Diagrammatic (∞, n)-categories

C. Chanavat & A. Hadzihasanovic

TalTech

SYCO 13 @UCL

24 April 2025

What is an (∞,n)-category?

- We want $(\infty,0)$-categories to be precisely ∞-groupoids i.e. classical homotopy types.
- We want all categories to be $(\infty,1)$-categories and bicategories to be $(\infty,2)$-categories.
- An $(\infty,n+1)$-category should have an (∞,n)-category of cells between any two points.

What is an (∞, n)-category? (continued)

- There are by now fully established models of $(\infty, 1)$-categories, often called just "∞-categories".
- It is commonly expected that strict n-categories should be a subclass of (∞, n)-categories

I DO NOT THINK SO EXCEPT IN A WEAK SENSE

What does a model of (∞, n)-categories consist of?

1. A model of k-cells for each k.
2. A model of structural homotopies existing between (diagrams of) cells.
3. A notion of what cells are internal equivalences, s.t. all k-cells for $k > n$ are internal equivalences.

What does a model of (∞, n)-categories consist of? ctd.

4. A class of composable diagrams, i.e. "composable arrangements" of cells.

5. A notion of what it means for a cell to be a composite of a composable diagram.

6. A notion of functor of (∞, n)-categories, and a characterisation of (weak) equivalences among them.

Example: Categories as a model of $(\infty,1)$-categories

1. $\bullet$, $\bullet \longrightarrow \bullet$, no k-cells for $k>1$
2. Identity 1-cells, no other structural homotopies
3. Isomorphisms as internal equivalences
4. Chains of arrows $\bullet$, $\bullet \longrightarrow \bullet$, $\bullet \longrightarrow \bullet \longrightarrow \bullet$, $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$, ...
5. Algebraic operation of composition
6. Functors are functors, equivalences are equivalences

A rough classification of models

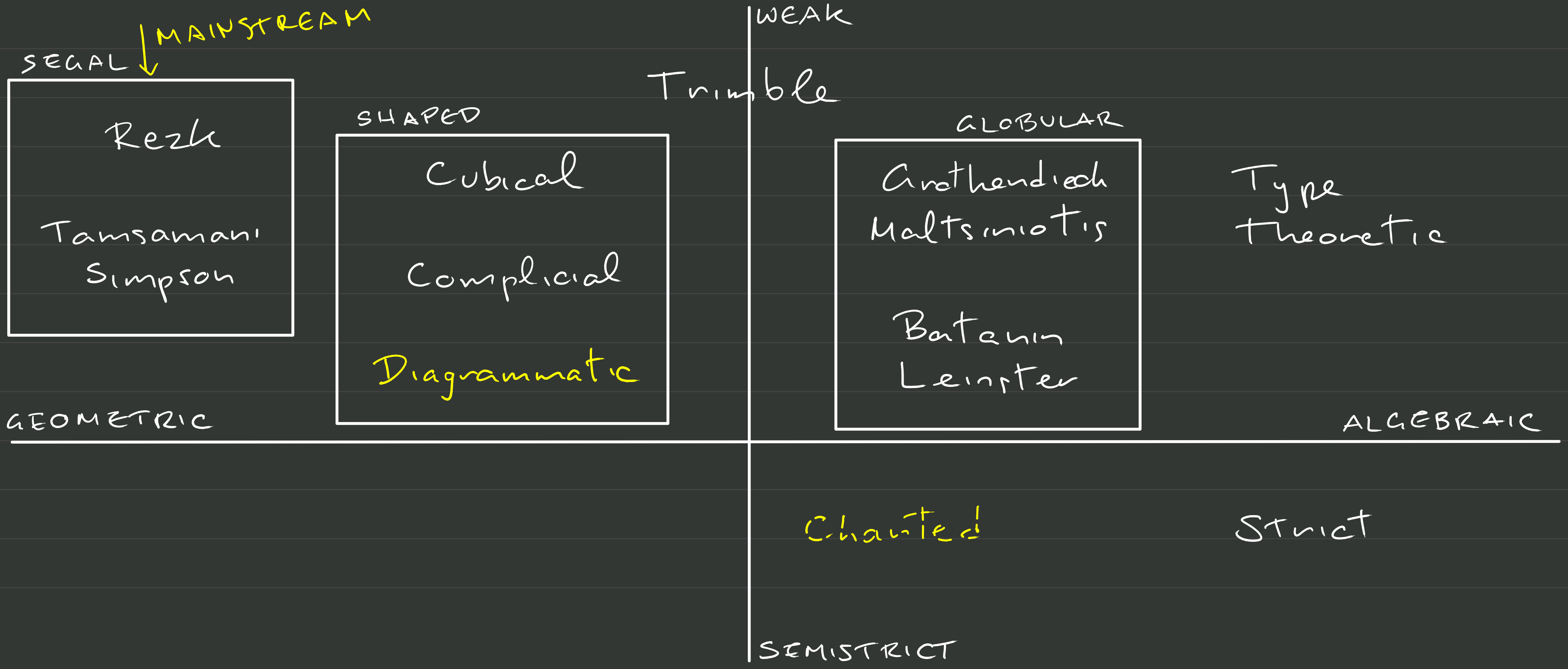
- Are structural homotopies, resp. composites determined explicitly by operations (rather than asserting their mere existence?)

  YES : ALGEBRAIC NO : GEOMETRIC

- Does composition satisfy nontrivial equations, strictly?

  YES : SEMISTRICT NO : WEAK

The current landscape

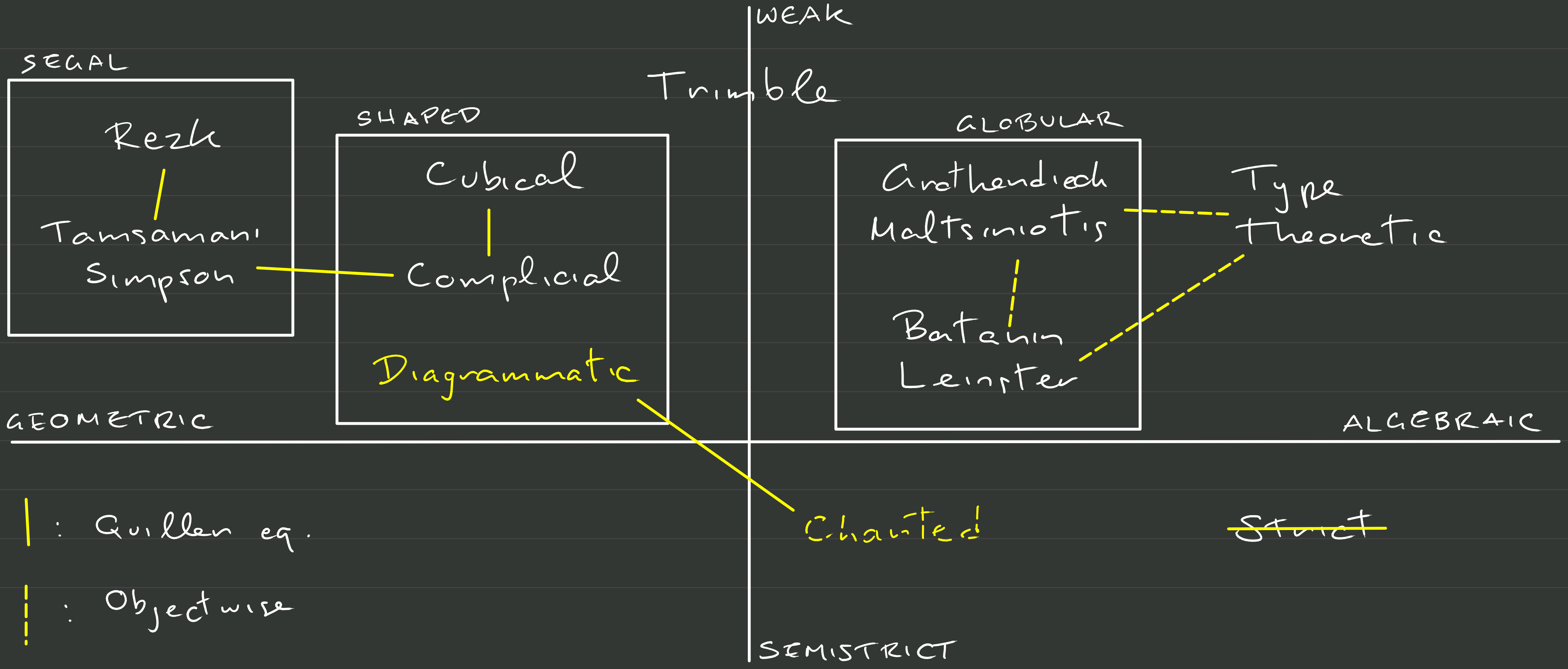


When are two models equivalent?

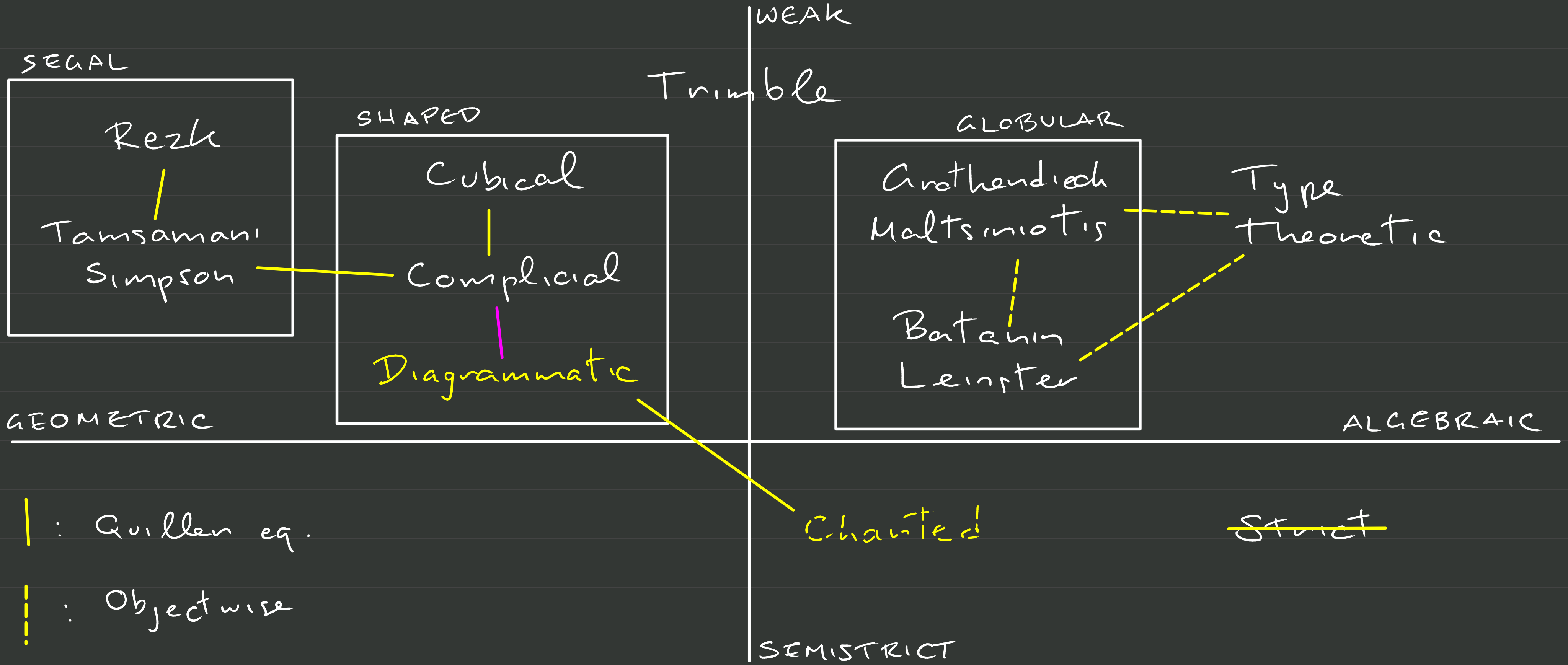
- The current gold standard: define model structures whose bifibrant objects are (∞, n)-categories & w. eq. between bifibrants are the w. eqs of (∞, n)-categories, then establish a Quillen equivalence

(THIS DOESN'T GUARANTEE THAT, E.G., WE HAVE THE SAME "LAX TRANSFORMATIONS", BUT IT'S THE BEST WE HAVE FOR NOW)

The current landscape



The current landscape FOR $n=\omega$



How does it feel to do (∞, n)-category theory?

With "mainstream" models: more like model category theory than, say, 2-category theory!

- Models are not "self-contained" — basic constructions take you outside the model, and you are always navigating some network of Quillen functors ...

The mainstream answer: Just work synthetically!

I.e. "univalent"-style —— never look inside a model, just find the right equivalence-invariant language...

- Who this leaves stranded: us SYCO-folks (& representation theorists, low-dim topologists...) who need explicit diagrammatic presentations!

Towards a self-contained model

- The diagrammatic model is closed under the most important constructions (higher functor categories, slices ...)

- It has an in-built, powerful diagrammatic language, such that e.g. diagrammatic arguments in 2-categories easily adapt to $(\infty, 2)$-categories

① The model of cells

- Is based on the combinatorial theory of atoms, molecules & regular directed complexes (cfr. my book, Combinatorics of higher-categorical diagrams)

- Cells can be pasted together globularly to form pasting diagrams; pasting satisfies strict associativity & interchange

② Structural homotopies

- They are specified algebraically by the category $\odot$ (atom) of atoms & cartesian maps

- The underlying structure of an (∞, n)-category is a diagrammatic set, a presheaf on $\odot$

③ Internal equivalences

- Cells in a diagrammatic set are an instance of a more general notion of round diagram.

- One can instantiate the notion of coinductive weak invertibility (cfr. Cheng, Rice, ...) at the level of round diagrams — these are the equivalences.

(C.C., A.A., Equivalences in diagrammatic sets, 2024)

④ Composable diagrams

- They are precisely the round diagrams.

- Every globular pasting of cells can be "padded" with structural homotopies (in a simple, systematic way) to turn it into a round diagram.

⑤ Composition

- An (∞,∞)-category is precisely a diagrammatic set X s.t.,

  $\forall$ round diagram u in X

  $\exists$ a cell $\langle u \rangle$ in X, and

  WEAK COMPOSITE

  an equivalence $c_u : u \Rightarrow \langle u \rangle$

  COMPOSITOR

"EVERY ROUND DIAGRAM IS EQUIVALENT TO A SINGLE CELL"

⑥ Functors & weak equivalences

- A functor is just a morphism of the underlying presheaves.

- It is an ω-equivalence precisely when it is "essentially surjective" on cells of every dimension.

Main results

(C.C., A.H., Model structures for diagrammatic (∞, n)-categories)

An (∞,∞)-category is an (∞,n)-category when all k-cells for $k>n$ are equivalences.

Thm There is a model structure on $\odot\underline{Set}$ whose

1. (bi) fibrants are the (∞,n)-categories,
2. w.eqs. between fibrants are the ω-equivalences.

Main results (continued)

(C.C., A.H., Model structures for diagrammatic (∞, n)-categories)

HOMOTOPY HYPOTHESIS

Thm) There is a triangle

$$\begin{tikzcd} & \odot\underline{Set} \arrow[dl, shift left=1.5] \arrow[dr, shift left=1.5] & \\ s\underline{Set} \arrow[ur, shift left=1.5] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left=1.5] \arrow[rr, phantom, "\perp"] & & s\underline{Set} \arrow[ul, shift left=1.5] \arrow[ul, phantom, "\perp"] \arrow[ll, shift left=1.5] \end{tikzcd}$$

of Quillen equivalences between

a) the classical model structure on $s\underline{Set}$,

b) the $(\infty, 0)$-model structure on $\odot\underline{Set}$.

THANKS FOR LISTENING!

www.ioc.ee/~amar