Naturality for higher-dimensional path types

SYCO 13, London

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University of Cambridge

Functoriality and naturality in 2-categories

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- These pieces of data also exist in bicategories.

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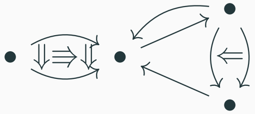
  - 2 other actions on morphisms

Weak ω-categories

What are ω-categories

- A globular set



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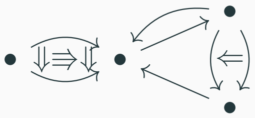

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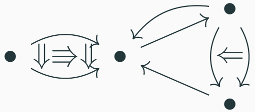

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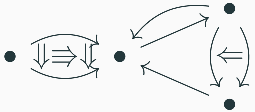

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- Weakly !
  Up to a higher cell : witnessing an equivalence

A quick overview of the panorama

catt
Finster, Mimram
dependent type theory

computads
description by DFMRV [1]
inductively defined

1. Dean, Finster, Markakis, Reutter, Vicary

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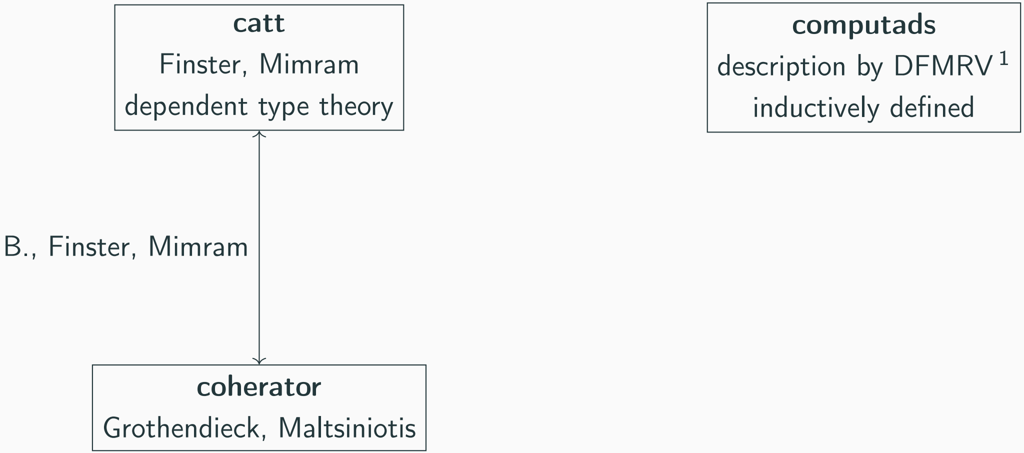

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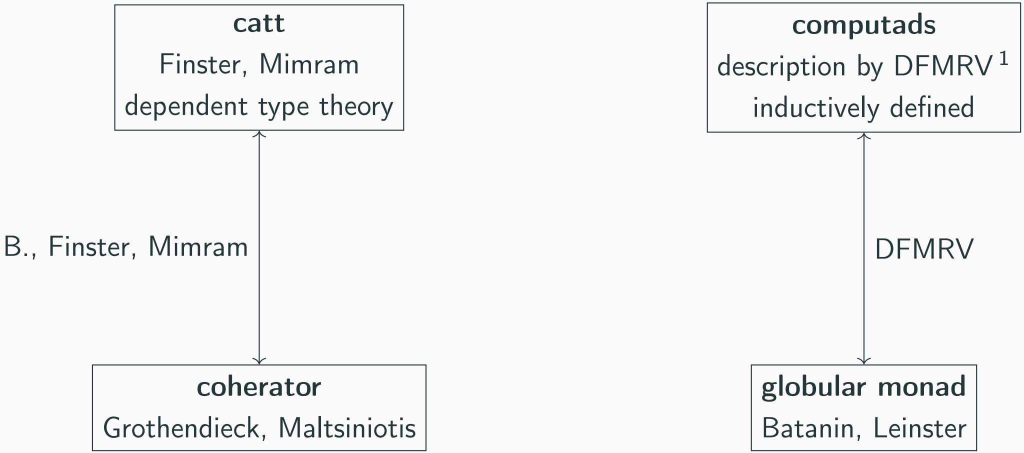

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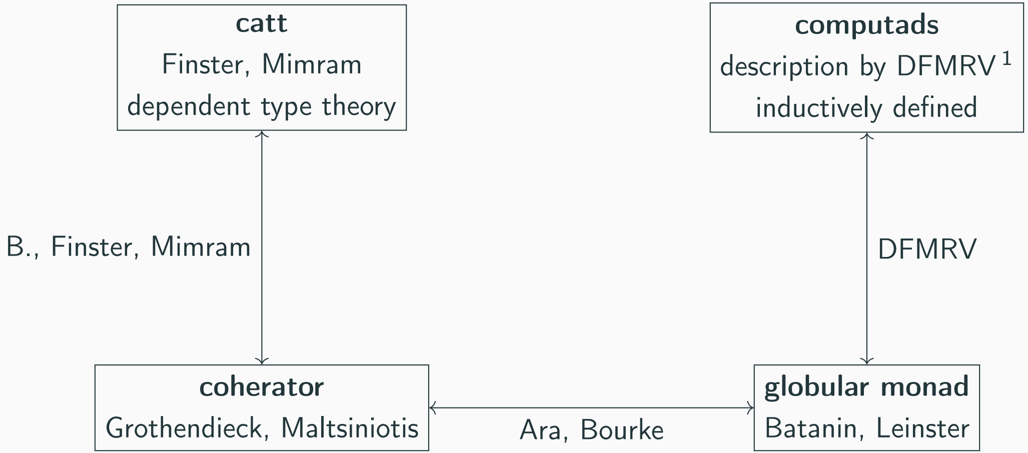

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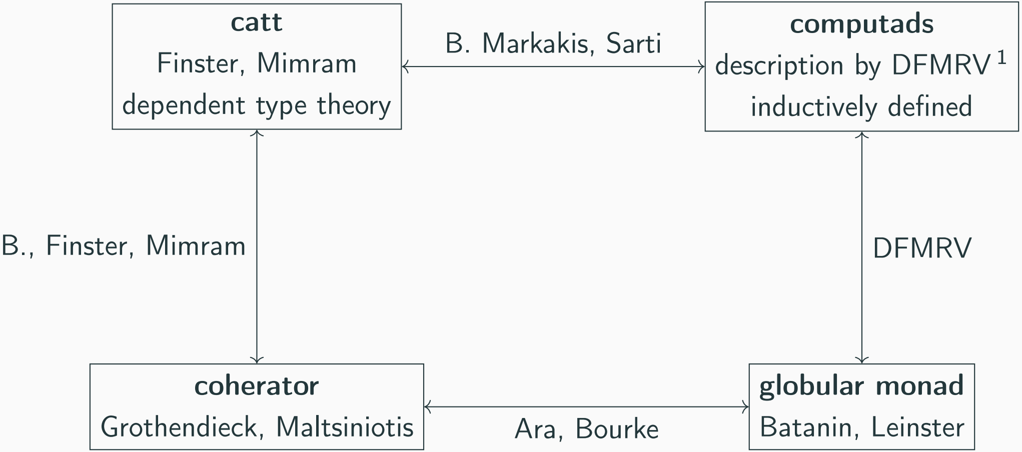

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A semi formal definition of weak ω-categories

This is a simplified presentation combining the different approaches

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  - Weakly the same : a cell $D^\bullet \to X$ whose source is one of the composite and whose target is the other.

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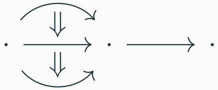

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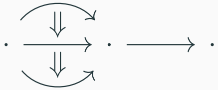

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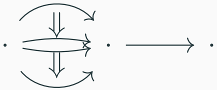

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- Plenty of descriptions : Globular sums, Batanin trees, Dyck words,. . .

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- Any two parallel cells u and v of dimension k that compose P should be related by a higher-dimensional cell
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Functoriality in ω-categories

What does functoriality means for weak ω-categories ?

- Recall – for 2-categories :

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \longmapsto \qquad A \xrightarrow{f *_0 g} C$$

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- Same holds in ω-categories :

$$c = \mathrm{coh}(\; A \xrightarrow{f} B \xrightarrow{g} C \;, A \to C)$$

$$\mathrm{coh}(\; A \;\overset{f}{\underset{f'}{\Downarrow_{\alpha}}}\; B \xrightarrow{g} C \;, c(f, g) \to c(f', g))$$

General picture – Functorialisation of pasting schemes

- Choose a pasting scheme P and a set X of maximal dimensional positions

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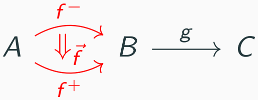

- $P \uparrow X$ is a pasting scheme, with 2 inclusions :

$$\mathrm{in}^-, \mathrm{in}^+ : P \to P \uparrow X$$

General picture – Functorialisation of compositions

- Choose a composition $c' = \mathrm{comp}(P, u \to v)$ and a set X of maximal dimensional positions in P

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- This extends to arbitrary terms in the theory of ω-categories.

Naturality in ω-categories

Why not coherences ?

- Consider a coherence :

$$a = \mathrm{coh}(\ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D\ , c(f, c(g, h)) \to c(c(f, g), h))$$

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- Try to apply the same recipe :

$$c(f^{-}, c(g, h)) \xrightarrow{\text{a}(\text{in}^{-})} c(c(f^{+}, g), h)$$

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- This looks a lot like the naturality square for the associator in bicategories.

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Why only maximal variables ?

- Consider the following composition :

$$w = \mathrm{comp}\Big(A \underset{f'}{\overset{f}{\Downarrow_\alpha}} B \xrightarrow{\color{red}g} C \;, c(f,g) \to c(f',g)\Big)$$

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- This gives the interchange law !

Our result

Theorem (B., Markakis, Offord, Sarti, Vicary)

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- Upwards closed : If $x \in X$, then X also contains all variables that have x in their source or target.
- Depth 1 : All variables contained in X are of dimension at least 1 less than the dimension of the term.

  This second condition makes everything square shaped.

One more unusual example

- For the composition of 1-cells

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- This theorem lets us construct a filler of the outer square out of the following data

$$\begin{array}{ccccc} A^- & \xrightarrow{f^-} & B^- & \xrightarrow{g^-} & C^- \\ \vec{A}\big\downarrow & \Downarrow_{\vec{f}} & \big\downarrow\vec{B} & \Downarrow_{\vec{g}} & \big\downarrow\vec{C} \\ A^+ & \xrightarrow[f^+]{} & B^+ & \xrightarrow[g^+]{} & C^+ \end{array} \mapsto \begin{array}{ccc} A^- & \xrightarrow{c(f^-,g^-)} & C^- \\ \vec{A}\big\downarrow & \Downarrow_{c\uparrow X} & \big\downarrow\vec{C} \\ A^+ & \xrightarrow[c(f^+,g^+)]{} & C^+ \end{array}$$

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- We may think of this square composite as the naturality of the composition with respect to all its variables.

Applications

Compositions of cylinders

- We have just illustrated that naturality produces a filler

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
{\scriptstyle h}\downarrow & \Downarrow_{\alpha} & \downarrow{\scriptstyle k} & \Downarrow_{\beta} & \downarrow{\scriptstyle l} \\
E & \xrightarrow{p} & F & \xrightarrow{q} & G
\end{array}
\;\mapsto\;
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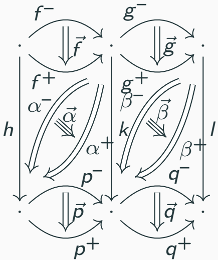
- Treat this as a blackbox and choose $X = \{f, p, \alpha, g, q, \beta\}$
- We get a filler for the cylindrical composition :



Composition of cylinders

Theorem (B., Markakis, Offord, Sarti, Vicary)

In weak ω-categories, one can construct a composition for cylinders of any dimension glued along a face of any dimension.

Implementation

- We have a proof-assistant CaTTfor working with the theory of weak ω-categories [2]

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- We have implemented automatic computation of the fillers provided by our theorem
- Construction in CaTTcan be exported as computation on identity types in homotopy type theory.
- We thus can obtain definition of the cylindrical composition of equalities.

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Thank you