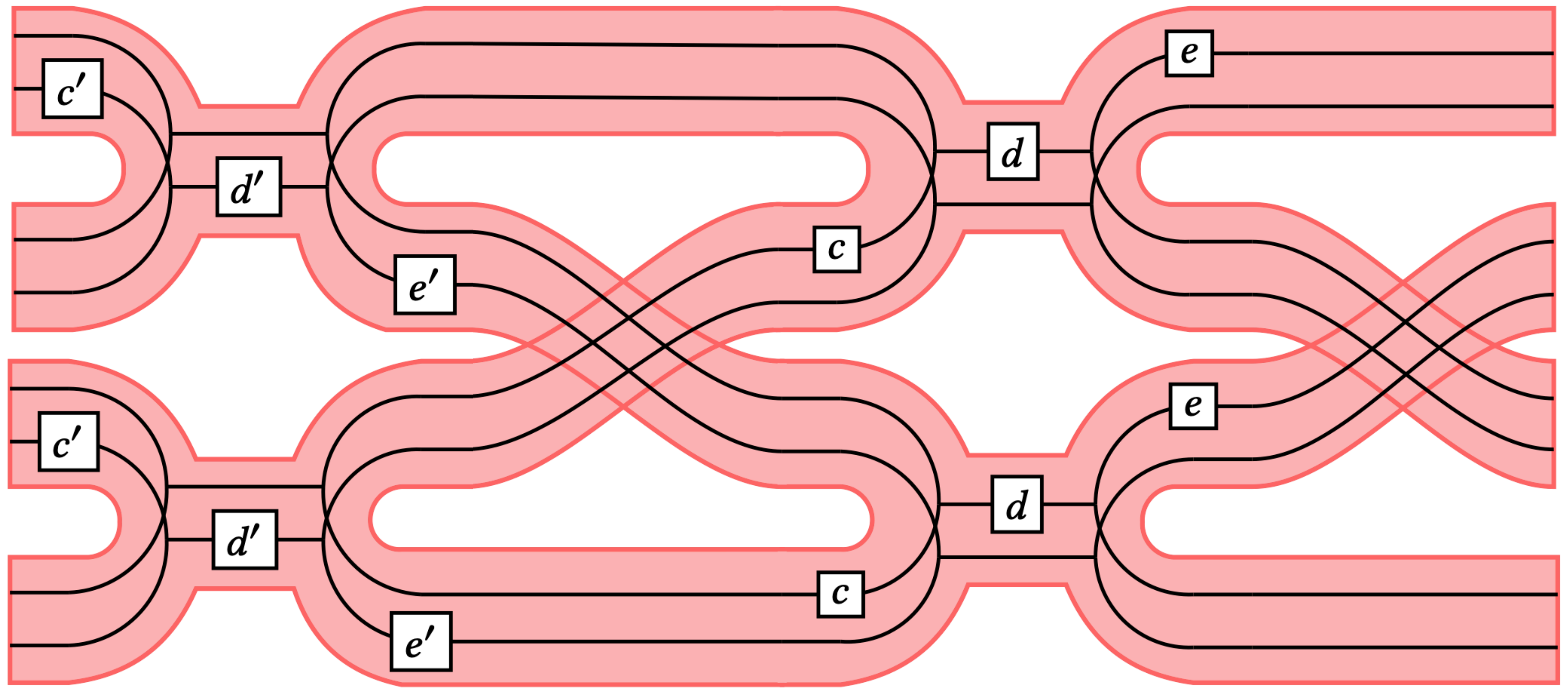
Tape Diagrams for Monoidal Monads

Alessandro Di Giorgio

j.w.w. Filippo Bonchi, Cipriano Junior Cioffo, Elena Di Lavore

Tape Diagrams are…

string diagrams of string diagrams



There are two monoidal products

$\otimes$ inside tapes

$\oplus$ outside tapes

Bonchi, Di Giorgio, Santamaria, 2023 Di Giorgio, 2024

Tape Diagrams are…

a universal language for *rig categories with finite biproducts*

biproduct

A category C with *two monoidal structures* $(\mathsf{C}, \otimes, 1)$ and $(\mathsf{C}, \oplus, 0)$ and *natural isomorphisms*

$$\delta^l_{X,Y,Z} \colon X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z) \qquad \lambda^\bullet_X \colon 0 \otimes X \to 0$$

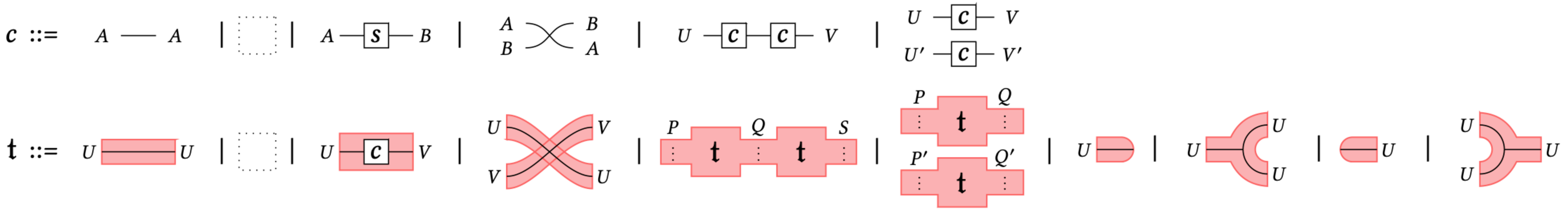
$$\delta^r_{X,Y,Z} \colon (X \oplus Y) \otimes Z \to (X \otimes Z) \oplus (Y \otimes Z) \qquad \rho^\bullet_X \colon X \otimes 0 \to 0$$

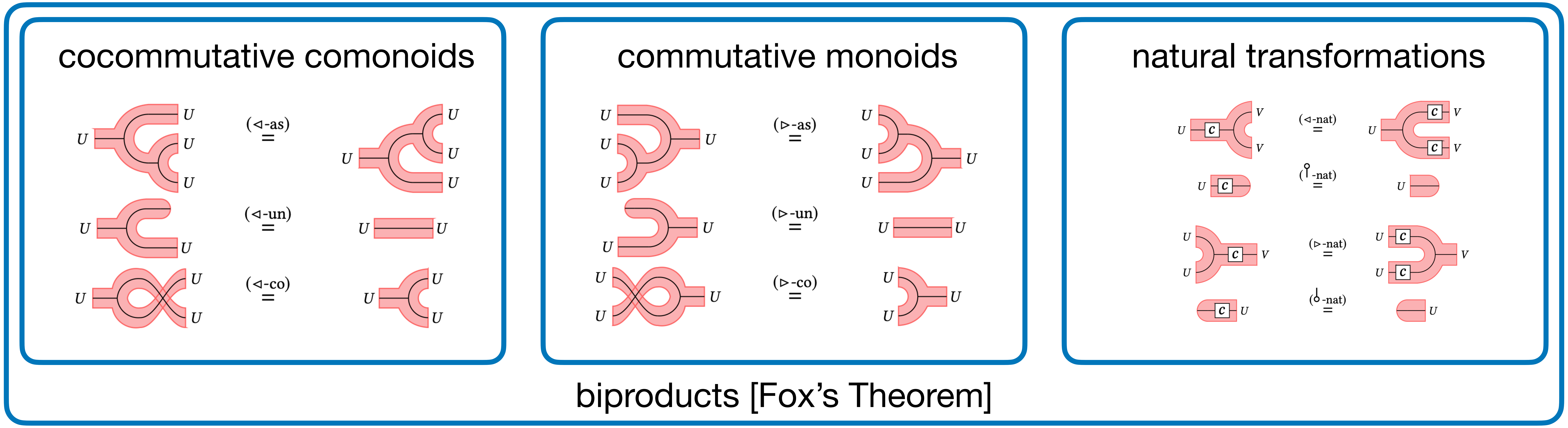
satisfying certain coherence conditions.

Laplaza, 1971

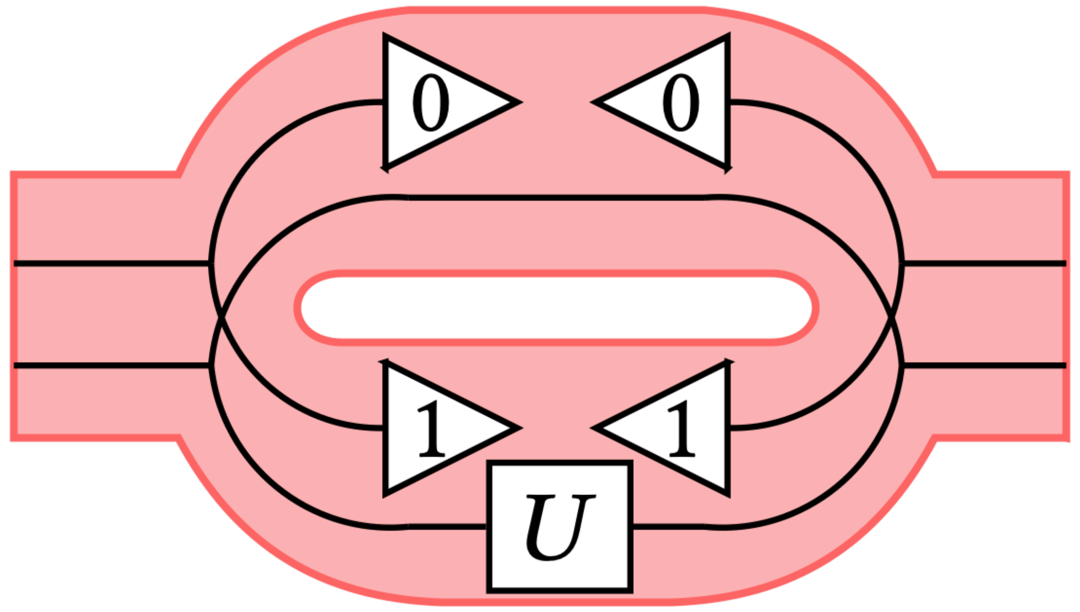
Tape Diagrams are…

the *free sesquistrict fb-rig category* generated by a monoidal signature $(\mathcal{S}, \Gamma)$





Example: Quantum Control

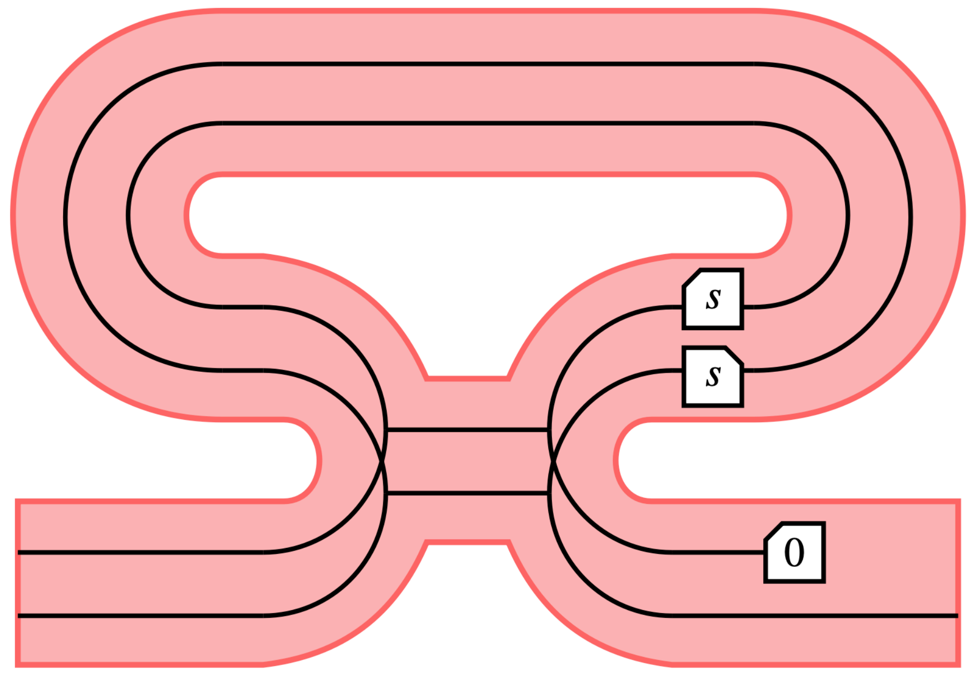


$$|00\rangle\langle 00| + |01\rangle\langle 01| + (|1\rangle \otimes U|0\rangle)\langle 10| + (|1\rangle \otimes U|1\rangle)\langle 11|$$

Takes semantics in the fb-rig category FdHilb with

- $\otimes$ tensor product
- $\oplus$ direct sum

Example: Control Flow




```
while (x>0)
   { x:=x-1; y:=y+1 };
return y
```

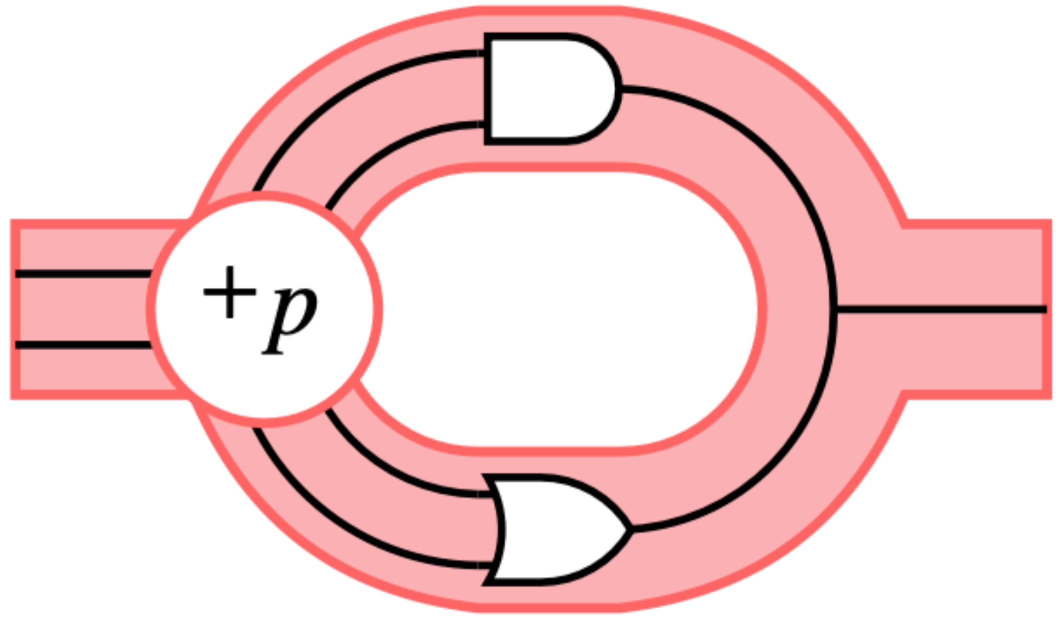
Takes semantics in the fb-rig category **Rel** with

$\otimes$ cartesian product

$\oplus$ disjoint union

Bonchi, Di Giorgio, Di Lavore, 2025

Quasi-example: Probabilistic Control



$$x \wedge y \quad +_p \quad x \vee y$$

Takes semantics in $Kl(\mathrm{D}_{\leq})$ (substochastic matrices)
which is not quite a fb-rig category

$\oplus$ is only a coproduct

$+_p$ is an additional operation on tapes

A Closer Look to $Kl(\mathsf{D}_{\leq})$

Objects are sets. **Morphisms** $f\colon X \to Y$ are functions $X \to \mathsf{D}_{\leq}(Y)$. **Composition** is $f;g(z \mid x) = \Sigma_{y \in Y} f(y \mid x) \cdot f(z \mid y)$

A Closer Look to $Kl(\mathsf{D}_{\leq})$

Objects are sets. **Morphisms** $f\colon X \to Y$ are functions $X \to \mathsf{D}_{\leq}(Y)$. **Composition** is $f; g(z \mid x) = \Sigma_{y \in Y} f(y \mid x) \cdot f(z \mid y)$

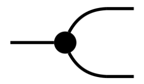
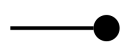
There are **two monoidal structures**:

$(Kl(\mathsf{D}_{\leq}), \otimes, \{\bullet\})$

$X \otimes Y = X \times Y$

$f \otimes g(y, y' \mid x, x') \overset{\text{def}}{=} f(y \mid x) \cdot g(y' \mid x')$

Cocommutative comonoids

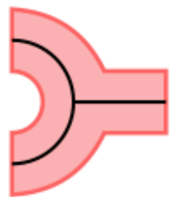
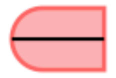
 $: X \to X \times X$, $x \mapsto \delta_{(x,x)}$  $: X \to 1$, $x \mapsto \delta_{\bullet}$

$(Kl(\mathsf{D}_{\leq}), \oplus, \{\})$

$X \oplus Y = X + Y$

$$f \oplus g(v \mid u) \overset{\text{def}}{=} \begin{cases} f(y \mid x) & \text{if } u = (x,0) \text{ and } v = (y,0) \\ g(y' \mid x') & \text{if } u = (x',1) \text{ and } v = (y',1) \\ 0 & \text{otherwise} \end{cases}$$

Natural commutative monoids

 $: X + X \to X$, $(x, i) \mapsto \delta_x$  $: 0 \to X$

A Closer Look to $Kl(\mathsf{D}_{\leq})$

Objects are sets. **Morphisms** $f\colon X \to Y$ are functions $X \to \mathsf{D}_{\leq}(Y)$. **Composition** is $f;g(z \mid x) = \Sigma_{y \in Y} f(y \mid x) \cdot f(z \mid y)$

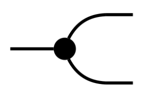
There are **two monoidal structures**:

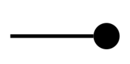
$(Kl(\mathsf{D}_{\leq}), \otimes, \{\bullet\})$

$X \otimes Y = X \times Y$

$f \otimes g(y, y' \mid x, x') \overset{\text{def}}{=} f(y \mid x) \cdot g(y' \mid x')$

Cocommutative comonoids

 $: X \to X \times X, \quad x \mapsto \delta_{(x,x)}$

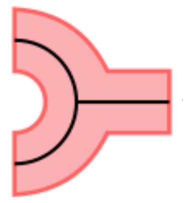
 $: X \to 1, \quad x \mapsto \delta_{\bullet}$

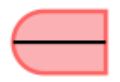
$(Kl(\mathsf{D}_{\leq}), \oplus, \{\,\})$

$X \oplus Y = X + Y$

$$f \oplus g(v \mid u) \overset{\text{def}}{=} \begin{cases} f(y \mid x) & \text{if } u = (x,0) \text{ and } v = (y,0) \\ g(y' \mid x') & \text{if } u = (x',1) \text{ and } v = (y',1) \\ 0 & \text{otherwise} \end{cases}$$

Natural commutative monoids

 $: X + X \to X, \quad (x, i) \mapsto \delta_x$

 $: 0 \to X$

There is additional structure given by

$(\!|\, +_p \,|\!)_X \colon X \to X + X$ mapping x into $(x,0) \mapsto p, \quad (x,1) \mapsto 1 - p$

$(\!|\, \star \,|\!)_X \colon X \to 0$ mapping x into the null subdistribution

A Closer Look to $Kl(\mathsf{D}_{\leq})$

Objects are sets. **Morphisms** $f\colon X \to Y$ are functions $X \to \mathsf{D}_{\leq}(Y)$. **Composition** is $f;g(z \mid x) = \Sigma_{y \in Y} f(y \mid x) \cdot f(z \mid y)$

There are **two monoidal structures**:

$(Kl(\mathsf{D}_{\leq}), \otimes, \{\bullet\})$

$X \otimes Y = X \times Y$

$f \otimes g(y, y' \mid x, x') \stackrel{\text{def}}{=} f(y \mid x) \cdot g(y' \mid x')$

Cocommutative comonoids

$$: X \to X \times X, \quad x \mapsto \delta_{(x,x)} \qquad : X \to 1, \quad x \mapsto \delta_{\bullet}$$

$(Kl(\mathsf{D}_{\leq}), \oplus, \{\})$

$X \oplus Y = X + Y$

$$f \oplus g(v \mid u) \stackrel{\text{def}}{=} \begin{cases} f(y \mid x) & \text{if } u = (x,0) \text{ and } v = (y,0) \\ g(y' \mid x') & \text{if } u = (x',1) \text{ and } v = (y',1) \\ 0 & \text{otherwise} \end{cases}$$

Natural commutative monoids

$$: X + X \to X, \quad (x,i) \mapsto \delta_x \qquad : 0 \to X$$

symmetric monoidal monad

There is additional structure given by

$(\!| +_p |\!)_X \colon X \to X + X$ mapping x into $(x,0) \mapsto p, \quad (x,1) \mapsto 1-p$

$(\!| \star |\!)_X \colon X \to 0$ mapping x into the null subdistribution

A Closer Look to $Kl(\mathsf{D}_{\leq})$

Objects are sets. **Morphisms** $f\colon X \to Y$ are functions $X \to \mathsf{D}_{\leq}(Y)$. **Composition** is $f;g(z \mid x) = \Sigma_{y \in Y} f(y \mid x) \cdot f(z \mid y)$

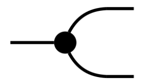
There are **two monoidal structures**:

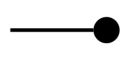
$(Kl(\mathsf{D}_{\leq}), \otimes, \{\bullet\})$

$X \otimes Y = X \times Y$

$f \otimes g(y, y' \mid x, x') \overset{\text{def}}{=} f(y \mid x) \cdot g(y' \mid x')$

Cocommutative comonoids

 $: X \to X \times X$, $x \mapsto \delta_{(x,x)}$

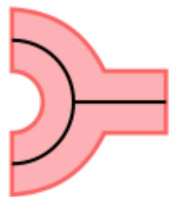
 $: X \to 1$, $x \mapsto \delta_{\bullet}$

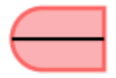
$(Kl(\mathsf{D}_{\leq}), \oplus, \{\})$

$X \oplus Y = X + Y$

$$f \oplus g(v \mid u) \overset{\text{def}}{=} \begin{cases} f(y \mid x) & \text{if } u = (x,0) \text{ and } v = (y,0) \\ g(y' \mid x') & \text{if } u = (x',1) \text{ and } v = (y',1) \\ 0 & \text{otherwise} \end{cases}$$

Natural commutative monoids

 $: X + X \to X$, $(x, i) \mapsto \delta_x$

 $: 0 \to X$

symmetric monoidal monad

There is additional structure given by

$(\!| +_p |\!)_X : X \to X + X$ mapping x into $(x,0) \mapsto p, \quad (x,1) \mapsto 1 - p$

$(\!| \star |\!)_X : X \to 0$ mapping x into the null subdistribution

presented by an algebraic theory

$(\mathbb{PCA})$

Finite Coproduct-Copy Discard Rig Categories

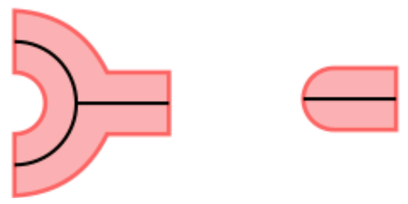
A rig category $(\mathsf{C}, \otimes, 1, \oplus, 0)$ such that

$(\mathsf{C}, \otimes, 1)$ is a copy-discard category

$(\blacktriangleleft_X, !_X)$

$(\mathsf{C}, \oplus, 0)$ is a finite coproduct category

$(\triangleright_X, \text{¡}_X)$



and the monoids and comonoids interact according to the following coherence conditions

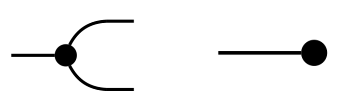
$$
\begin{tikzcd}
X \oplus Y \arrow[rr, "\blacktriangleleft_{X \oplus Y}"] \arrow[dd, "\rho_X^{-\oplus} \oplus \lambda_Y^{-\oplus}"'] & & (X \oplus Y) \otimes (X \oplus Y) \\
& & (X \otimes (X \oplus Y)) \oplus (Y \otimes (X \oplus Y)) \arrow[u, "(\delta^{-r}_{X,Y,X\oplus Y})"'] \\
(X \oplus 0) \oplus (0 \oplus Y) \arrow[rr, "(\blacktriangleleft_X \oplus \text{¡}_{X \otimes Y}) \oplus (\text{¡}_{Y \otimes X} \oplus \blacktriangleleft_Y)"'] & & ((X \otimes X) \oplus (X \otimes Y)) \oplus ((Y \otimes X) \oplus (Y \otimes Y)) \arrow[u, "(\delta^{-l}_{X,X,Y} \oplus \delta^{-l}_{Y,X,Y})"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \oplus Y \arrow[r, "!_X \oplus !_Y"] \arrow[dr, "!_{X \oplus Y}"'] & 1 \oplus 1 \arrow[d, "\triangleright_1"] \\
& 1
\end{tikzcd}
$$

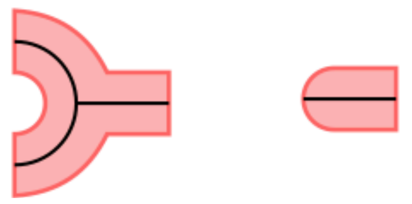
Finite Coproduct-Copy Discard Rig Categories

A rig category $(\mathsf{C}, \otimes, 1, \oplus, 0)$ such that

$(\mathsf{C}, \otimes, 1)$ is a copy-discard category

$(\blacktriangleleft_X, !_X)$ 

$(\mathsf{C}, \oplus, 0)$ is a finite coproduct category

$(\triangleright_X, \text{¡}_X)$ 

and the monoids and comonoids interact according to the following coherence conditions

$$\begin{tikzcd}
X \oplus Y \arrow[r, "\blacktriangleleft_{X \oplus Y}"] \arrow[dd, "\rho_X^{-\oplus} \oplus \lambda_Y^{-\oplus}"'] & (X \oplus Y) \otimes (X \oplus Y) \\
& (X \otimes (X \oplus Y)) \oplus (Y \otimes (X \oplus Y)) \arrow[u, "(\delta^{-r}_{X,Y,X \oplus Y})"'] \\
(X \oplus 0) \oplus (0 \oplus Y) \arrow[r, "(\blacktriangleleft_X \oplus \text{¡}_{X \otimes Y}) \oplus (\text{¡}_{Y \otimes X} \oplus \blacktriangleleft_Y)"'] & ((X \otimes X) \oplus (X \otimes Y)) \oplus ((Y \otimes X) \oplus (Y \otimes Y)) \arrow[u, "(\delta^{-l}_{X,X,Y} \oplus \delta^{-l}_{Y,X,Y})"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \oplus Y \arrow[r, "!_X \oplus !_Y"] \arrow[dr, "!_{X \oplus Y}"'] & 1 \oplus 1 \arrow[d, "\triangleright_1"] \\
& 1
\end{tikzcd}$$

Let $(\mathsf{C}, \otimes, 1, \oplus, 0)$ be a fc-cd rig category and T a symmetric monoidal monad over $(\mathsf{C}, \otimes, 1)$. Then the Kliesli category $Kl(T)$ is a fc-cd rig category.

Adding the Algebraic Structure: $\mathbb{T}$-CD Rig Categories

Let $\mathbb{T} = (\Sigma, E)$ be an algebraic theory. A $\mathbb{T}$-cd rig category is a

fc-cd rig category C together with a functor of fc categories $i \colon \mathsf{L}_{\mathbb{T}}^{op} \to \mathsf{C}$

such that

$$\begin{tikzcd}
\mathbf{L}_{\mathbb{T}}^{\mathsf{op}} \arrow[rr, "i"] & & \mathbf{C} \\
& \aleph_0 \arrow[ul, "d"] \arrow[ur, "c"'] &
\end{tikzcd}$$

commutes.

Adding the Algebraic Structure: $\mathbb{T}$-CD Rig Categories

Let $\mathbb{T} = (\Sigma, E)$ be an algebraic theory. A $\mathbb{T}$-cd rig category is a

$\boxed{\text{fc-cd rig category } \mathsf{C}}$ together with $\boxed{\text{a functor of fc categories } i\colon \mathsf{L}_{\mathbb{T}}^{op} \to \mathsf{C}}$

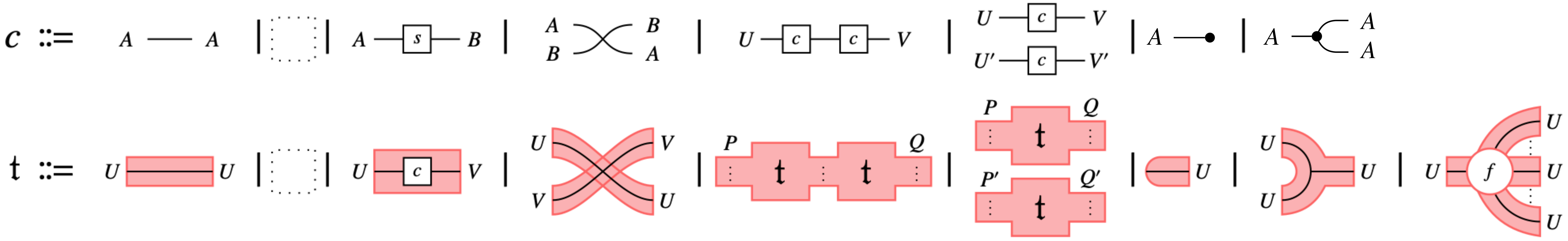
such that

$$\begin{tikzcd}
\mathbf{L}_{\mathbb{T}}^{\mathrm{op}} \arrow[rr, "i"] & & \mathbf{C} \\
& \aleph_0 \arrow[ul, "d"] \arrow[ur, "c"'] &
\end{tikzcd}$$

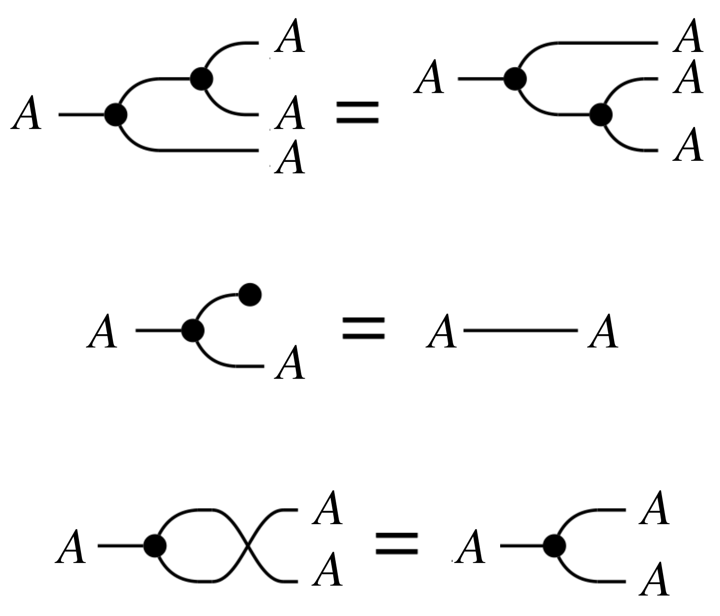
commutes.

For each $t\colon n \to 1 \in \Sigma$, there are natural transformations $(\!|t|\!)_X \stackrel{\text{def}}{=} X \xrightarrow{\lambda_X^{-\otimes}} 1 \otimes X \xrightarrow{i(t)\otimes id_X} (\bigoplus_{i=1}^{n} 1) \otimes X \xrightarrow{\delta^r_{n,X}} \bigoplus_{i=1}^{n} X$

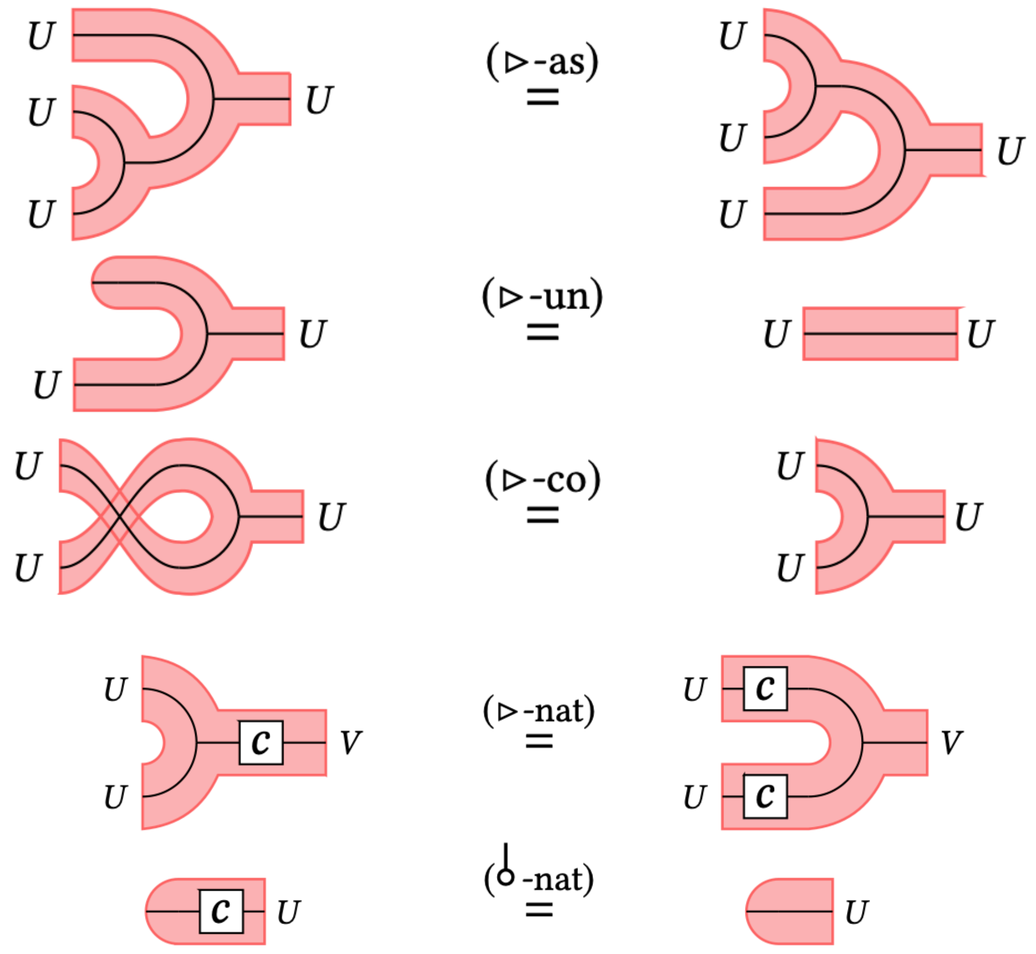
Tape Diagrams for 𝕋-CD Rig Categories



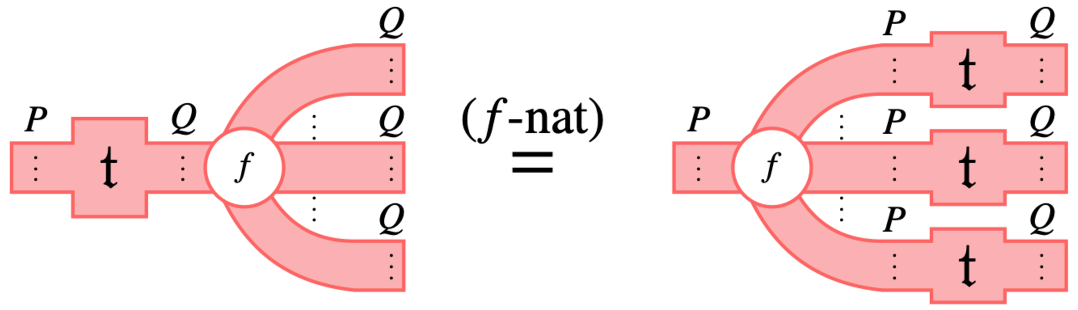
cocommutative comonoids



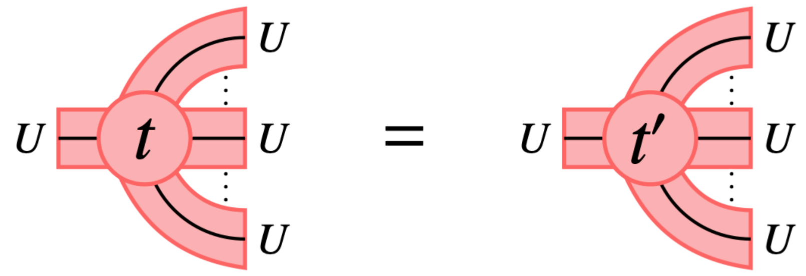
natural commutative monoids



f is a natural transformation



$(t, t') \in \mathbb{T}$



Compositional Semantics

$\mathbb{T}$-tapes are the *free sesquistrict* $\mathbb{T}$*-cd rig category* generated by a monoidal signature $(\mathcal{S}, \Gamma)$

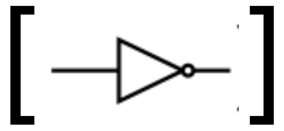
$$\begin{tikzcd}
\text{string diagrams} \arrow[r, "{[\,\cdot\,]}"] \arrow[d, "i"'] & \mathbb{T}\text{-cd rig category} \\
\text{tape diagrams} \arrow[ur, "{[\,\cdot\,]^\sharp}"'] &
\end{tikzcd}$$

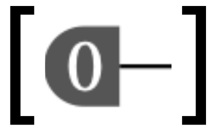
Back to $Kl(\mathsf{D}_{\leq})$

Boolean Circuits

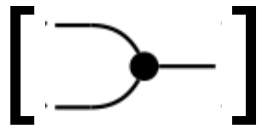
[]: $2 \times 2 \to 2$, $(x, y) \mapsto \delta_{x \wedge y}$

[]: $2 \times 2 \to 2$, $(x, y) \mapsto \delta_{x \vee y}$

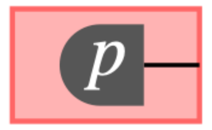
[]: $2 \to 2$, $x \mapsto \delta_{\neg x}$

[]: $1 \to 2$, $\bullet \mapsto \delta_0$

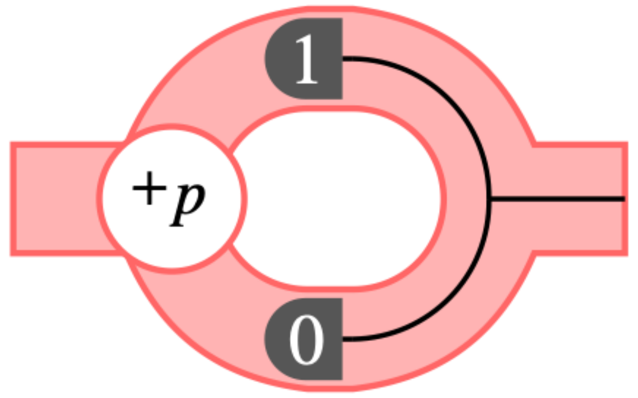
[]: $1 \to 2$, $\bullet \mapsto \delta_1$

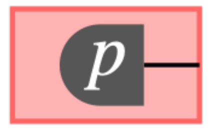
[]: $2 \times 2 \to 2$, $(x, x) \mapsto \delta_x$

Embedded in $\mathbb{PCA}$-tapes



:=



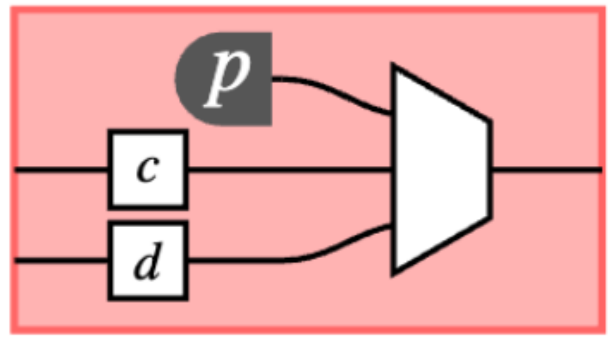
[]$^{\#}$ $: 1 \to 2$

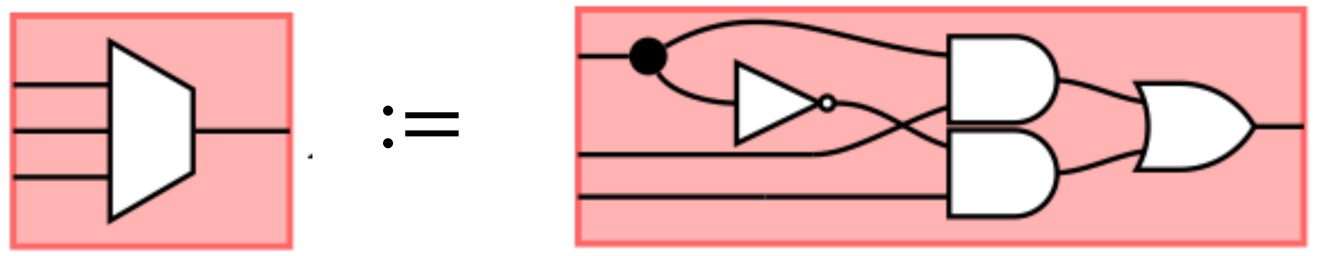
$$\bullet \mapsto \begin{cases} 1 \mapsto p \\ 0 \mapsto (1-p) \end{cases}$$

Back to $Kl(\mathsf{D}_{\leq})$

Probabilistic Control

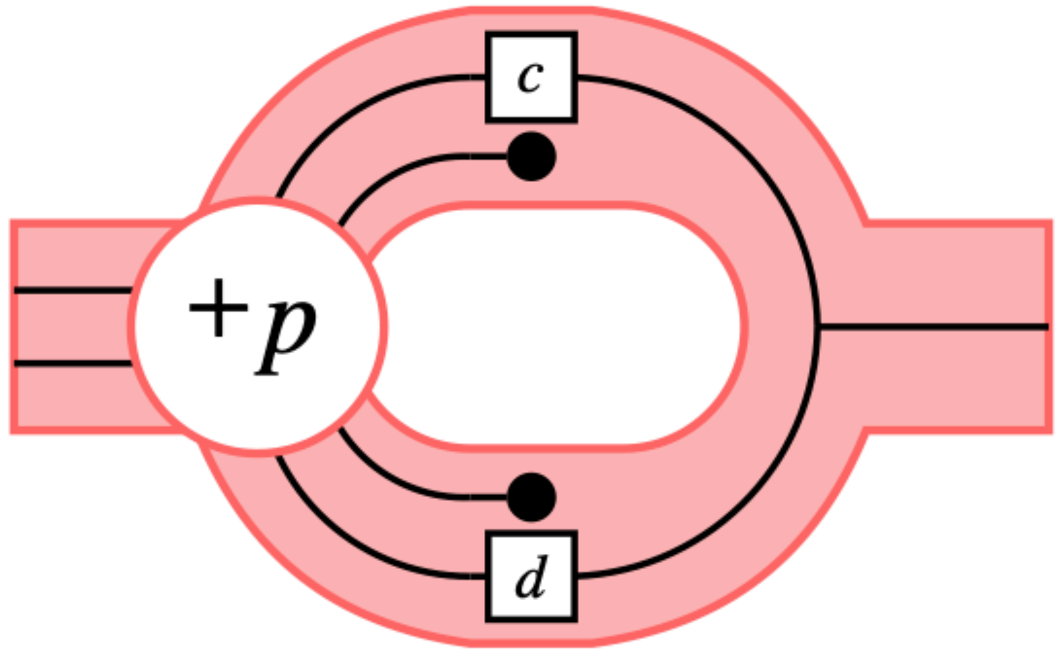
multiplexer ($\otimes$)



where 

vs.

convex sum ($\oplus$)

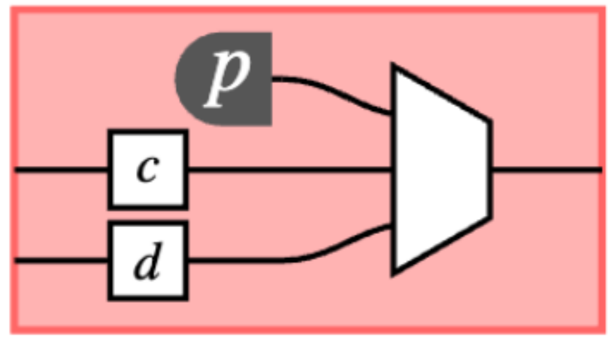


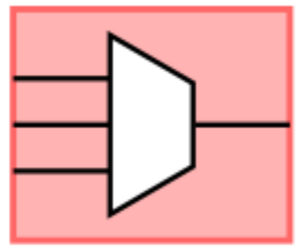
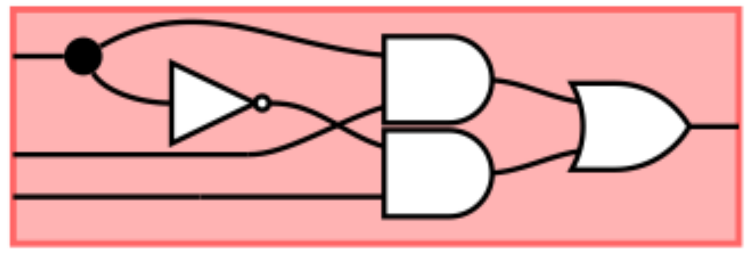
Piedeleu, Torres-Ruiz, Silva, Zanasi, 2025

Back to $Kl(\mathrm{D}_{\leq})$

Probabilistic Control

multiplexer ($\otimes$)

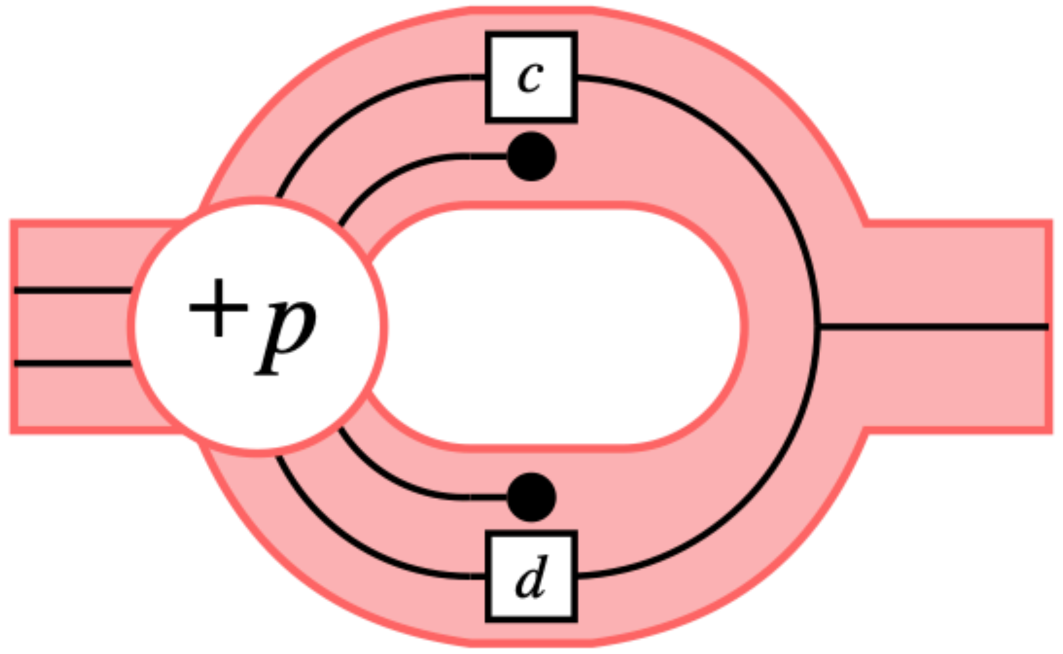


where  := 

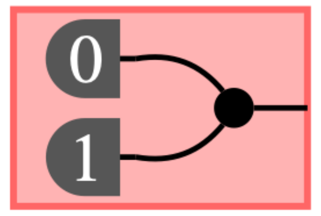
Piedeleu, Torres-Ruiz, Silva, Zanasi, 2025

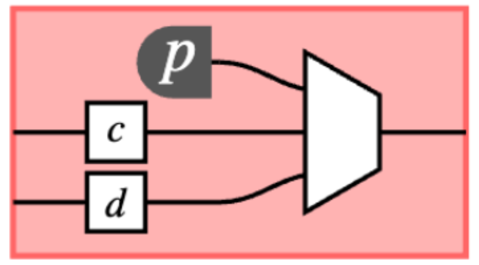
vs.

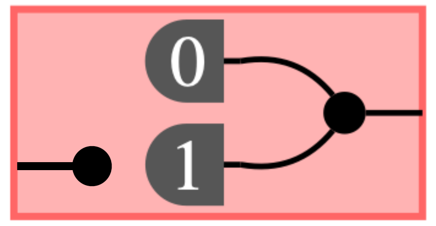
convex sum ($\oplus$)



Different behaviours w.r.t *failure*

when c (or d) = 

the multiplexer always fails 

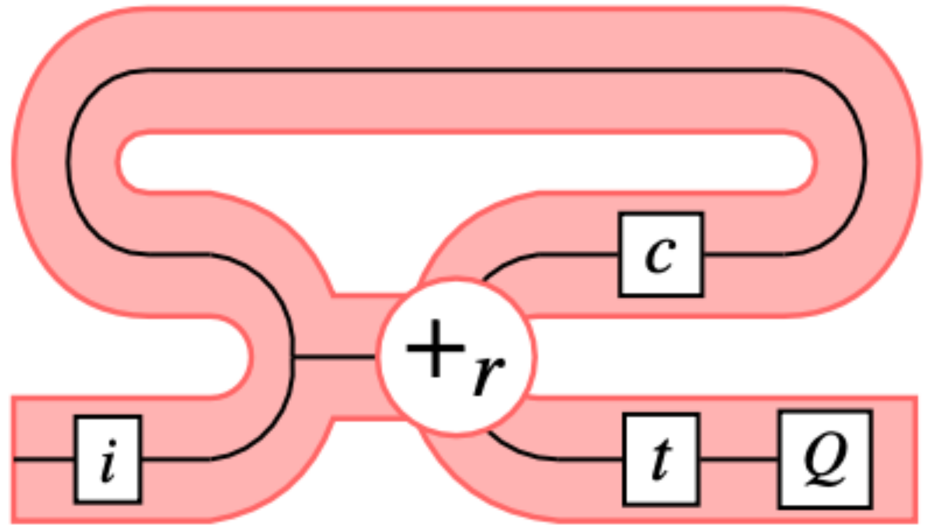
= 

Future Work

- Complete axiomatisation for $Kl(\mathsf{D}_{\leq})$?

  Possibly exploiting existing completeness theorems

- Traces? Probabilistic diagrammatic program logic?



- Other interesting examples?

Thank you!