Surface Diagrams for Grothendieck-Verdier Categories

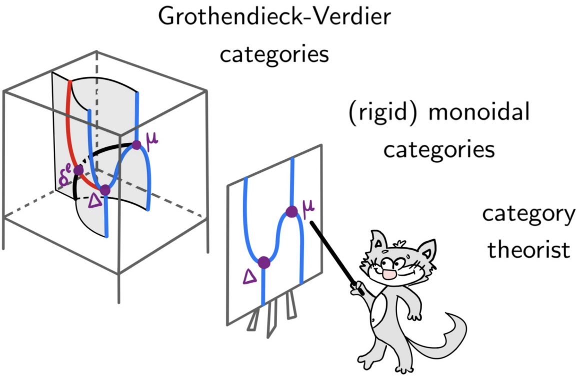
Max Demirdilek (he/him)
University of Hamburg

joint with Christoph Schweigert

arXiv:2503.13325

tinyurl.com/SurfacesSYCO

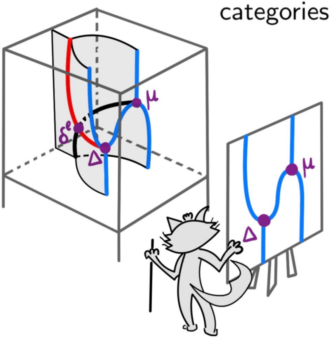
Outlook


Grothendieck-Verdier
categories
(rigid) monoidal
categories
category
theorist
μ
δ^e
Δ
μ
Δ

Grothendieck-Verdier categories

(rigid) monoidal categories

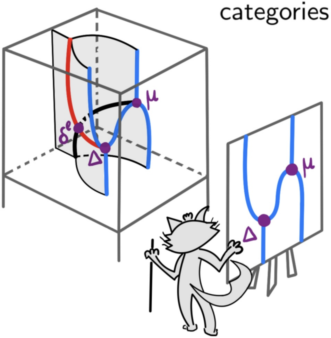
category theorist



Grothendieck-Verdier
categories

(rigid) monoidal
categories

category
theorist



‘Coherence morphisms live in a third dimension.’

Let's begin with the protagonist:
Grothendieck–Verdier categories.

A toy example

Bimodules: Monoidal structure

A	$\rightsquigarrow$	a k-algebra.
$A\text{-bimod}$	$\rightsquigarrow$	category of **A-bimodules**.

$A\text{-bimod}$ carries a **monoidal structure** $(\otimes_A, A)$:

$$M \otimes_k A \otimes_k N \rightrightarrows M \otimes_k N \to M \otimes_A N \quad (M, N \in A\text{-bimod}).$$

The tensor product $\otimes_A$ is right-exact.

Bimodules: Monoidal structure

A $\rightsquigarrow$ a finite-dimensional k-algebra.

$A\text{-bimod}_{\text{f.d.}}$ $\rightsquigarrow$ category of finite-dimensional **A-bimodules**.

$A\text{-bimod}_{\text{f.d.}}$ carries a **monoidal structure** $(\otimes_A, A)$:

$$M \otimes_k A \otimes_k N \rightrightarrows M \otimes_k N \to M \otimes_A N \quad (M, N \in A\text{-bimod}_{\text{f.d.}}).$$

The tensor product $\otimes_A$ is right-exact.

Bimodules: Duality

For $M \in A\text{-bimod}_{\text{f.d.}}$, the k-linear **dual** $DM := \mathrm{Hom}_k(M, k)$ becomes a f.d. A-bimodule via

$$(x.f.y)(m) \;:=\; f(y.m.x),$$

for $x, y \in A$ and $f \in DM$.

This yields an **antiequivalence**

$$D\colon A\text{-bimod}_{\text{f.d.}} \xrightarrow{\simeq} (A\text{-bimod}_{\text{f.d.}})^{\text{op}},$$

with quasi-inverse $D^{-1} = D$.

Bimodules: De Morgan Duality

$$D\colon A\text{-bimod}_{\mathrm{f.d.}} \xrightarrow{\simeq} (A\text{-bimod}_{\mathrm{f.d.}})^{\mathrm{op}}$$

induces a **second monoidal structure** $(\otimes^A, DA)$ on $A\text{-bimod}_{\mathsf{f.d.}}$:

$$M \otimes^A N := D(D^{-1}N \otimes_A D^{-1}M) \quad (M, N \in A\text{-bimod}_{\mathrm{f.d.}}).$$

The tensor product $\otimes^A$ is left-exact.

What is the appropriate *categorical* duality structure on $A\text{-bimod}_{\mathsf{f.d.}}$?

Rigidity?

$A\text{-bimod}_{\mathsf{f.d.}}$ is generally not rigid, since ...

$A\text{-bimod}_{\mathsf{f.d.}}$ is generally not rigid, since ...

... $\otimes_A$ is not exact ...

A second attempt ...

Grothendieck-Verdier categories

Duality beyond rigidity

Definition.

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. $K \in \mathcal{C}$ is *dualizing* if

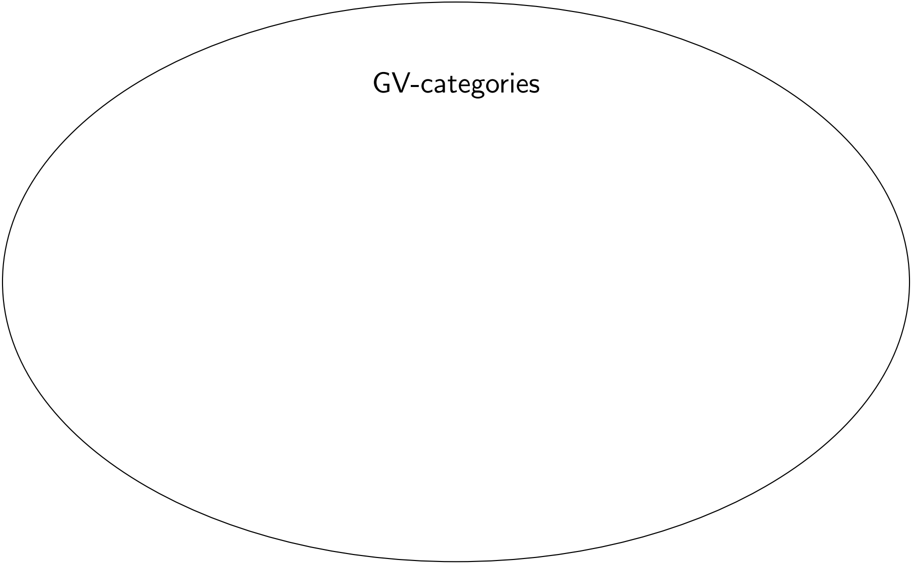
1. $\forall$ Y$\in \mathcal{C}$, the functor

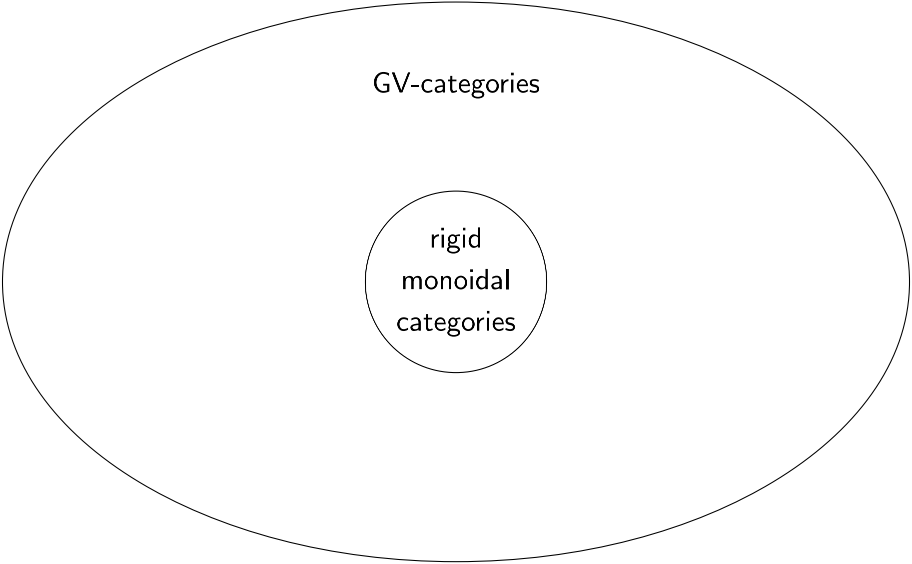
$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K)$$

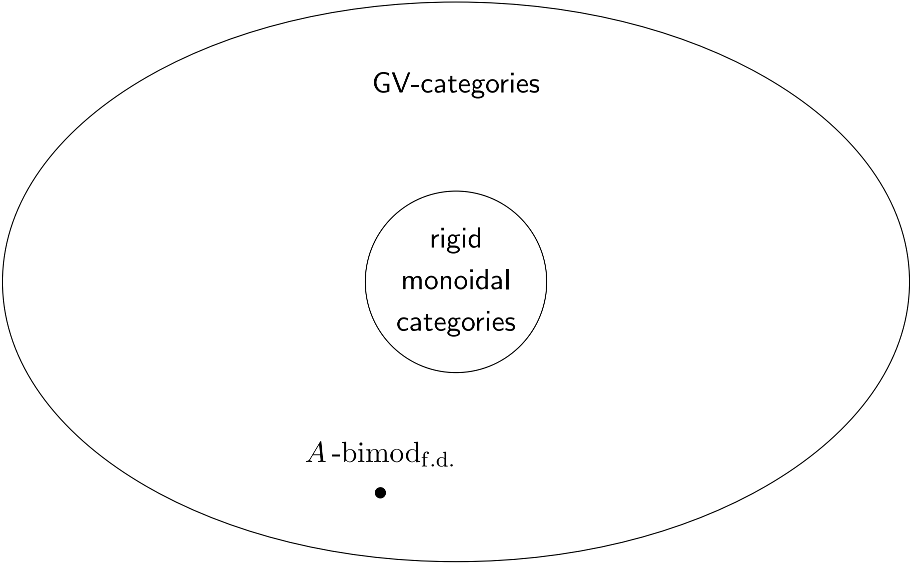
   is representable, i.e.
   $\exists$ DY$\in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(- \otimes Y, K) \cong \mathrm{Hom}_{\mathcal{C}}(-, DY)$.

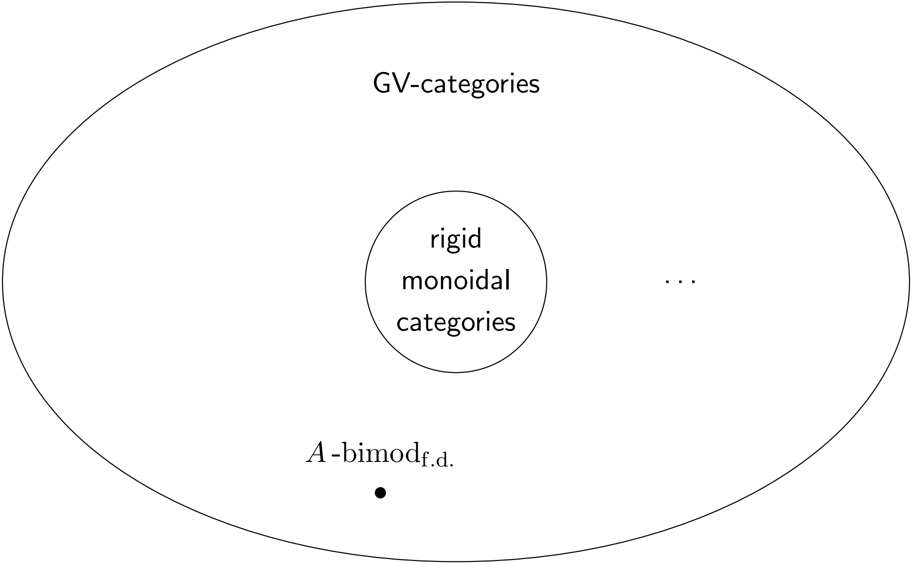
2. The contravariant functor D from 1. is an antiequivalence.

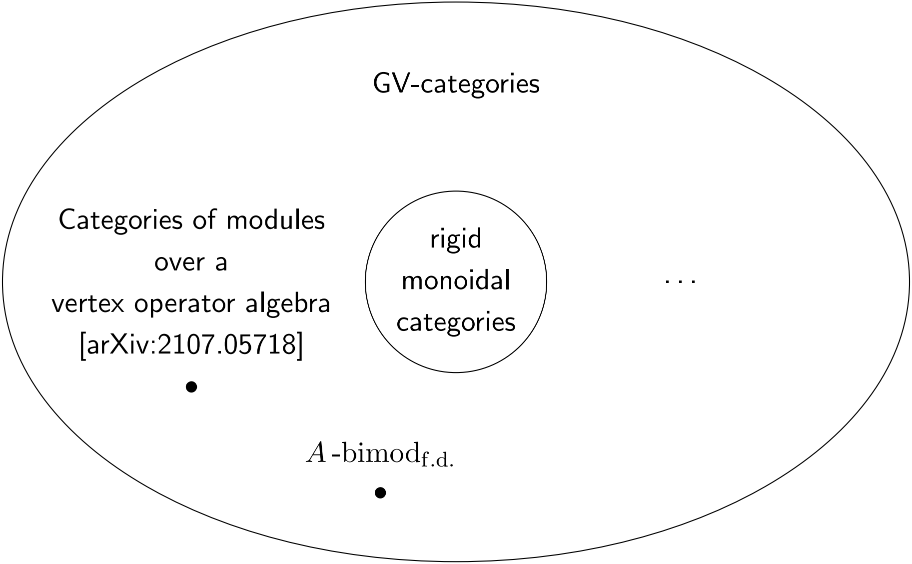
$(\mathcal{C}, K)$ is called a *Grothendieck-Verdier (GV)* category.
Also known as $*$-*autonomous categories*.

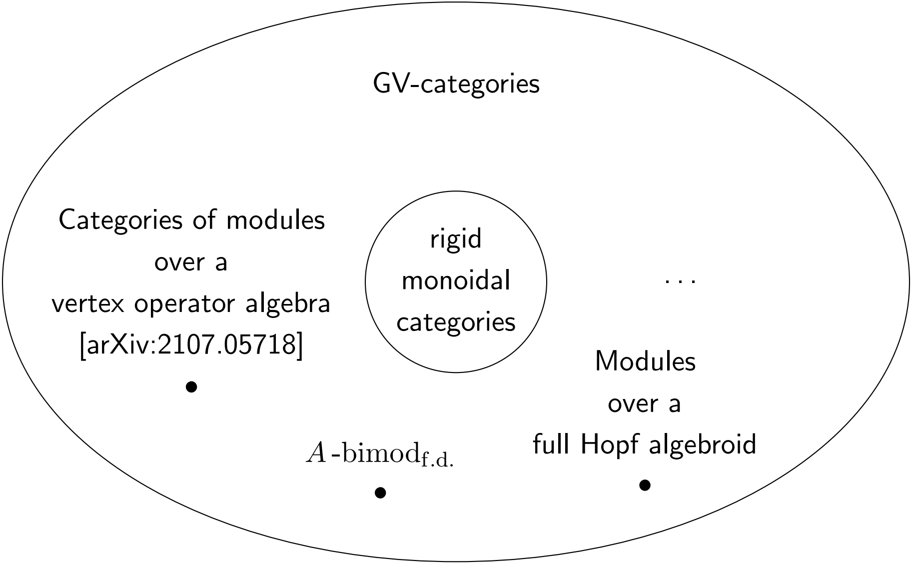

GV-categories


GV-categories
rigid
monoidal
categories


GV-categories
rigid
monoidal
categories
A-bimod$_{\text{f.d.}}$


GV-categories
rigid monoidal categories
· · ·
A-bimod$_{\text{f.d.}}$


GV-categories
Categories of modules
over a
vertex operator algebra
[arXiv:2107.05718]
rigid
monoidal
categories
· · ·
A-bimod$_{\mathrm{f.d.}}$


GV-categories
Categories of modules over a vertex operator algebra [arXiv:2107.05718]
rigid monoidal categories
· · ·
Modules over a full Hopf algebroid
A-bimod$_{\mathrm{f.d.}}$

Recall that

$A\text{-bimod}_{\mathrm{f.d.}}$ carries a second monoidal structure $(\otimes^A, DA)$.

Linearly distributive categories

Definition (Cockett-Seely '97).
A *linearly distributive (LD)* category is a category $\mathcal{C}$ with:

- Two monoidal structures $(\otimes, 1)$ and $(\parr, K)$ on $\mathcal{C}$.
- Two natural transformations

$$\delta^l \colon \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \parr) \Longrightarrow \parr \circ (\otimes \times \mathrm{id}_{\mathcal{C}}),$$

$$\delta^r \colon \otimes \circ (\parr \times \mathrm{id}_{\mathcal{C}}) \Longrightarrow \parr \circ (\mathrm{id}_{\mathcal{C}} \times \otimes),$$

  satisfying coherence axioms.

  The distributors are not required to be invertible.

Compatibility with ...

unitors:

$$\begin{array}{ccc}
1 \otimes (A \parr B) & \xrightarrow{\delta^l_{1,A,B}} & (1 \otimes A) \parr B \\
 & \underset{l^{\otimes}_{A \parr B}}{\overset{\sim}{\searrow}} & \wr \big\downarrow l^{\otimes}_A \parr B \\
 & & A \parr B
\end{array}$$

$+$ 3 more triangle diagrams

Compatibility with ...

associators:

$$
\begin{array}{ccc}
(A \otimes B) \otimes (C \parr D) & \xrightarrow[\sim]{\alpha^{\otimes}_{A,B,C \parr D}} & A \otimes (B \otimes (C \parr D)) \\
 & & \Big\downarrow {\scriptstyle A \otimes \delta^{l}_{B,C,D}} \\
\delta^{l}_{A \otimes B,C,D} \Big\downarrow & & A \otimes ((B \otimes C) \parr D) \\
 & & \Big\downarrow {\scriptstyle \delta^{l}_{A,B \otimes C,D}} \\
((A \otimes B) \otimes C) \parr D & \xrightarrow[\alpha^{\otimes}_{A,B,C} \parr D]{\sim} & (A \otimes (B \otimes C)) \parr D
\end{array}
$$

$+$ 5 more pentagon diagrams

A correspondence

$$\left\{ \text{GV-categories} \right\} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \text{LD-category} \right\}$$

(Cockett-Seely '97)

A correspondence

$$\left\{\ \text{GV-categories}\ \right\} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{\ \begin{array}{c}\text{LD-category}\\ \text{with negation}\end{array}\ \right\}$$

(Cockett-Seely '97)

Problem

A lot of coherence axioms!

$\rightsquigarrow$ Difficult and tedious calculations.

A no-go theorem

LD-categories are generally not strictifiable.

Theorem (Cockett-Seely '97).
Let $\mathcal{C}$ be an LD-category with invertible distributors.
Then $\mathcal{C}$ is suitably equivalent to a *shift monoidal category*,
i.e. an LD-category $(\mathcal{D}, \otimes, 1, \parr, K)$, where

$$X \parr Y = Y \otimes (S \otimes X),$$

for an $\otimes$-invertible object $S \in \mathcal{D}$.

Another no-go

LD-categories are generally not coherent.

Consider, for instance, the $\mathbb{C}$-algebra

$$A := \mathbb{C}[x, y]/\langle x^2, y^2, xy\rangle.$$

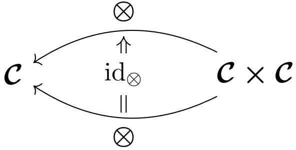
There exists a formal diagram in the LD-category $A\text{-bimod}_{\mathrm{f.d.}}$ that does not commute.

Surface diagrams

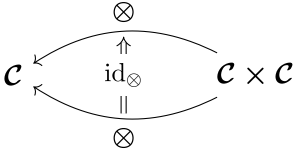
Fix an LD-category $\mathcal{C}$.

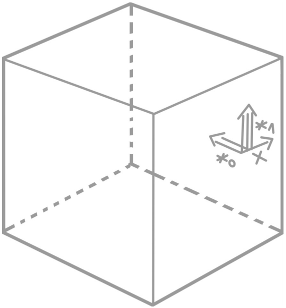
By definition, $\mathcal{C}$ lives internal to the monoidal 2-category Cat.
Monoidal 2-categories admit a three-dimensional graphical calculus.

Monoidal tuning fork

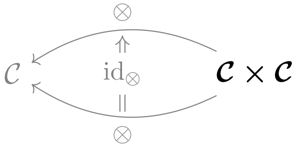


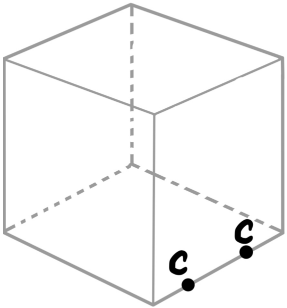
Monoidal tuning fork



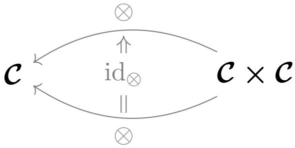


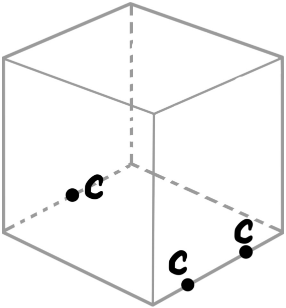
Monoidal tuning fork



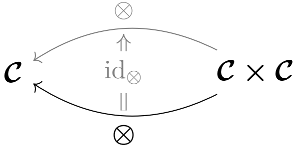


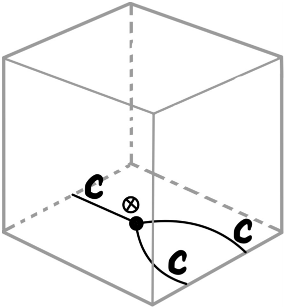
Monoidal tuning fork



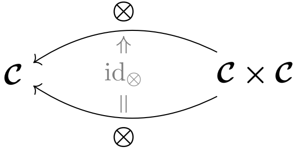


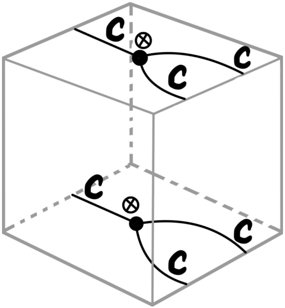
Monoidal tuning fork



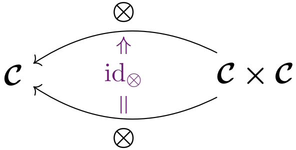


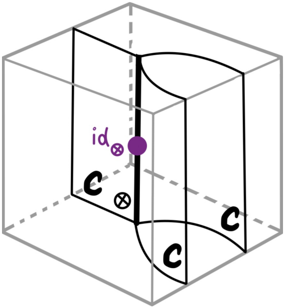
Monoidal tuning fork



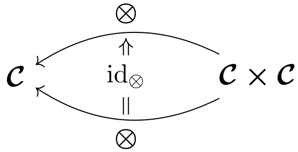


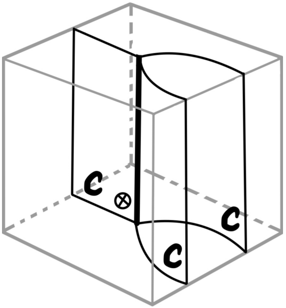
Monoidal tuning fork



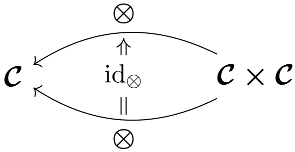


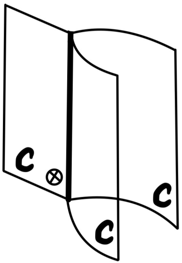
Monoidal tuning fork



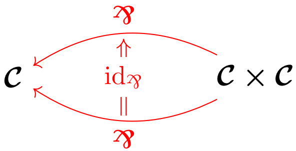


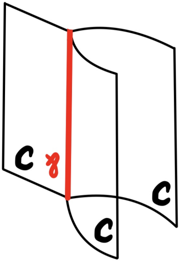
Monoidal tuning fork





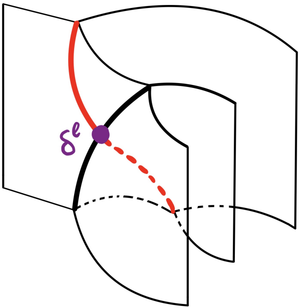
Monoidal tuning fork





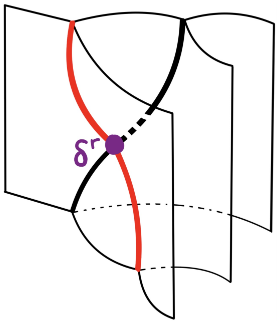
Left distributor

$$\otimes \circ (\mathrm{id}_{\mathcal{C}} \times \parr) \xRightarrow{\delta^l} \parr \circ (\otimes \times \mathrm{id}_{\mathcal{C}})$$

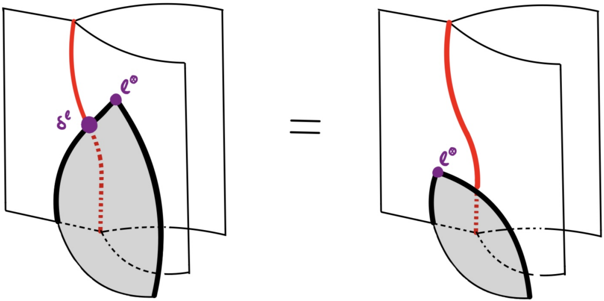


Right distributor

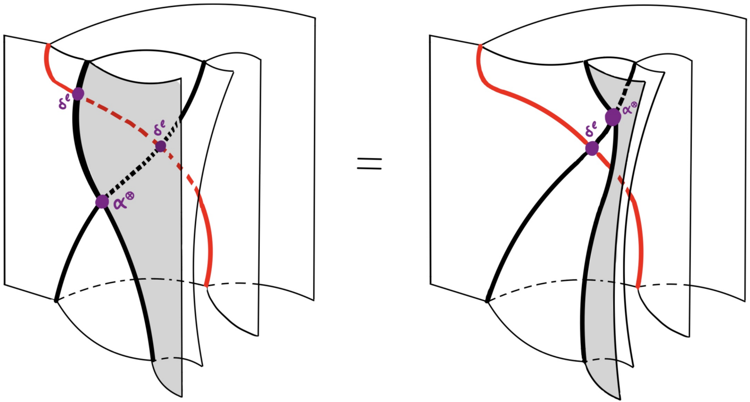
$$\otimes \circ (\parr \times \mathrm{id}_{\mathcal{C}}) \overset{\delta^r}{\Longrightarrow} \parr \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$$



A triangle diagram



A pentagon diagram



Frobenius algebras

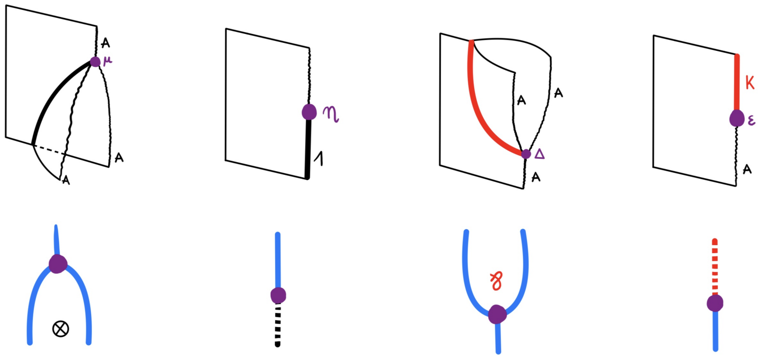
Definition.
An *LD-Frobenius algebra* in $\mathcal{C}$ consists of

- a unital associative algebra (A, μ, η) in $(\mathcal{C}, \otimes, 1)$,
- a counital coassociative coalgebra (A, Δ, ϵ) in $(\mathcal{C}, \parr, K)$,

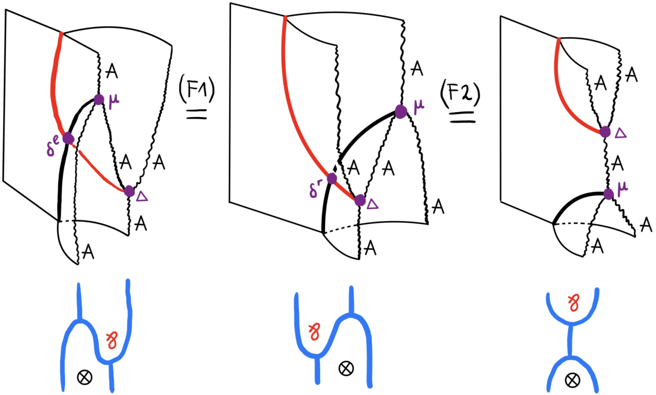
such that the following two LD-Frobenius relations hold

$$(\mu \parr A) \circ \delta^l_{A,A,A} \circ (A \otimes \Delta) \overset{\text{(F1)}}{=} (A \parr \mu) \circ \delta^r_{A,A,A} \circ (\Delta \otimes A) \overset{\text{(F2)}}{=} \Delta \circ \mu.$$

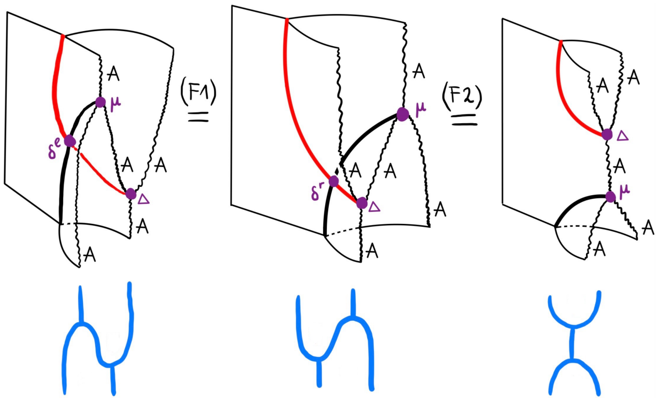
(Co)algebras



Frobenius relations



Frobenius relations



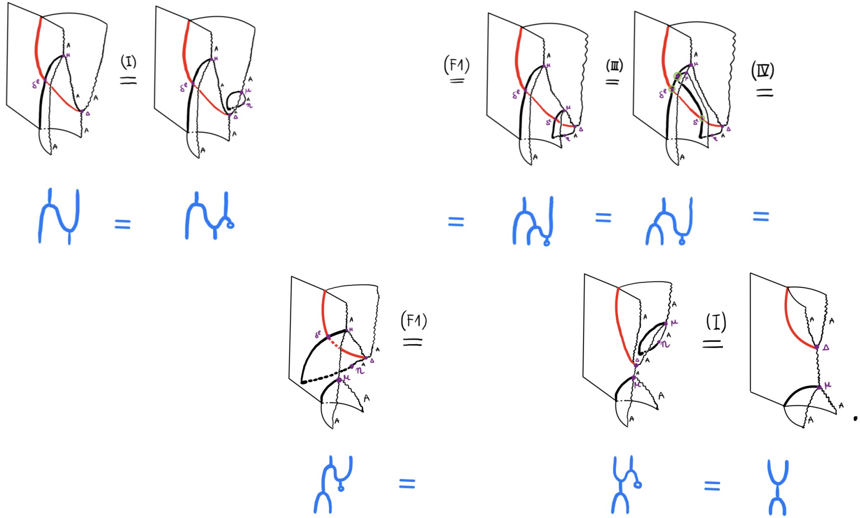
Proposition (D, Schweigert '24).

Let (A, μ, η) be a unital associative algebra in $(\mathcal{C}, \otimes, 1)$.
Let (A, Δ, ϵ) be a counital coassociative coalgebra in $(\mathcal{C}, ⅋, K)$.

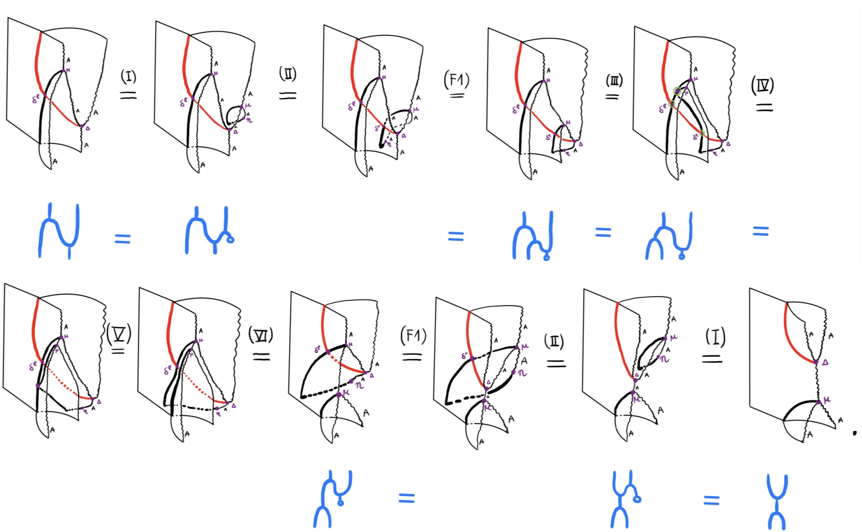
If $(A, \mu, \eta, \Delta, \epsilon)$ satisfies Frobenius relation (F1),
then it also satisfies Frobenius relation (F2).

Proof

Proof



Proof



Frobenius forms

Proposition (D, Schweigert '24).
Let $\mathcal{C}$ be an abelian LD-category with negation. An LD-Frobenius algebra in $\mathcal{C}$ amounts to an algebra (A, μ, η) in $(\mathcal{C}, \otimes, 1)$ with a morphism $\lambda \in \mathrm{Hom}_{\mathcal{C}}(A, K)$ whose kernel contains non non-zero left ideals of A.

(Co)modules and more

Proposition (D, Schweigert '24).
Let $(A, \mu, \eta, \Delta, \epsilon)$ be an LD-Frobenius algebra. The category of left A-modules is isomorphic to the category of left A-comodules.

Proposition.
Morphisms of LD-Frobenius algebras are invertible.

Proposition.
Frobenius LD-functors preserve LD-Frobenius algebras.

Proposition.
LD-Frobenius algebras are self-dual.

...

Higher Frobenius-Schur indicators

Ordinary *Higher Frobenius-Schur indicators*

are

invariants of k-linear pivotal rigid monoidal categories.

Generalize to:

Higher Frobenius-Schur indicators

are

invariants of k-linear pivotal GV-categories.

Pivotality

Definition.
A *pivotal structure* on a rigid monoidal category $(\mathcal{C}, \otimes, 1)$ is an isomorphism of monoidal functors

$$\rho\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} D^2,$$

where D is the duality functor on $\mathcal{C}$.

Pivotality

Definition.

A *pivotal structure* on a GV-category $(\mathcal{C}, \otimes, 1, K)$ is an isomorphism of Frobenius-LD functors

$$\rho\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} D^2,$$

where D is the duality functor on $\mathcal{C}$.

Pivotality

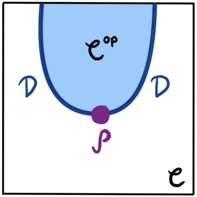
Definition.

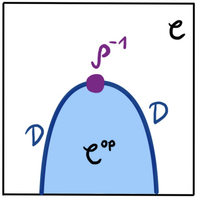
A *pivotal structure* on a GV-category $(\mathcal{C}, \otimes, 1, K)$ is an isomorphism of Frobenius-LD functors

$$\rho\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} D^2,$$

where D is the duality functor on $\mathcal{C}$.

Graphically:





.

Fix a field k.

Fix a k-linear pivotal GV-category

with finite-dimensional hom-spaces $(\mathcal{C}, \rho^{\mathcal{C}})$.

Fix $V \in \mathcal{C}$.

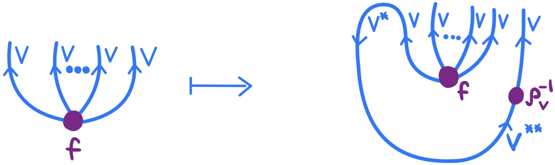
Definition

For every integer $n \geq 1$, define a k-linear endomorphism

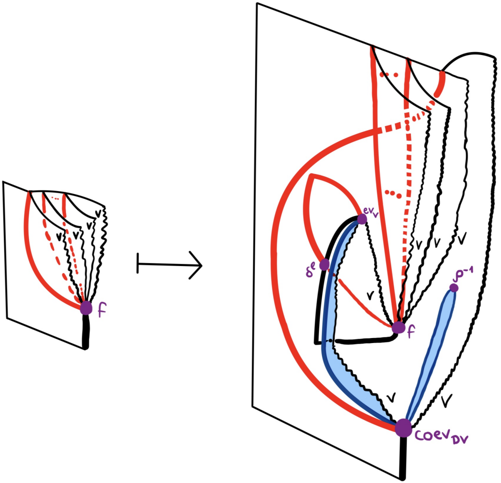
$$E_V^{(n)} \colon \operatorname{Hom}_{\mathcal{C}}(1, V^{\parr n}) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(1, V^{\parr n})$$

as follows ...

‘Definition’ of $E_V^{(n)}$



Definition of $E_V^{(n)}$



Definition of $\nu_{n,r}(V)$

Definition.

For integers $n, r \geq 1$, we call the k-linear trace

$$\nu_{n,r}(V) := \mathrm{Tr}((E_V^{(n)})^r) \in k$$

the (n, r)*-th Frobenius-Schur indicator* of $V \in \mathcal{C}$.

Properties

Theorem (D, Schweigert '24).
For all $n, r \geq 1$, the (n,r)-th Frobenius-Schur indicator $\nu_{n,r}$ is invariant under k-linear pivotal Frobenius LD-equivalence.

The theorem has a surface diagrammatic proof.

Proposition (D, Schweigert '24).
We have $(E_V^{(n)})^n = \mathrm{id}\,.$
In particular, $\nu_{n,n}(V) = 1$. Thus, for an algebraically closed field k of characteristic zero, $\nu_{n,r}(V)$ is a cyclotomic integer in $\mathbb{Q}_n \subset k$.

Future work

Compute higher Frobenius-Schur indicators for $A\text{-}\mathrm{bimod}_{\mathrm{f.d.}}$ and for representation categories of vertex operator algebras.

Thank you!

Details: arXiv:2503.13325

Contact: max.demirdilek@uni-hamburg.de

Website: maxdemirdilek.github.io