perp \\ \perp & \text{if } f(a) = \perp \end{cases}$$

- monoidal product is

$$f \times f'\,(a, a') := \begin{cases} (f(a), f'(a')) & \text{if } f(a) \neq \perp \text{ and } f'(a') \neq \perp \\ \perp & \text{if } f(a) = \perp \text{ or } f'(a') = \perp \end{cases}$$

PREDICATES, DOMAINS, RESTRICTIONS

Morphisms $q : A \to 1$ in Par are predicates.

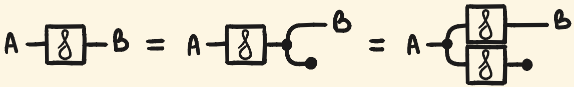


$(a) = \begin{cases} * & \text{if } a \text{ satisfies } q \\ \perp & \text{if } a \text{ does not satisfy } q \end{cases}$

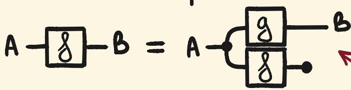
The domain of 

is the predicate 

.



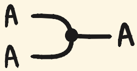
The restriction preorder on morphisms is

$f \leq g$ iff 

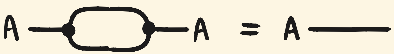
g restricted to the domain of f

EQUALITY CHECK

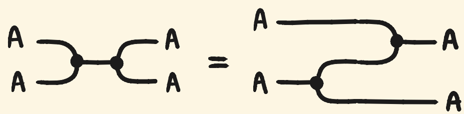
Par has equality checks.

 $(a, a') := \begin{cases} a & \text{if } a = a' \\ \perp & \text{if } a \neq a' \end{cases}$

Equality checks interact with the comonoid structure.

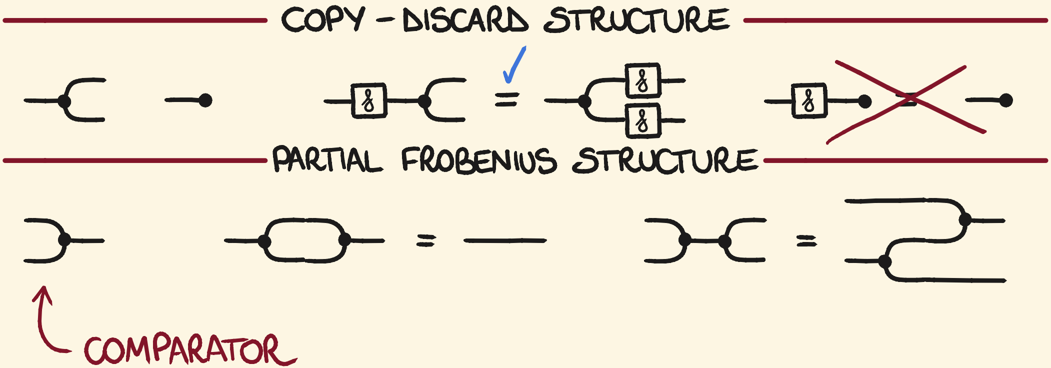


and



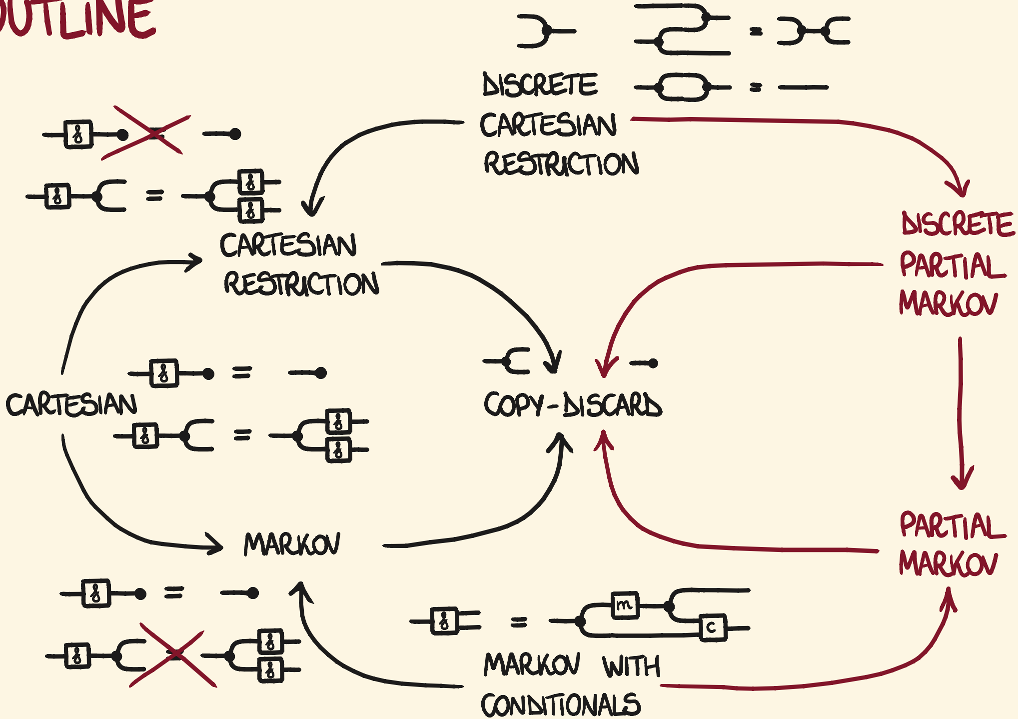
CONSTRAINTS VIA PARTIAL FROBENIUS

A discrete cartesian restriction category is a copy-discard category with comparators where all morphisms are deterministic.



[Cockett, Guo & Hofstra 2012, Di Liberti, Loregian, Nester & Sobociński 2020]

OUTLINE


DISCRETE CARTESIAN RESTRICTION
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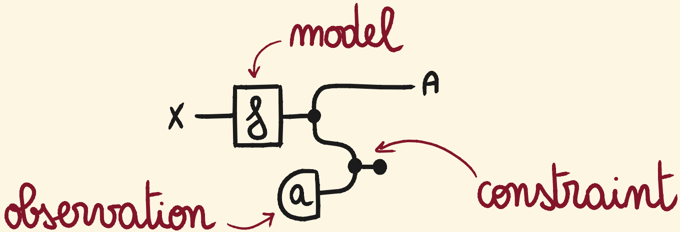
MARKOV WITH CONDITIONALS

PARTIALITY FOR OBSERVATIONS

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.



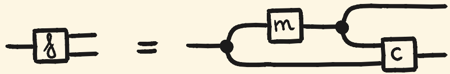
constraints  cannot be total computations because  $\neq$  .

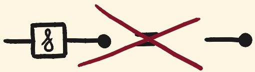
OVERVIEW

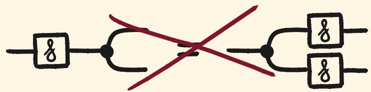
Combine Markov and cartesian restriction categories to express partial stochastic processes.

cartesian restriction

Markov with conditionals

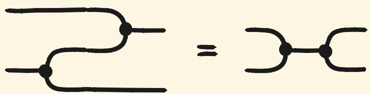


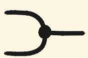




Add the discrete structure to express equality checking.

discrete cartesian restriction



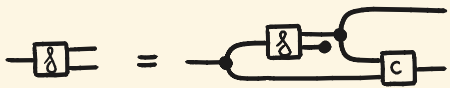


PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category with quasi-total conditionals.

COPY - DISCARD STRUCTURE

CONDITIONALS



[EDL, Román (2023); EDL, Román, Sobociński (2025)]

SUBDISTRIBUTIONS

A <u>subdistribution</u> σ on A is a distribution on $A+1$:

$\sigma \in \mathcal{D}_{\leq 1}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is at most 1, $\sum_{a \in A} \sigma(a) \leq 1$.

A morphism $X - \boxed{f} - A$ in $\mathrm{Kl}\mathcal{D}_{\leq 1}$ is a function $X \to \mathcal{D}_{\leq 1}(A)$

$f(a \mid x)$ = "probability of a given x"

$f(\perp \mid x)$ = "probability of failure"

Composition is

$$X - \boxed{f} - \boxed{g} - B \ (b \mid x) := \sum_{a \in A} f(a \mid x) \cdot g(b \mid a)$$

$$X - \boxed{f} - \boxed{g} - B \ (\perp \mid x) := \sum_{a \in A} f(a \mid x) \cdot g(\perp \mid a) + f(\perp \mid x)$$

EXAMPLES : PARTIAL STOCHASTIC PROCESSES

A <u>partial stochastic process</u> is a stochastic process that may fail.

↳ Maybe monad

a Markov category with conditionals

Partial stochastic processes form a partial Markov category.

PROPOSITION

The Kleisli category of the Maybe monad on a distributive Markov category with conditionals is partial Markov.

EXAMPLES

- $\mathsf{Kl}\mathcal{D}_{\leq} = \mathsf{Kl}(\cdot + 1)$ on $\mathsf{Kl}\mathcal{D}$
- $\mathsf{Kl}\mathcal{G}_{B\leq} = \mathsf{Kl}(\cdot + 1)$ on $\mathsf{Kl}\mathcal{G}_B$
- $\mathsf{Par} = \mathsf{Kl}(\cdot + 1)$ on Set
- (non-example) Rel

CONDITIONALS IN SUBDISTRIBUTIONS

The marginal of $f : X \to A \otimes B$ is

$$X - \boxed{m} - A\,(a|x) = X - \boxed{f} \begin{smallmatrix} - A \\ - \bullet \end{smallmatrix}\,(a|x) = \sum_{b \in B} f(a,b|x)$$

$$X - \boxed{m} - A\,(\perp|x) = X - \boxed{f} \begin{smallmatrix} - A \\ - \bullet \end{smallmatrix}\,(\perp|x) = f(\perp|x)$$

a conditional of f is :

$$\begin{smallmatrix} A \\ X \end{smallmatrix} \boxed{c} - B\,(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ 0 & m(a|x) = 0 \end{cases}$$

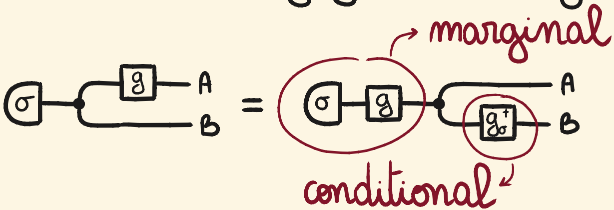
$$\begin{smallmatrix} A \\ X \end{smallmatrix} \boxed{c} - B\,(\perp|a,x) = \begin{cases} 0 & m(a|x) \neq 0 \\ 1 & m(a|x) = 0 \end{cases}$$

BAYES INVERSION

The Bayes inversion of a channel $g: B \to A$ with respect to a distribution $\sigma: I \to B$ is classically defined as

$$g^\dagger_\sigma(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b' \in B} g(a|b')\,\sigma(b')}$$

In a partial Markov category, it is a $g^\dagger_\sigma: A \to B$ such that



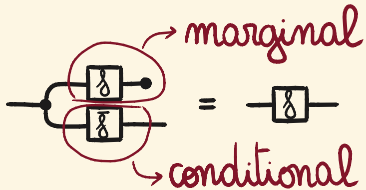
Bayes inversions are instances of conditionals.

NORMALISATION

The normalisation of a partial channel $f : X \to A$ is classically defined as

$$\bar{f}(a|x) := \frac{f(x|a)}{1 - f(\perp|a)}$$

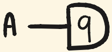
In a partial Markov category, it is a $\bar{f} : X \to A$ such that



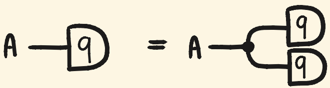
Normalisations are instances of conditionals.

PREDICATES & DOMAINS

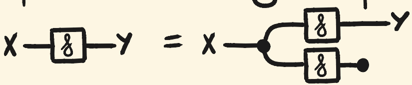
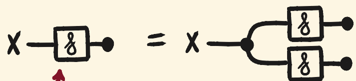
Morphisms $q : A \to 1$ in $\mathrm{Kl}(\mathcal{D}_{\leq})$ are 'fuzzy' predicates.

 $(* \mid a)$ ⇝ probability of a being true

Deterministic predicates are classical predicates.

 ⇒ q is a classical predicate

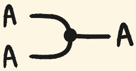
Self-normalising morphisms have a domain.

 ⟺ 

probability of success of f ⇒ domain of f

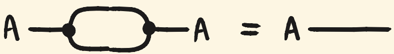
EQUALITY CHECK

$\mathrm{Kl}\,\mathcal{D}_{\leq}$ has equality checks.

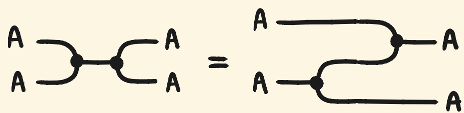


$$(a, a') := \begin{cases} \delta_a & \text{if } a = a' \\ \delta_\perp & \text{if } a \neq a' \end{cases}$$

Equality checks interact with the comonoid structure.



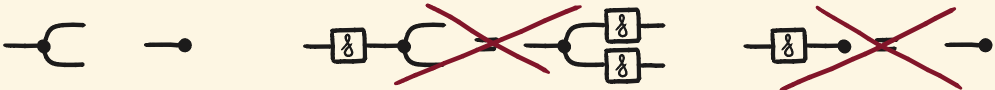
and



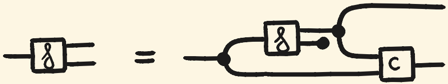
DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.

COPY-DISCARD STRUCTURE



CONDITIONALS

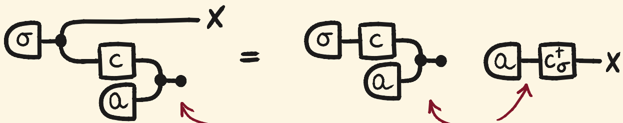


PARTIAL FROBENIUS STRUCTURE

[EDL, Román (2023); EDL, Román, Sobociński (2025)]

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.

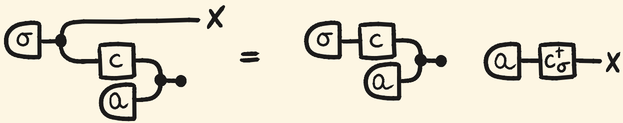


$$P(X=x \mid A=a) = \frac{P(A=a \mid X=x) \cdot P(X=x)}{\sum_{y \in X} P(A=a \mid X=y) \cdot P(X=y)}$$

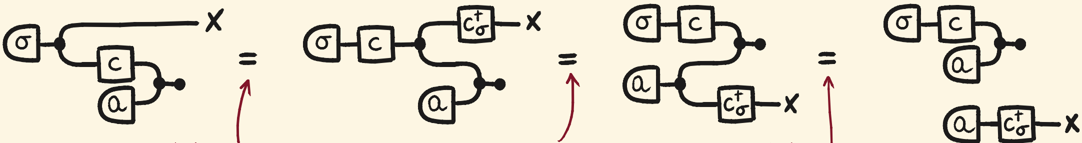
classical formula
for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.



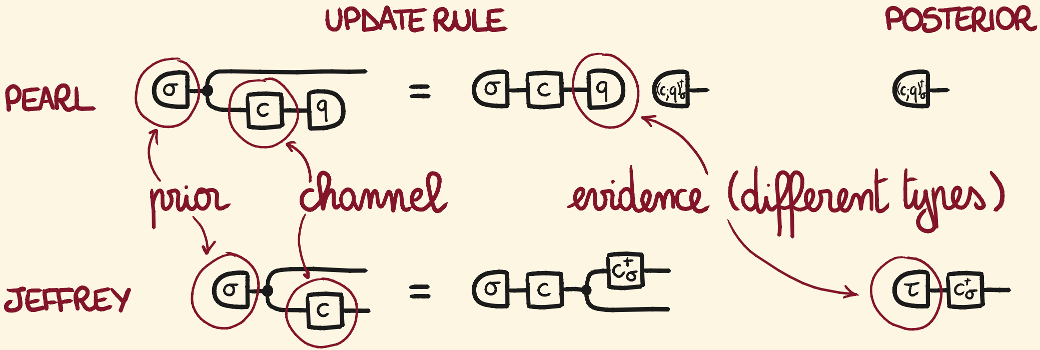
PROOF

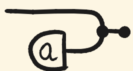
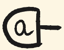
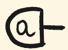


conditionals Frobenius determinism

□

PEARL'S VS JEFFREY'S UPDATES



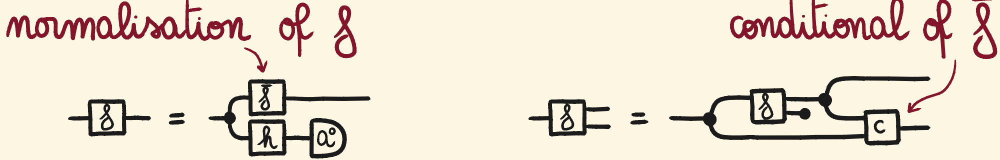
Pearl's update on  coincides with Jeffrey's update on , whenever  is deterministic.

[Jacobs (2019)]

PROCESSES WITH EXACT OBSERVATIONS

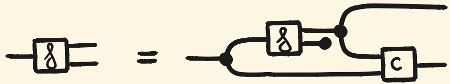
We can add exact observations to any Markov category:

Conditionals and normalisations are computed in 𝒞

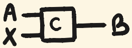


⇒ exOb(𝒞) is a partial Markov category.

MINIMAL CONDITIONALS IN SUBDISTRIBUTIONS



The conditionals of f are:



$$(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ \sigma(b) & m(a|x) = 0 \end{cases}$$

$$(\perp|a,x) = \begin{cases} 0 & m(a|x) \neq 0 \\ \sigma(\perp) & m(a|x) = 0 \end{cases}$$

for some $\sigma \in \mathcal{D}(B+1)$.

$\Rightarrow$ The 'minimal' choice for σ is $\begin{cases} \sigma(b) = 0 & \text{for } b \in B \\ \sigma(\perp) = 1 \end{cases}$.

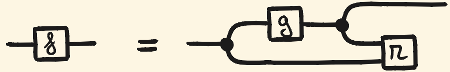
$\rightsquigarrow$ fail on unexpected observations

PARTIAL MARKOV CATEGORIES ARE PREORDERED

$f, g : X \to A$ in a partial Markov category

CONDITIONAL INEQUALITY

$f \leq g$ if $\exists\, r : A \otimes X \to I$



THEOREM

Any partial Markov category is preorder-enriched.

EXAMPLES

- In $\mathrm{Kl}\mathcal{D}_{\leq}$, $f \leq g$ is pointwise order: $\forall y, x \;\; f(y|x) \leq g(y|x)$.
- In $\mathrm{Kl}\mathcal{G}_{B\leq}$, $f \leq g$ is pointwise order: $\forall T, x \;\; f(T|x) \leq g(T|x)$.

[EDL, Román, Sobociński, Széles (2025)]

SUBUNITAL PREORDERS

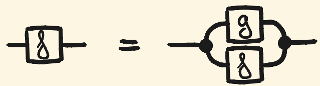
The conditional preorder is the minimal subunital preorder.

truth is top

PROPOSITION

$\mathscr{C}$ preorder-enriched partial Markov category s.t. $\forall p$  .
If $f \leq g$, then $f \subseteq g$.

EXAMPLES

- In Par (any cartesian restriction), $f \subseteq g$ iff  .
  Then, $f \subseteq g \Leftrightarrow f \leq g$.
- In Rel (any cartesian bicategory), $f \subseteq g$ iff  .
  Then, $f \subseteq g \Leftrightarrow f \leq g$.

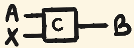
LEAST CONDITIONALS

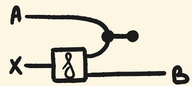
A partial Markov category has <u>least conditionals</u> if, for every $f : X \to A \otimes B$, the preorder of its conditionals has a least element.

EXAMPLES

- $\mathrm{Kl}\mathcal{D}_{\leq}$, 

$(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ 0 & m(a|x) = 0 \end{cases}$.

- Par (any discrete cartesian restriction), 

$=$ 

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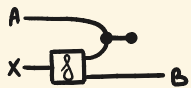
- Rel (any cartesian bicategory), 

$=$ 

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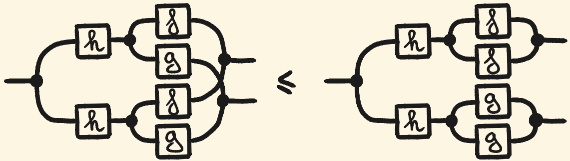
UPDATING INCREASES VALIDITY

CAUCHY-SCHWARZ INEQUALITY

- classical: $\left(\sum_i u_i \cdot v_i\right)^2 \leq \left(\sum_i u_i^2\right) \cdot \left(\sum_j v_j^2\right)$
- parametric: $\left(\sum_i h_i(x) \cdot f_i(y) \cdot g_i(z)\right)^2 \leq \left(\sum_i h_i(x) \cdot f_i(y)^2\right) \cdot \left(\sum_j h_j(x) \cdot g_j(z)^2\right)$
- synthetic:



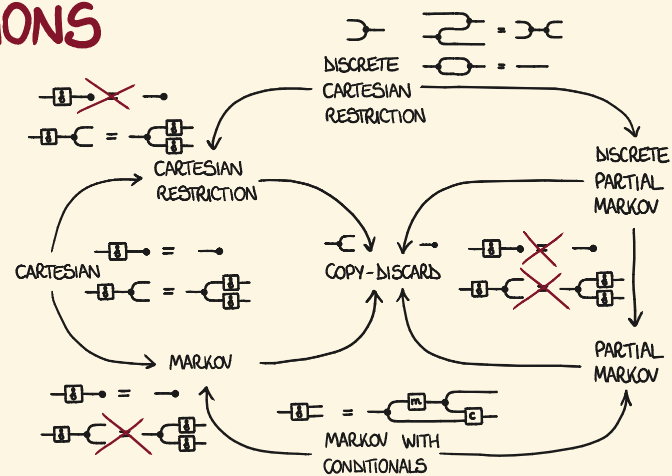
THEOREM

In a partial Markov category that satisfies the Cauchy-Schwarz inequality and has non-zero cancellative scalars, updating a prior $\sigma : I \to X$ with a predicate $p : X \to I$ increases the validity of p:

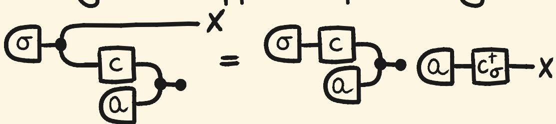
$$\sigma - p \;\leq\; p^{\dagger}_{\sigma} - p \quad .$$

[EDL, Román, Sobociński, Szeles (2025); cf. Jacobs (2019)]

CONCLUSIONS



- Partial Markov categories support updating on observations.



Synthetic Bayes theorem