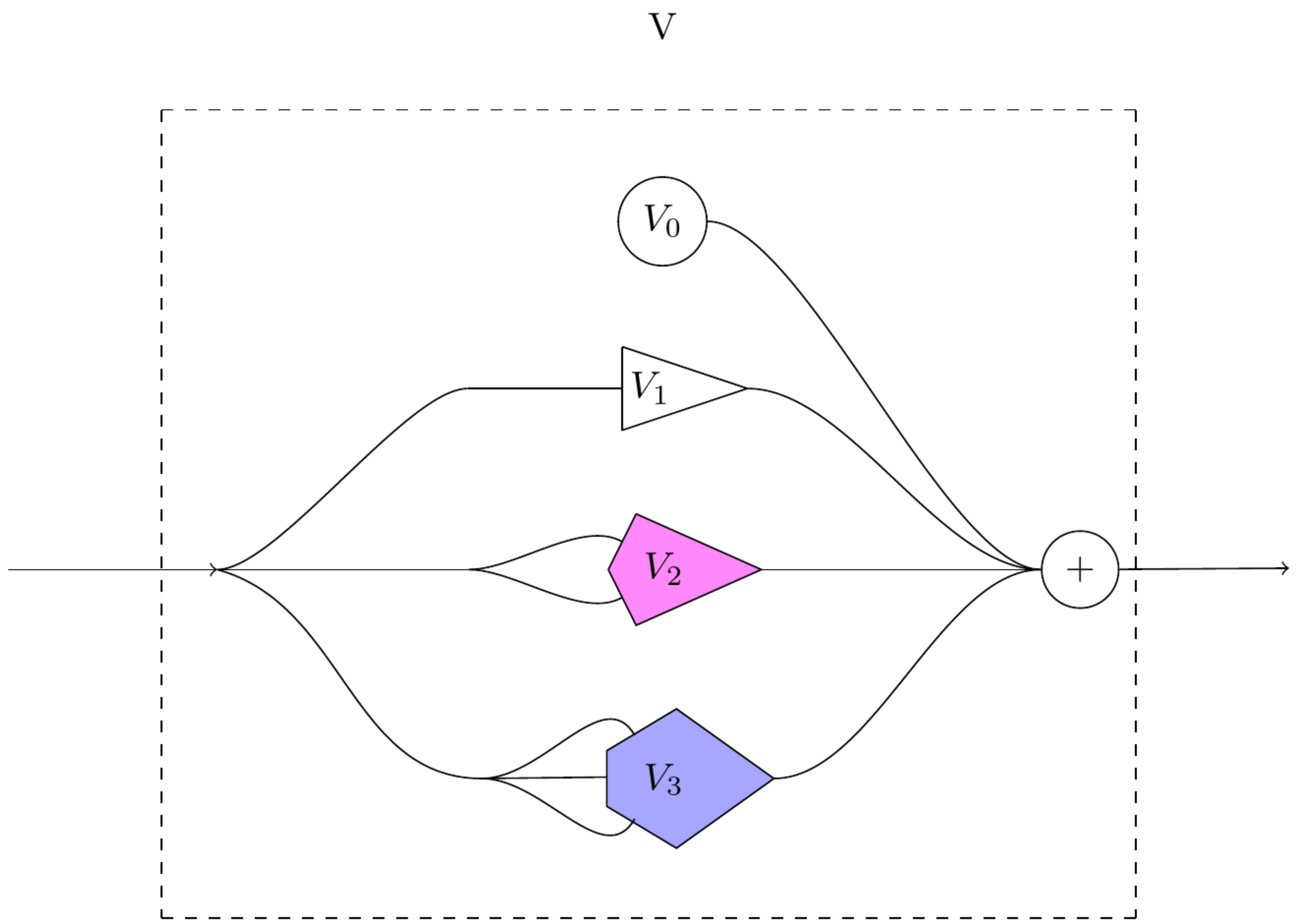
Compositional nonlinear (audio) signal processing with Volterra series



Jake Araujo-Simon

Overview of the talk

Background on Signal Processing:

1. Linear Time-Invariant (LTI) systems

2. Introduction to Volterra series

Categorification

3. The base category, $S'(\mathbb{R})$

4. Functoriality of Volterra series

5. Morphisms of Volterra series

6. The category *Volt*

7. If time: Time-Frequency Analysis

What I will not cover: nonlinear system identification; Volterra Neural Networks

Background: (Linear) Signal Processing

Linear Time-Invariant (LTI) systems

LTI systems obeys superposition and scaling, and commute with translations. They are in bijection with convolution-type operators.

$$V_1(s)(t) = \int_{\tau \in \mathbb{R}} v(\tau)s(t-\tau)d\tau.$$

They're defined by their *impulse response*, v. The Fourier transform of v, $\hat{v}$, is called the *frequency response*.

LTI systems don't add new frequencies - they just scale and impart a phase shift to each existing one.

Fourier transform

Transforms between the time and frequency domains.

(1-D) $$F(s)(\omega) = \hat{s}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} s(t) dt$$

(n-D) $$F(s)(\boldsymbol{\Omega}) = \hat{s}(\boldsymbol{\Omega}) = \int_{\mathbb{R}^n} e^{-i\boldsymbol{\Omega}\cdot\mathbf{t}} s(\mathbf{t}) d\mathbf{t}$$

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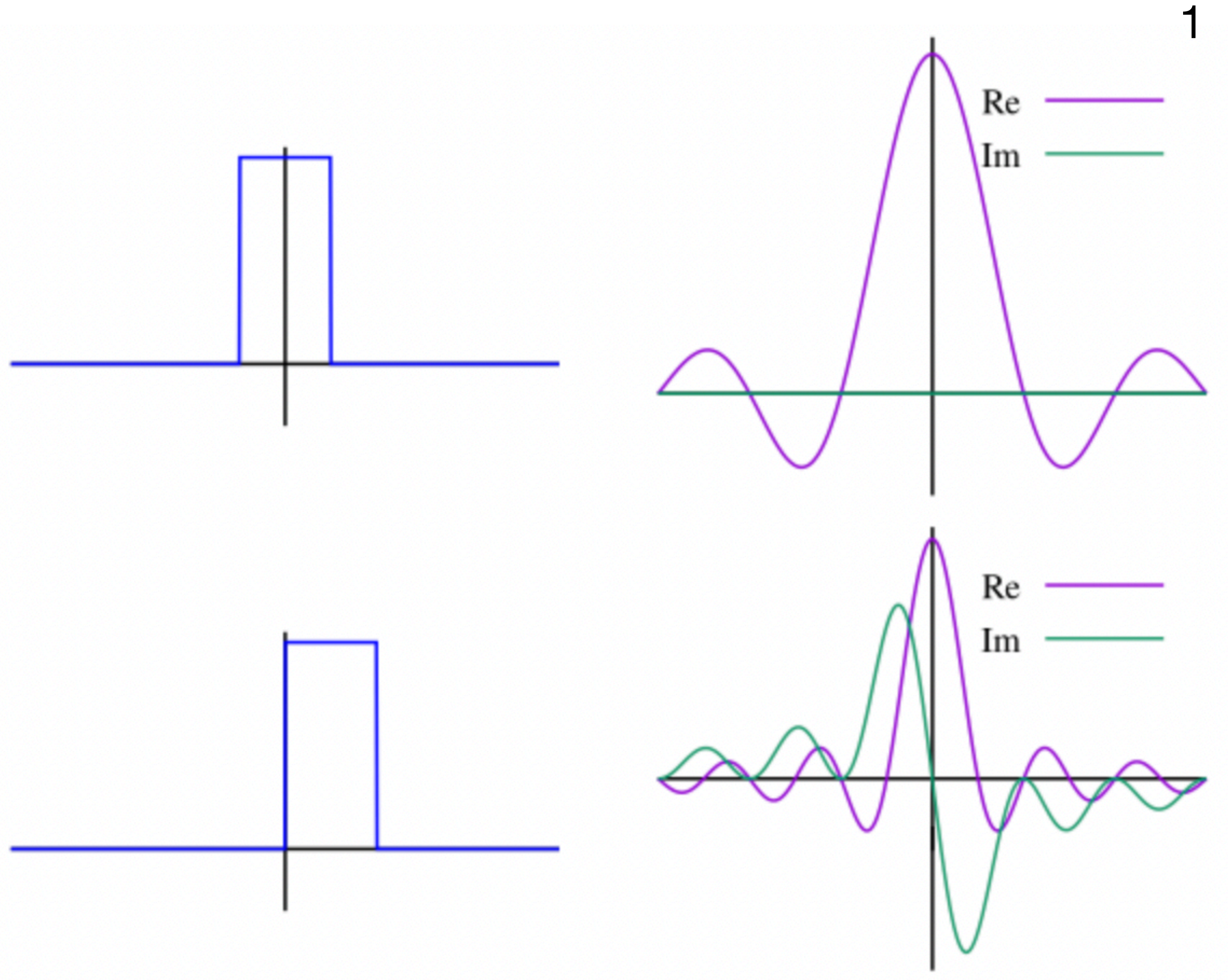

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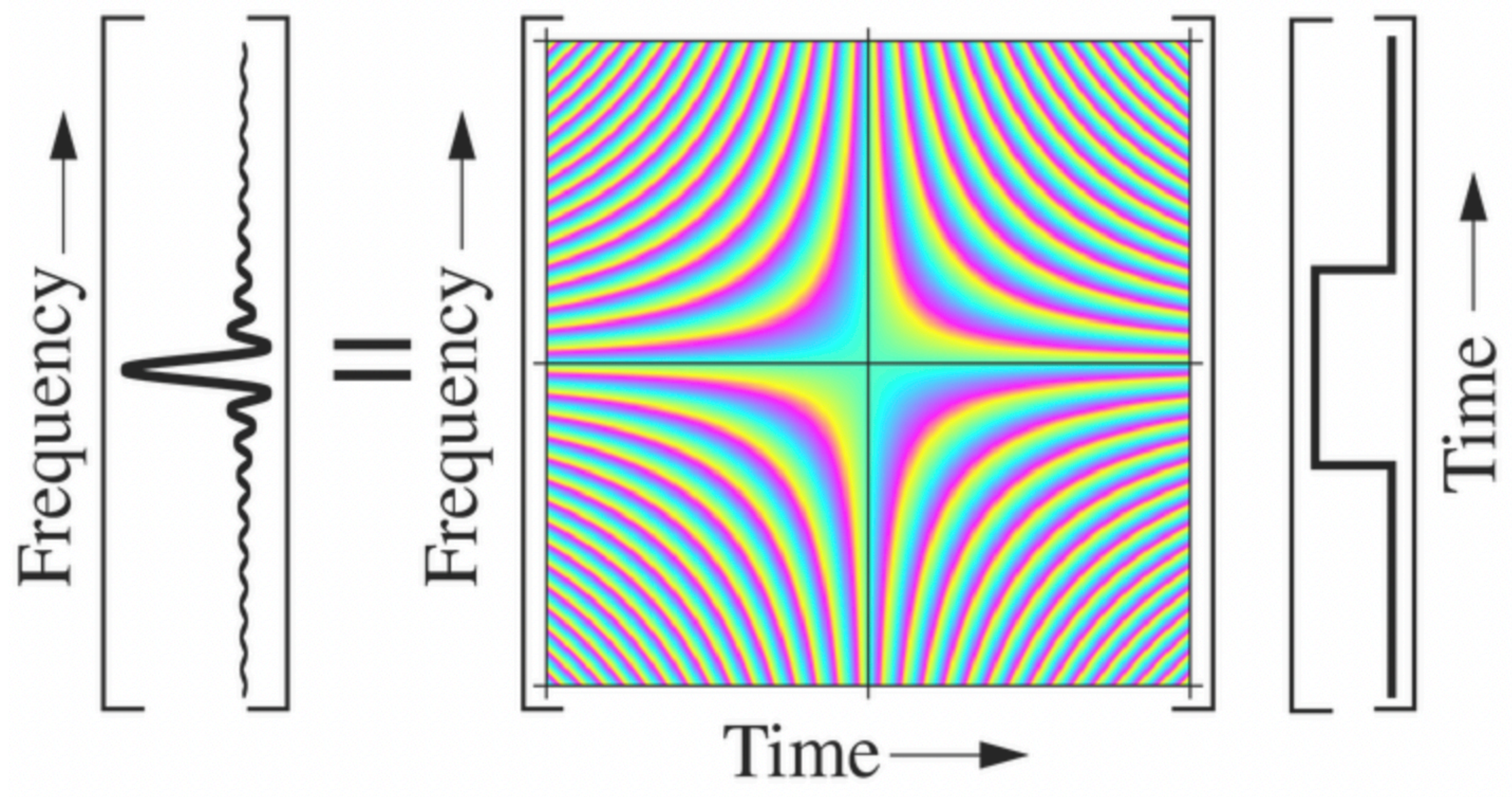
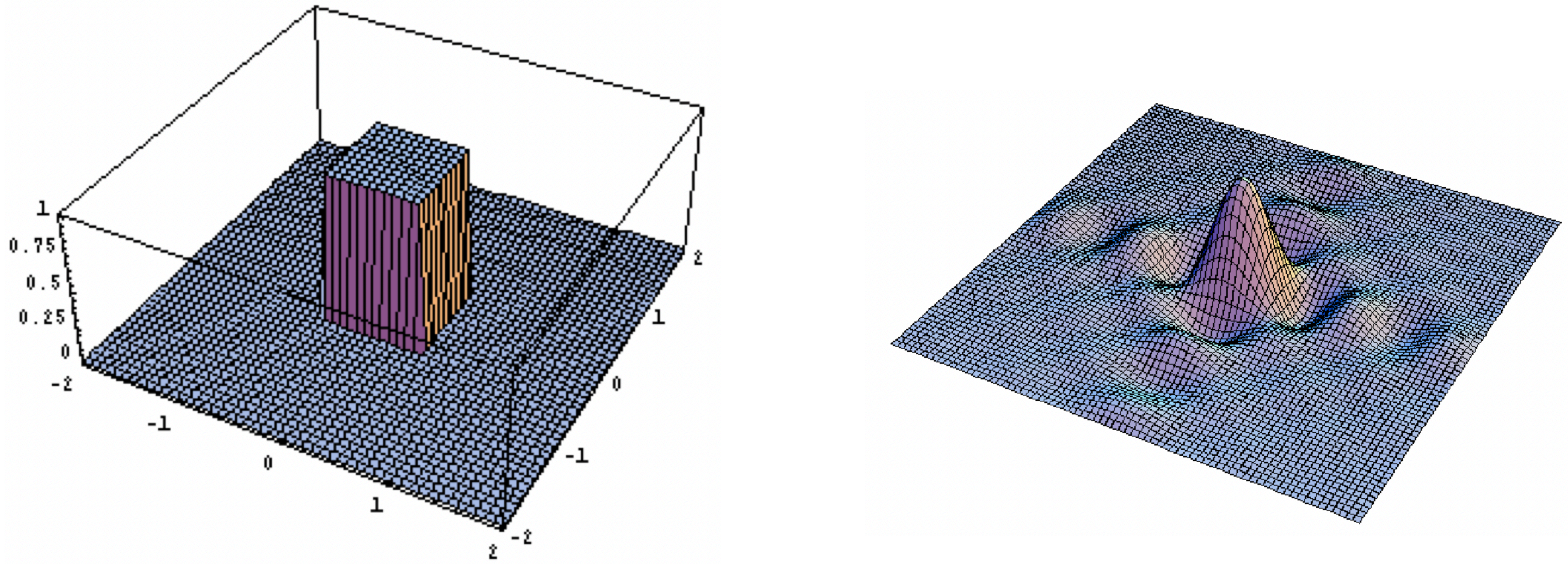




2

1 wikipedia.org/wiki/Fourier_transform

3 wikipedia.org/wiki/Fourier_operator

2 see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf

Convolution and Modulation[1]

$$(u * v)(t) = \int u(t-\tau)v(\tau)d\tau$$

$$(\hat{u} \odot \hat{v})(\xi) = \hat{u}(\xi)\hat{v}(\xi)$$

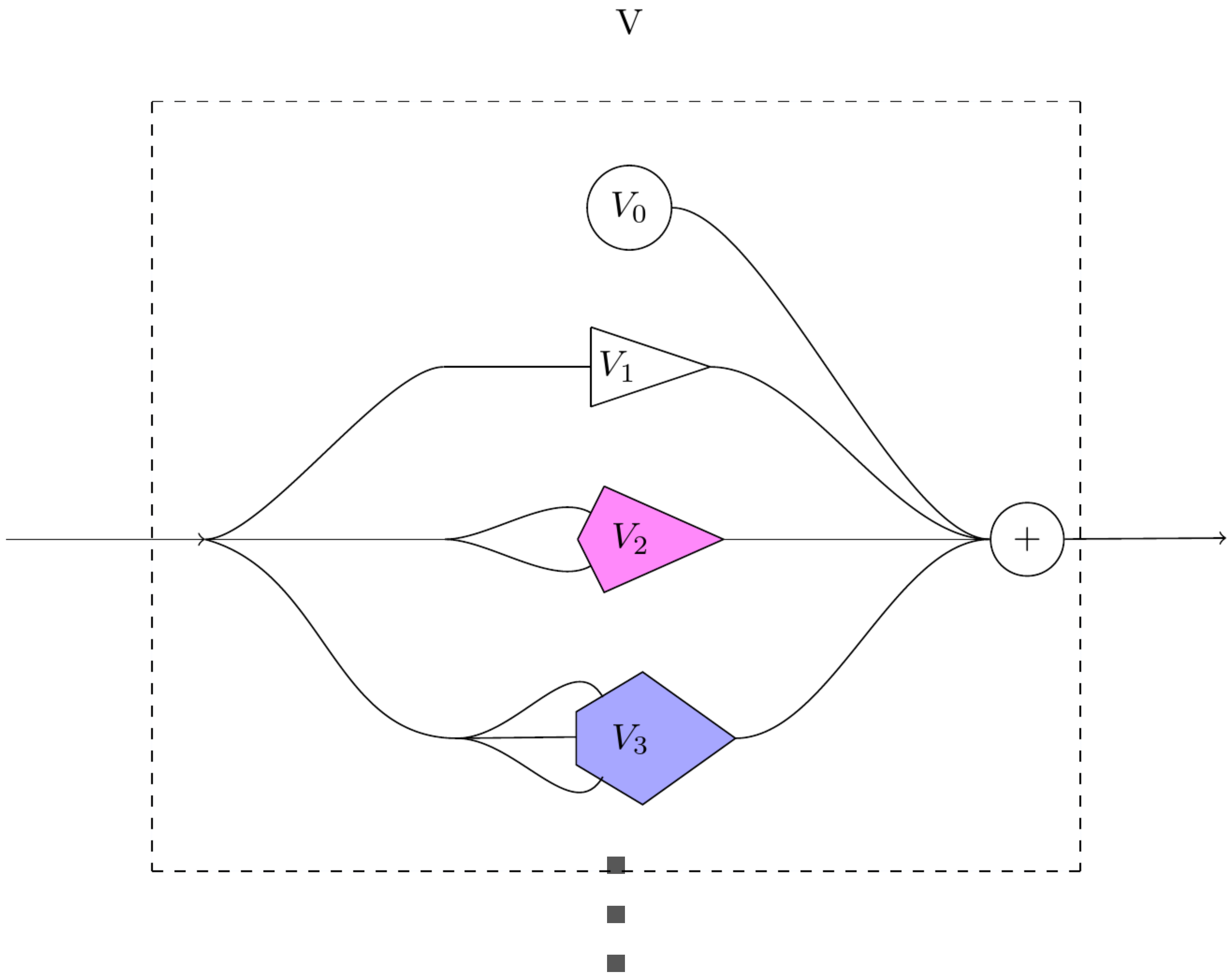
$$\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi) \qquad \text{(Convolution Thm.)}$$

$$\begin{tikzcd}
u \arrow[r, "F"] \arrow[d, "*v"'] & \hat{u} \arrow[d, "\cdot\hat{v}"] \\
u * v & \hat{u} \cdot \hat{v} \arrow[l, "F^{-1}"]
\end{tikzcd}$$

[1]'Modulation' is a synonym for point-wise multiplication.

the Volterra series

A universal model* that generalizes the LTI concept to the nonlinear regime



*for systems with *fading memory*

Fading Memory and the Problem of Approximating Nonlinear Operators with Volterra Series, Boyd and Chua

Volterra series (time-domain)

A Volterra series $V : S(\mathbb{R}) \to S(\mathbb{R})$ is a sum of homogeneous operators, the V_j,

$$y(t) = V[s](t) := \sum_{j=0}^{\infty} V_j[s](t)$$

(For now, think of $S(\mathbb{R})$ as the space of signals $\mathbb{R} \to \mathbb{C}$.)

which convolve the tensor power of their input by a kernel function, $v_j : \mathbb{R}^j \to \mathbb{C}$, then slice along the diagonal

$$= V_0 + \sum_{j=1}^{\infty} \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) \prod_{r=1}^{j} s(t - \tau_r) d\tau_r$$

The j^{th}-order output is

$$y_j(t) = V_j[s](t) = \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) \prod_{r=1}^{j} s(t - \tau_r) d\tau_r$$

The supports of the v_j constrain the system *memory*.

Volterra series (frequency-domain)

The spectrum of the output at order j is given (by the projection-slice theorem) by point-wise multiplication followed by projection

$$\hat{y}_j(\omega) = \int_{\mathbf{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \mathbf{\Omega}_j = \omega} \hat{v}_j(\mathbf{\Omega}_j) \prod_{q=1}^{j} \hat{s}(\omega_q) d\omega_q$$

$$= \int_{t \in \mathbb{R}} e^{-i\omega t} y_j(t) dt.$$

Volterra series, tensor power form

$$V_j(s)(t) = \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) s^{\otimes j}(t\mathbf{1}_j - \boldsymbol{\tau}_j) d\boldsymbol{\tau}_j$$

$$V_j(\hat{s})(\omega) = \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \boldsymbol{\Omega}_j = \omega} (\hat{v}_j \odot \hat{s}^{\otimes j})(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j$$

$$\begin{tikzcd}
S(\mathbb{R})^j \arrow[r, "\otimes j"] \arrow[dr, "\tilde{h}"'] & T^j(S(\mathbb{R})) \arrow[d, "*v_j"] \\
S(\mathbb{R}) \arrow[u, "(-)^j"] & T^j(S(\mathbb{R})) \arrow[l, "\mathrm{diag}(-)"]
\end{tikzcd}$$

Recall that F distributes over $\otimes$: $\mathcal{F}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \mathcal{F}f_1 \otimes \mathcal{F}f_2 \otimes \cdots \otimes \mathcal{F}f_n$

Key idea: a Volterra series represents nonlinear effects by filtering *intermodulation components* of frequencies from the input signal.

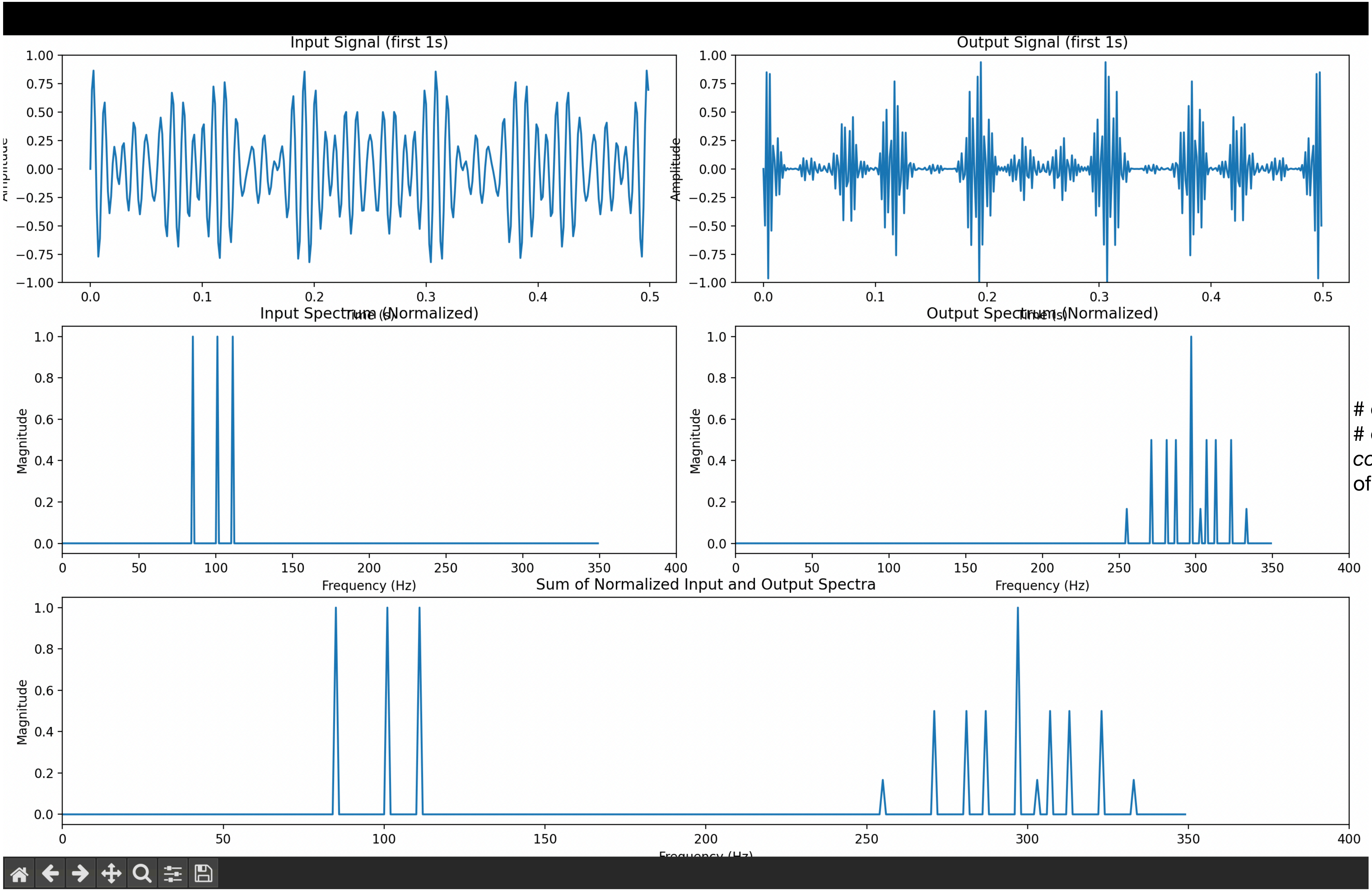
A first-order Volterra operator is just an LTI system: $V_1(s)(t) = \int_{\tau\in\mathbb{R}} v(\tau)s(t-\tau)d\tau$

Volterra series representations of some simple systems

- translation: $$T_\tau(s)(t) = s(t-\tau) \quad \Longrightarrow \quad \hat{v}_1(\omega) = e^{i\omega\tau}$$
- differential: $$D^r(s) = \tfrac{d^r}{dt^r} s \quad \Longrightarrow \quad \hat{v}_1(\omega) = (i\omega)^r$$
- memoryless polynomial: $$P[s] = \textstyle\sum_{j=0}^{\infty} a_j s^j \quad \Longrightarrow \quad \hat{v}_j(\Omega_j) = a_j$$

Audio demo

- demo harmonics generation (3rd-order VS)
- demo cello distortion



# of peaks = # of *weak* 3-*compositions* of 3

Categorification, level 1: the category $S'(\mathbb{R})$

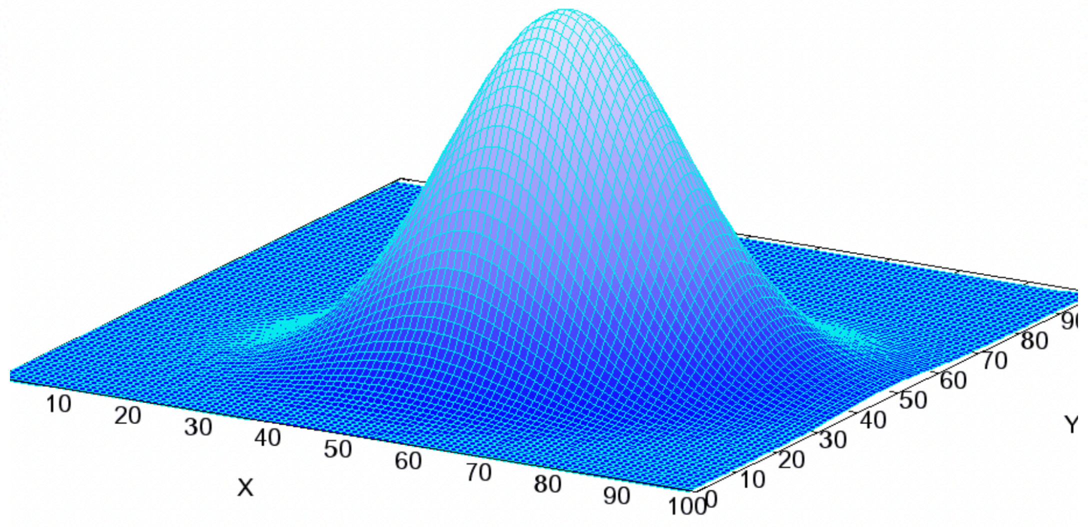
For a reference on these spaces, see *Time-frequency analysis on* $\mathbb{R}^n$*,* by Vuojamo et al.

Key points:

- objects are signals (resp., spectra)
- morphisms are convolutors (resp., multipliers) between them
- the Fourier transform is well-defined

Definition (Schwartz space) The *Schwartz space* $\mathscr{S}(\mathbb{R})$ of rapidly decreasing smooth functions, or *test functions*, is the subspace of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R})$ for which

$$\sup_{x\in\mathbb{R}} |x^\beta \partial_x^\alpha \varphi(x)| < \infty,$$



[1]

Definition (Tempered distributions) The space $\mathscr{S}'(\mathbb{R})$ of *tempered distributions* is the space of continuous linear functionals on $\mathscr{S}(\mathbb{R})$.

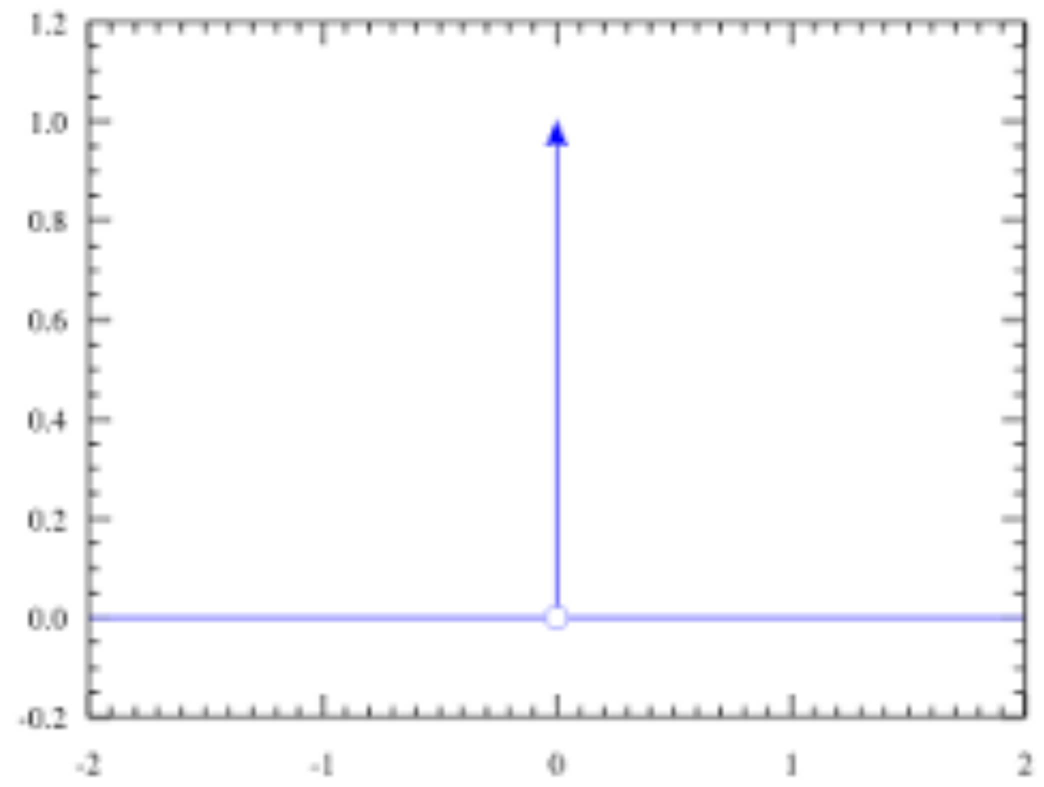
1 wikipedia.org/wiki/Multivariate_normal_distribution

Example: delta function

The Dirac delta 'function' is the distribution,

$$\delta \in \mathscr{S}'(\mathbb{R}), \;\; \delta(\phi) = \phi(0).$$

1

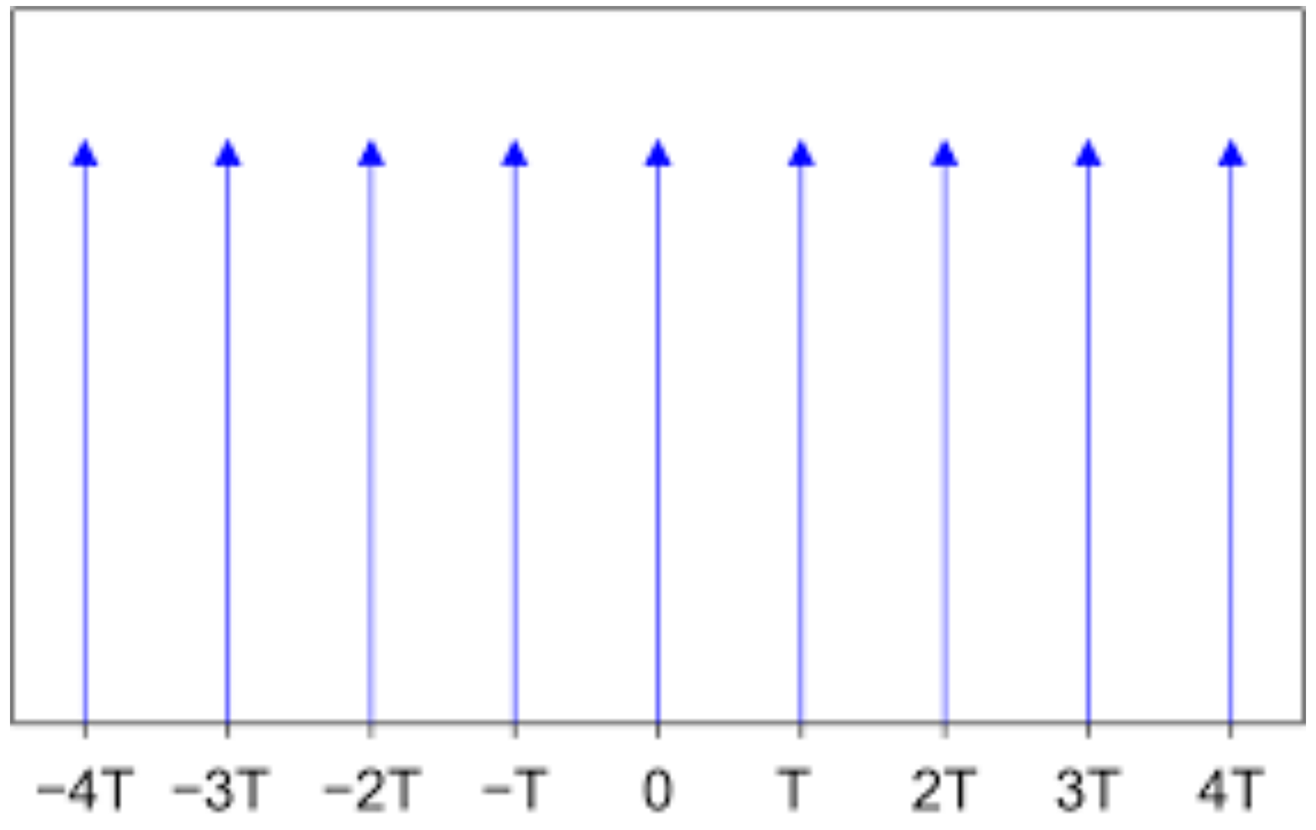


Example: Dirac comb function

The *Dirac Comb Ш* is the distribution given by

$$Ш_T \in \mathscr{S}'(\mathbb{R}^n)$$
$$Ш_T(\phi) = \begin{cases} \phi(x) & x = nT, n \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

2



2 wikipedia.org/wiki/Dirac_comb

1 wikipedia.org/wiki/Dirac_delta_function

Definition: Multipliers The space of *multipliers*, $\mathscr{O}(\mathbb{R})$, is the space of functions $\varphi \in \mathscr{S}(\mathbb{R})$ such that, for every $\alpha \in \mathbb{N}$, there is a polynomial P_α such that, $\forall x \in \mathbb{R}$,

$$|\partial_\alpha \varphi(x)| \leq |P(x)|,$$

i.e., whose derivatives are polynomially bounded. Pointwise multiplication of a Schwartz function by a multiplier results in another Schwartz function.

Definition: Convolutors The space of *convolutors*, $\mathscr{O}'(\mathbb{R})$ is the space of tempered distributions Λ for which, for any integer $h \geq 0$, there is a finite family of continuous functions, $f_\alpha : \mathbb{R} \to \mathbb{C}$, with index $\alpha \in \mathbb{N}_0$, such that

$$\Lambda = \sum_{|\alpha| \leq h} \partial_\alpha f_\alpha,$$

and such that

$$\lim_{|x| \to \infty} (1 + |x|)^h |f_\alpha(x)| = 0$$

for all $|\alpha| \leq h$.

The convolutors are precisely those tempered distributions which map Schwartz functions to Schwartz functions under convolution. The Fourier Transform is a linear bijection between the spaces $\mathscr{O}_M(\mathbb{R})$ and $\mathscr{O}_C'(\mathbb{R})$.

the base category, $S'(\mathbb{R})$

Definition: The category $S'(\mathbb{R})$ is the category with objects, tempered distributions (including Schwartz functions), and morphisms, convolutors between them.

The category $S'(\mathbb{R})$ has a filtered structure, with convolutions in the time-domain weakly contracting spectral bandwidth.

Categorification, level 2: the Volterra series as functor

If we morph the input to a Volterra series, how does the output change?

Action of V on a morphism of signals

$V(l)$ applies l to each copy of s occurring at order j. (Can think of this as post-composition.)

$$V(l)(V(s))(t) = \sum_{j=0}^{\infty} \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) \prod_{r=1}^{j} l(s)(t - \tau_r) d\boldsymbol{\tau}_j$$

$$\begin{tikzcd}
s \arrow[r, "l"] \arrow[d, mapsto, "V"'] & s' \arrow[d, mapsto, "V"] \\
V(s) \arrow[r, "V(l)"'] & V(s')
\end{tikzcd}$$

Equivalently, it filters $\hat{v}_j \odot \hat{s}^{\otimes j}$ by the tensor power of the multiplier weight function:

$$V(m)(\hat{s})(\omega) = \sum_{j=0}^{\infty} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \boldsymbol{\Omega}_j = \omega} \hat{v}_j(\boldsymbol{\Omega}_j) \prod_{q=1}^{j} m(\hat{s})(\omega_q) d\boldsymbol{\Omega}_j$$

$$= \sum_{j=0}^{\infty} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \boldsymbol{\Omega}_j = \omega} \hat{v}_j(\boldsymbol{\Omega}_j) \prod_{q=1}^{j} \gamma(\omega_q) \hat{s}(\omega_q) d\boldsymbol{\Omega}_j$$

(γ is the weight function of the multiplier m related to l)

$$= \sum_{j=0}^{\infty} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \boldsymbol{\Omega}_j = \omega} (\hat{v}_j \odot \hat{s}^{\otimes i} \odot \gamma^{\otimes i})(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j,$$

Functoriality

Does V respect composition?

$$V(g \circ f)V(\hat{s})(\omega)$$

$$= \sum_{j=0}^{\infty} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \mid \Sigma \boldsymbol{\Omega}_j = \omega} (\hat{v}_j \odot \hat{s}^{\otimes i} \odot \widehat{g \circ f}^{\otimes i})(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j$$

$$= \sum_{j=0}^{\infty} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j \mid \Sigma \boldsymbol{\Omega}_j = \omega} (\hat{v}_j \odot \hat{s}^{\otimes i} \odot \hat{g}^{\otimes i} \odot \hat{f}^{\otimes i})(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j$$

$$= V(g)V(f)V(\hat{s})(\omega)$$

Identity?

$$V(\delta_s) = \delta_{V(s)}$$

Caveat: destructive interference

$$V(\hat{l})(\omega) := \frac{V(\widehat{l(s)})(\omega)}{V(\hat{s})(\omega)}$$

— $V(\hat{s})(\omega)$ could be zero when $V(\widehat{l(s)})(\omega)$ is not. But the former will almost always be richer than the latter, since l is a filter.

Examples of the action of a Volterra series on:

- translation
- modulation
- periodization
- sampling

Translation

Let $c : s \to s'$ be the translation-by-l map, given in the time-domain as $c(s)(t) = (s * \delta_l)(t) = \int_\tau s(\tau)\delta(t - l - \tau)d\tau$. Then c commutes with the action of a Volterra series; i.e., $V(c) : V(s) \to V(s')$ is defined by

$$V(c)V(s)(t) = V(c(s))(t)$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \int_{\mathbf{\Omega}_j\in\mathbb{R}^j|\Sigma\mathbf{\Omega}_j=\omega} \hat{v}_j(\mathbf{\Omega}_j) \prod_{r=1}^{j} e^{-i\omega_r\tau}\hat{s}(\omega_r)d\mathbf{\Omega}_j d\omega$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \int_{\mathbf{\Omega}_j\in\mathbb{R}^j|\Sigma\mathbf{\Omega}_j=\omega} e^{-i(\omega_1+\cdots+\omega_j)\tau}\hat{v}_j(\mathbf{\Omega}_j)\hat{s}^{\otimes j}(\mathbf{\Omega}_j)d\mathbf{\Omega}_j d\omega$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega(t-\tau)}\hat{y}_j(\omega)d\omega$$

$$= V(s)(t-\tau)$$

Modulation

Let $m : s \to s'$, $m(s)(t) = e^{i\xi t}s(t)$ be the modulation-by-ξ map. Then $V(m) : V(s) \to V(s')$ is written in the frequency domain as

$$\widehat{V(m)V}(s)(\omega)$$

$$=\sum_{j=0}^{\infty} \int_{\mathbf{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \mathbf{\Omega}_j = \omega} \hat{v}_j(\mathbf{\Omega}_j) \prod_{q=1}^{j} (\delta_\xi * \hat{s})(\omega_q) d\mathbf{\Omega}_j$$

$$=\sum_{j=0}^{\infty} \int_{\mathbf{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \mathbf{\Omega}_j = \omega} \hat{v}_j(\mathbf{\Omega}_j) \prod_{q=1}^{j} \hat{s}(\omega_q - \xi) d\mathbf{\Omega}_j$$

$$=\sum_{j=0}^{\infty} \int_{\mathbf{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \mathbf{\Omega}_j = \omega} \hat{v}_j(\mathbf{\Omega}_j) \hat{s}^{\otimes j}(\mathbf{\Omega}_j - \xi \mathbf{1}) d\mathbf{\Omega}_j.$$

and if $s = \mathbf{1}$,

$$V(m)V(\mathbf{1})(t) = \sum_j e^{ijt\xi}\, \hat{v}_j(\xi \mathbf{1})$$

Periodization

Let $c : s \to s'$ be the operation of convolution against the Dirac comb with period T: $c(s)(t) = (\text{Ш}_T * s)(t)$; such an operation 'periodizes' the signal s. Then $V(c) : V(s) \to V(s')$ is defined by

$$V(c)V(s)(t) = V(c(s))(t)$$

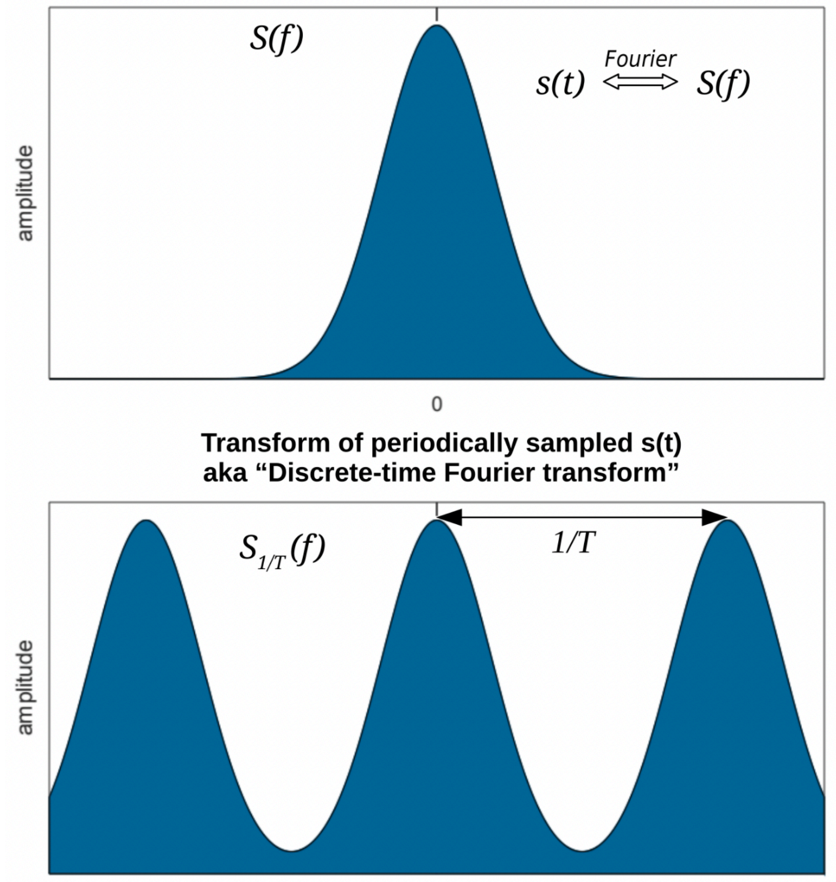
$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j | \Sigma\boldsymbol{\Omega}_j = \omega} \hat{v}_j(\boldsymbol{\Omega}_j) \prod_{r=1}^{j} \text{Ш}_{\frac{1}{T}}(\omega_r)\hat{s}(\omega_r) d\boldsymbol{\Omega}_j d\omega$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \int_{\boldsymbol{\Omega}_j \in \mathbb{R}^j | \Sigma\boldsymbol{\Omega}_j = \omega} \hat{v}_j(\boldsymbol{\Omega}_j)\, \text{Ш}_{\frac{1}{T}}{}^{\otimes j}(\boldsymbol{\Omega}_j)\hat{s}^{\otimes j}(\boldsymbol{\Omega}_r) d\boldsymbol{\Omega}_j d\omega$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \sum_{\boldsymbol{k}\in\mathbb{Z}^j \,|\, \Sigma T^{-1}\boldsymbol{k} = \omega} \hat{v}_j(T^{-1}\boldsymbol{k})\hat{s}^{\otimes j}(T^{-1}\boldsymbol{k}) d\omega$$

$$= \sum_{j=0}^{\infty} \int_{\omega\in\mathbb{R}} e^{i\omega t} \sum_{\boldsymbol{k}|\Sigma T^{-1}\boldsymbol{k} = \omega} \binom{j}{n(k_1), \ldots, n(k_p)} \hat{v}_j(T^{-1}\boldsymbol{k})\, \hat{s}^{\otimes j}(T^{-1}\boldsymbol{k}) d\omega.$$

[1]



1 wikipedia.org/wiki/Periodic_summation

Sampling

Let $m : s \to s'$ be the operation of multiplication against the Dirac comb with period T: $m(s)(t) = (\text{Ш}_T \cdot s)(t)$; such an operation 'samples' the signal s. Then $V(m) : V(s) \to V(s')$ is defined

$$
\begin{aligned}
V(m)V(s)(t) &= V(m(s))(t) \\
&= \sum_{j=0}^{\infty} \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) \prod_{r=1}^{j} \text{Ш}_T(t - \tau_r) s(t - \tau_r) d\boldsymbol{\tau}_j \\
&= \sum_{j=0}^{\infty} \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_j(\boldsymbol{\tau}_j) \, {\text{Ш}_T}^{\otimes j}(t\mathbf{1}_j - \boldsymbol{\tau}_j) \, s^{\otimes j}(t\mathbf{1}_j - \boldsymbol{\tau}_j) d\boldsymbol{\tau}_j \\
&= \sum_{j=0}^{\infty} \sum_{\boldsymbol{k} \in \mathbb{Z}^j \,|\, \Sigma n(k_i) = j} \binom{j}{n(k_1), \ldots, n(k_p)} v_j(t\mathbf{1} - T\boldsymbol{k}) \, s^{\otimes j}(t\mathbf{1} - T\boldsymbol{k})
\end{aligned}
$$

Morphisms of Volterra series

Systems change; thus, we need a notion of morphism of Volterra series.

notational aside

Denote by $[i]$ the order of nonlinearity of the ith operator;

and by $I_V = V(\mathbf{1})$, the *index set* (of homogeneous operators) of V.[1]

[1] I_V here is analogous to what Spivak and Niu call the set of *positions* of a polynomial functor, in *Polynomial Functors: A Mathematical Theory of Interaction.*

Definition: A *morphism* $\phi : V \to W$ *of Volterra series* is comprised of the following data:

- a function $\phi_1 : I_V \to I_W$ between index-sets;
- for each pair $(i, \phi_1(i))$, with $i \in I_V$ and $\phi_1(i) \in I_W$,
  - a linear map $\phi_i : \mathbb{R}^{[i]} \to \mathbb{R}^{[\phi_1(i)]}$
  - and a weight function, or *mask*, $\psi : \mathbb{R}^{[i]} \to \mathbb{C}$

from which we obtain the *weighted pullback*,

$$
{}_{\psi}\phi_i^{\#} : S(\mathbb{R}^{[\phi_1(i)]}) \to S(\mathbb{R}^{[i]})
$$

$$
{}_{\psi}\phi_i^{\#}(z)(\boldsymbol{x}) = \psi(\boldsymbol{x}) \cdot z(\phi_i(\boldsymbol{x}))
$$

where $z \in S'(\mathbb{R}^{\phi_1[i]})$ and $\boldsymbol{x} \in \mathbb{R}^{[i]}$.

Note that ϕ has the structure of a lens.

We then obtain *component morphisms*, $\phi_s : V(s) \to W(s)$, indexed by the objects of $S'(\mathbb{R})$,

$$\phi_s(V_i(\hat{s}))(\omega) =$$

$$\int_{\substack{\boldsymbol{\Omega}_i \in \mathbb{R}^i \\ \Sigma\boldsymbol{\Omega}_i = \omega}} (\psi \odot (\hat{v}_i \odot \hat{s}^{\otimes[i]}) \odot \phi_i^{\#}(\hat{w}_j))(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j =$$

$$\int_{\substack{\boldsymbol{\Omega}_i \in \mathbb{R}^i \\ \Sigma\boldsymbol{\Omega}_i = \omega}} (\psi \odot \hat{v}_i \odot \hat{s}^{\otimes[i]})(\boldsymbol{\Omega}_j) \hat{w}_j(\phi_i(\boldsymbol{\Omega}_j)) d\boldsymbol{\Omega}_j$$

and can ask, do they assemble into a natural transformation?

$$\begin{tikzcd}
V(s) \arrow[r, "V(f)"] \arrow[d, "\phi_s"'] & V(s') \arrow[d, "\phi_{s'}"] \\
W(s) \arrow[r, "W(f)"'] & W(s')
\end{tikzcd}$$

Naturality

$$
\begin{tikzcd}
V(s) \arrow[r, "V(f)"] \arrow[d, "\phi_s"'] & V(s') \arrow[d, "\phi_{s'}"] \\
W(s) \arrow[r, "W(f)"'] & W(s')
\end{tikzcd}
$$

$$\int_{\substack{\boldsymbol{\Omega}_i \in \mathbb{R}^i \\ \Sigma \boldsymbol{\Omega}_i = \omega}} (\hat{f}^{\otimes[i]} \odot (\psi \odot (\,\hat{v}_i \odot \hat{s}^{\otimes[i]}) \odot \phi_i^{\#}(\hat{w}_j)))(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j \qquad \text{(lower path)}$$

$$= \int_{\substack{\boldsymbol{\Omega}_i \in \mathbb{R}^i \\ \Sigma \boldsymbol{\Omega}_i = \omega}} (\psi \odot ((\,\hat{v}_i \odot \hat{s}^{\otimes[i]} \odot \hat{f}^{\otimes[i]}) \odot \phi_i^{\#}(\hat{w}_j)))(\boldsymbol{\Omega}_j) d\boldsymbol{\Omega}_j \qquad \text{(upper path)}$$

Note: a core fact in TFA is that modulation and convolution *do not* (generally) commute. This forces our choice of the mask ψ to be a convolutor in the time domain.

A technical restriction

We further impose the following condition on the map ϕ_i: that it preserve the weak compositions of frequencies, i.e. $\Sigma\Omega_i = \omega \implies \Sigma\phi_i(\Omega_i) = \omega$.

Intuition: the two systems must interact at the same frequencies, in order for the image of any component in the source to lie within the target spectrum; i.e., so the convolutor $\phi_s : V(s) \to W(s)$ is well-defined.

The category *Volt*

Definition: The category, *Volt*, of Volterra series is the category having, as objects, Volterra series, and as morphisms, natural transformations between them.

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "h"'] & W \arrow[d, "g"] \\
 & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
V(\mathbf{1}) \arrow[r, "f_1"] \arrow[dr, "h_1"'] & W(\mathbf{1}) \arrow[d, "g_1"] \\
 & X(\mathbf{1})
\end{tikzcd}
\qquad
\begin{tikzcd}
S(\mathbb{R}^{[i]}) & S(\mathbb{R}^{[f_1(i)]}) \arrow[l, "f_i^{\#}"'] \\
 & S(\mathbb{R}^{[h_1(i)]}) \arrow[u, "g^{\#}_{f_1(i)}"'] \arrow[ul, "h_i^{\#}"]
\end{tikzcd}
$$

Examples of Volterra morphisms

Autoconvolution The autoconvolution $\mathrm{aut}_V : V \to V$ is given by the pair $(\phi_1, \phi^{\#})$, where both ϕ_1 and all of the $\phi_i^{\#}$ are identity maps. This map results in the Volterra series whose VKF at each order i is the autoconvolution $(v_i * v_i)$ of v_i.

Identity morphism The identity morphism $\mathrm{id}_V : V \to V$ is given by the pair $(\phi_1, \phi^{\#})$, where $\phi_1 = \mathrm{id}_{V(\mathbf{1})}$ is the identity and where, for any $i \in V(\mathbf{1})$, $\phi_i^{\#} : S'(\mathbb{R}^{[i]}) \to S'(\mathbb{R}^{[i]})$ is the weighted pullback along the identity on $\mathbb{R}^{[i]}$ that scales by the reciprocal of the spectrum of v_i, i.e. $\psi(\mathbf{\Omega}_i) = \frac{1}{\hat{v}_i(\mathbf{\Omega}_i)}$ for $\hat{v}_i(\mathbf{\Omega}_i) \neq 0$. The definition of $\phi_i^{\#}$ follows from the fact that the spectrum of the autoconvolution $R_{v_i} = (v_i * v_i)$ is $\hat{R}_{v_i}(\mathbf{\Omega}_i) = \hat{v}_i(\mathbf{\Omega}_i)^2$.

This is why we need ψ: to have identity morphisms.

Translation Let $\boldsymbol{\tau} = [\tau_1, \boldsymbol{\tau}_2, \ldots, \boldsymbol{\tau}_j] \in \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^j$; then the translation-by-$\boldsymbol{\tau}$ morphism is the morphism with target the VS whose VKF at each order i is a multidimensional distribution centered at $\boldsymbol{\tau}_i$, with $\phi_1 = \mathrm{id}$ and all of the $\phi_i^{\#}$ also identities. The offsets can be varied and/or the morphism iterated.

Translation is an example of a *parameterized* morphism.

Categorification, level 3: *Volt* as a monoidal category

How can we wire nonlinear systems represented by Volterra series together?[1]

- sum (+)
- product (x)
- series composition (◁)

[1] These operations are well-known in the Volterra series literature; see *Modeling Nonlinear Systems by Volterra Series*, by Carassale and Kareem.

Coproduct: +

Sums the homogeneous operators level-wise,

$$
\begin{aligned}
&(V+W)(s)(t) \\
&= \sum_j (V+W)_j(s)(t) = \sum_j V_j(s)(t) + W_j(s)(t) \\
&= \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} (v_j(\boldsymbol{\tau}_j) + (w_j(\boldsymbol{\tau}_j)) \prod_{q=0}^{j} s(t-\tau_q) d\boldsymbol{\tau}_j.
\end{aligned}
$$

Doesn’t change the order of nonlinearity.

Does it satisfy the universal property of the coproduct?

$$
\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & V + W \arrow[d, dashed, "h"] & W \arrow[l, "\kappa"'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}
$$

$$\begin{tikzcd}
V(\mathbf{1}) \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & V(\mathbf{1}) + W(\mathbf{1}) \arrow[d, dashed, "h_1"] & W(\mathbf{1}) \arrow[l, "\kappa_1"'] \arrow[dl, "g_1"] \\
 & X(\mathbf{1}) &
\end{tikzcd}$$

$$\begin{tikzcd}
S(\mathbb{R}^{[i]}) & S(\mathbb{R}^{[(1,i)]}) \arrow[l, "\iota_i^{\#}"'] \\
 & S(\mathbb{R}^{[f_1(i)]}) \arrow[ul, "f_i^{\#}"] \arrow[u, dashed, "h^{\#}_{(1,i)}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S(\mathbb{R}^{[(2,j)]}) \arrow[r, "\kappa_j^{\#}"] & S(\mathbb{R}^{[j]}) \\
S(\mathbb{R}^{[g_1(j)]}) \arrow[u, dashed, "h^{\#}_{(2,j)}"] \arrow[ur, "g_j^{\#}"'] &
\end{tikzcd}$$

$$h^{\#}_{(1,i)} = f_i^{\#} \text{ and } h^{\#}_{(2,j)} = g_j^{\#}$$

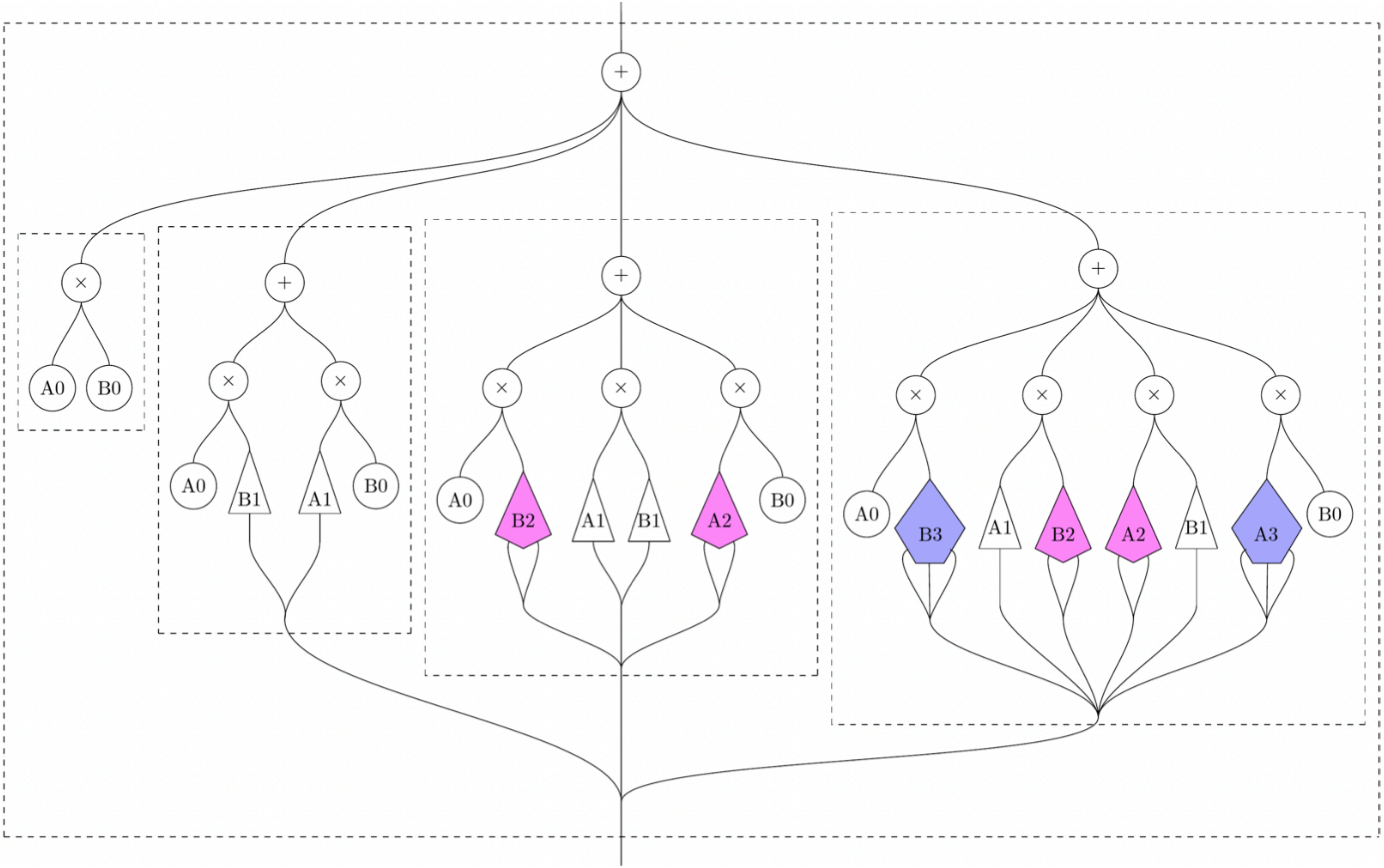
Cartesian product: $\times$

Kernel function given as product of kernels of the factors: $\widehat{v}_j(\Omega_j) = \sum_{p=1}^{j} \widehat{a}_{\alpha_1^p}(\theta_1^p)\, \widehat{b}_{\alpha_2^p}(\theta_2^p)$

$$V(s)(t) = (A \times B)(s)(t) = \sum_j \sum_{k=0}^{j} (A_k(s) B_{j-k}(s))(t)$$

$$V_j(s)(t) = \textstyle\sum_{k=0}^{j} \displaystyle\int_{\Omega_k \in \mathbb{R}^k} e^{i\Sigma\Omega_k t} \hat{a}_k(\Omega_k) \prod_{p=0}^{k} \hat{s}(\omega_p) d\omega_p$$
$$\times \int_{\Omega_{j-k} \in \mathbb{R}^{j-k}} e^{i\Sigma\Omega_{j-k} t} \hat{b}_{j-k}(\Omega_{j-k}) \prod_{q=0}^{j-k} \hat{s}(\omega_q) d\omega_q.$$

Orders of nonlinearity sum.



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Universal property of the product

$$\begin{tikzcd}
Y(\mathbf{1}) \arrow[r, "g_1"] \arrow[d, "f_1"'] \arrow[dr, dashed, "h_1"] & B(\mathbf{1}) \\
A(\mathbf{1}) & A(\mathbf{1}) \times B(\mathbf{1}) \arrow[l, "\pi_1"] \arrow[u, "\varphi_1"']
\end{tikzcd}$$

$$h_1(k) = (f_1(k), g_1(k))$$

$$\begin{tikzcd}
S'(\mathbb{R}^{[k]}) & S'(\mathbb{R}^{[g_1(k)]}) \arrow[l, "g_k^{\#}"'] \arrow[d, "\varphi^{\#}_{(f_1(k),g_1(k))}"] \\
S'(\mathbb{R}^{[f_1(k)]}) \arrow[u, "f_k^{\#}"] \arrow[r, "\pi^{\#}_{(f_1(k),g_1(k))}"'] & S'(\mathbb{R}^{[f_1(k)]}) \oplus S'(\mathbb{R}^{[g_1(k)]}) \arrow[ul, dashed, "h_k^{\#}"]
\end{tikzcd}$$

Series composition: $\triangleleft$

Outputs from the operators in A are fed as inputs to those in B; the B_k are *multivariate*.

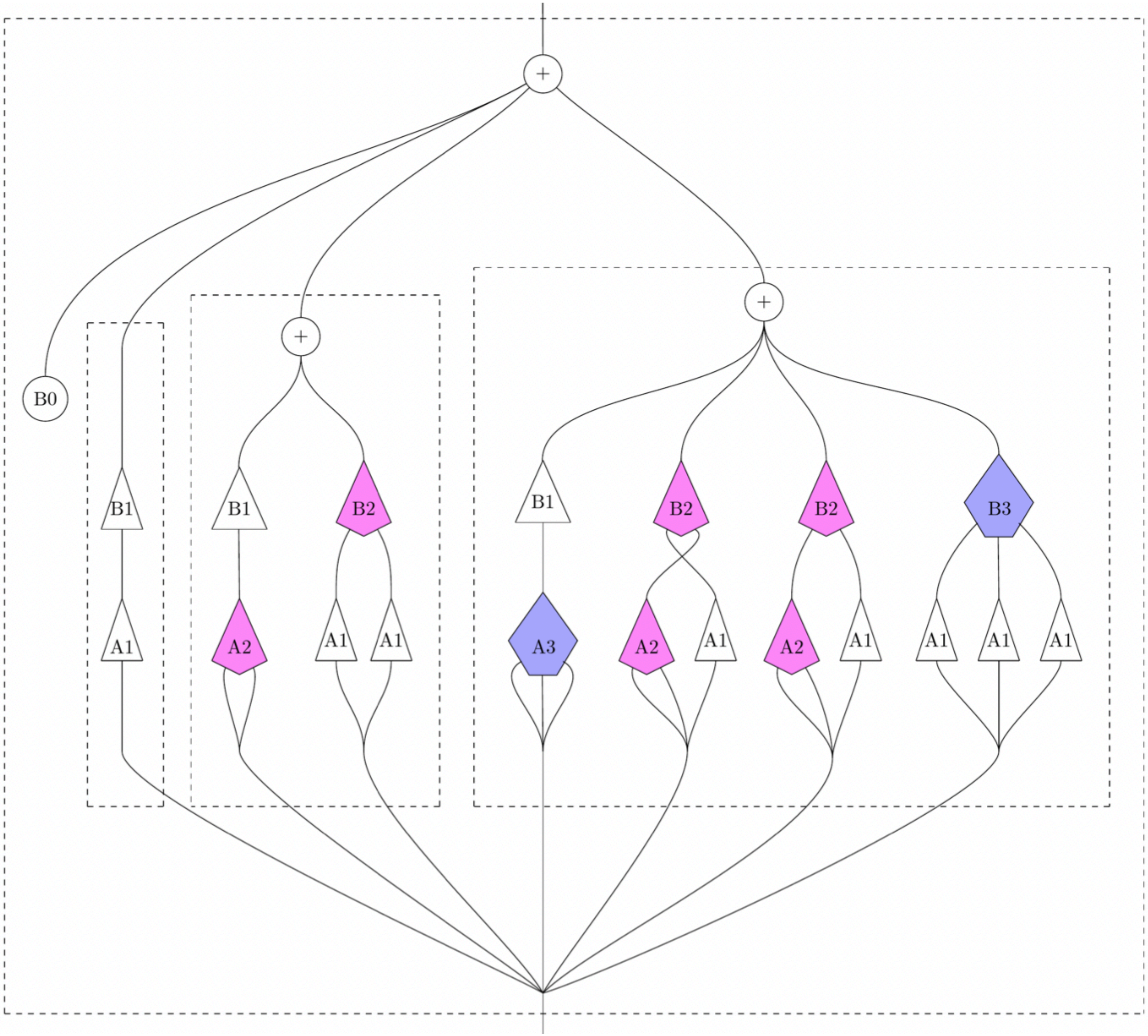
$$V(u) = \sum_{k=0}^{n_B} \sum_{p \mid \sum_{r=1}^{k} \alpha_r^p = j} B_k[y_{\alpha_1^p}, \ldots, y_{\alpha_k^p}]$$

$$v_j(\boldsymbol{\Omega}_j) = \sum_{k=0}^{n_B} \sum_{\{p \mid \sum_{r=1}^{k} \alpha_r^p = j\}} \widehat{b}_k(S_p^{(j,k)} \boldsymbol{\Omega}_j) \prod_{r=1}^{k} \widehat{a}_{\alpha_r^p}(\boldsymbol{\theta}_r^p)$$

Recall:

Orders of nonlinearity multiply.

$$\sum_{p \in \binom{j+k-1}{j}} \binom{j}{\alpha_1^p, \alpha_2^p, \ldots, \alpha_k^p} = k^j$$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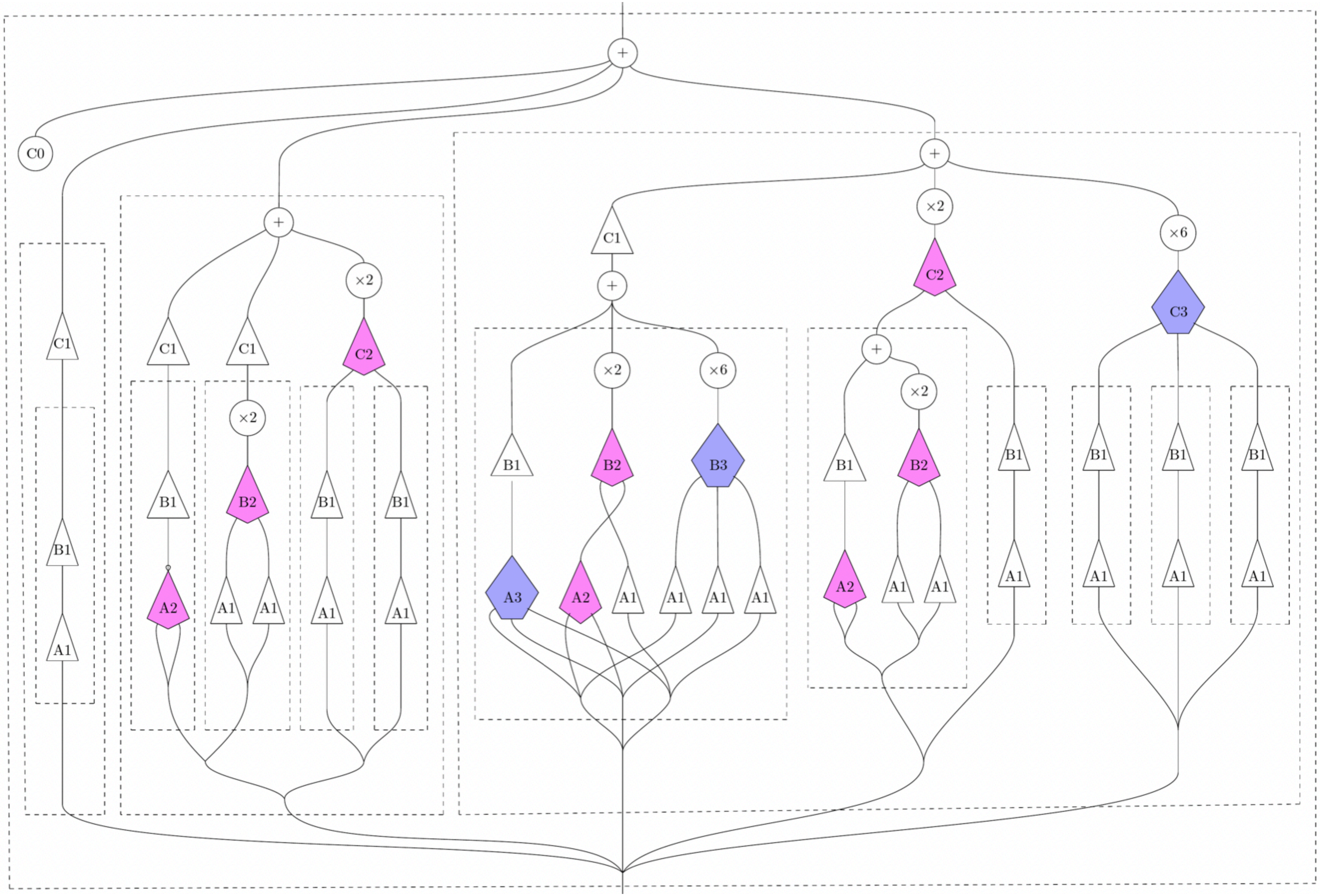
Associativity of $\triangleleft$

Theorem 1: The series composition, $\triangleleft$, of Volterra series is associative. I.e.,

$$(c \triangleleft (b \triangleleft a))_j \cong ((c \triangleleft b) \triangleleft a)_j$$

for all $0 \leq j \leq \infty$.



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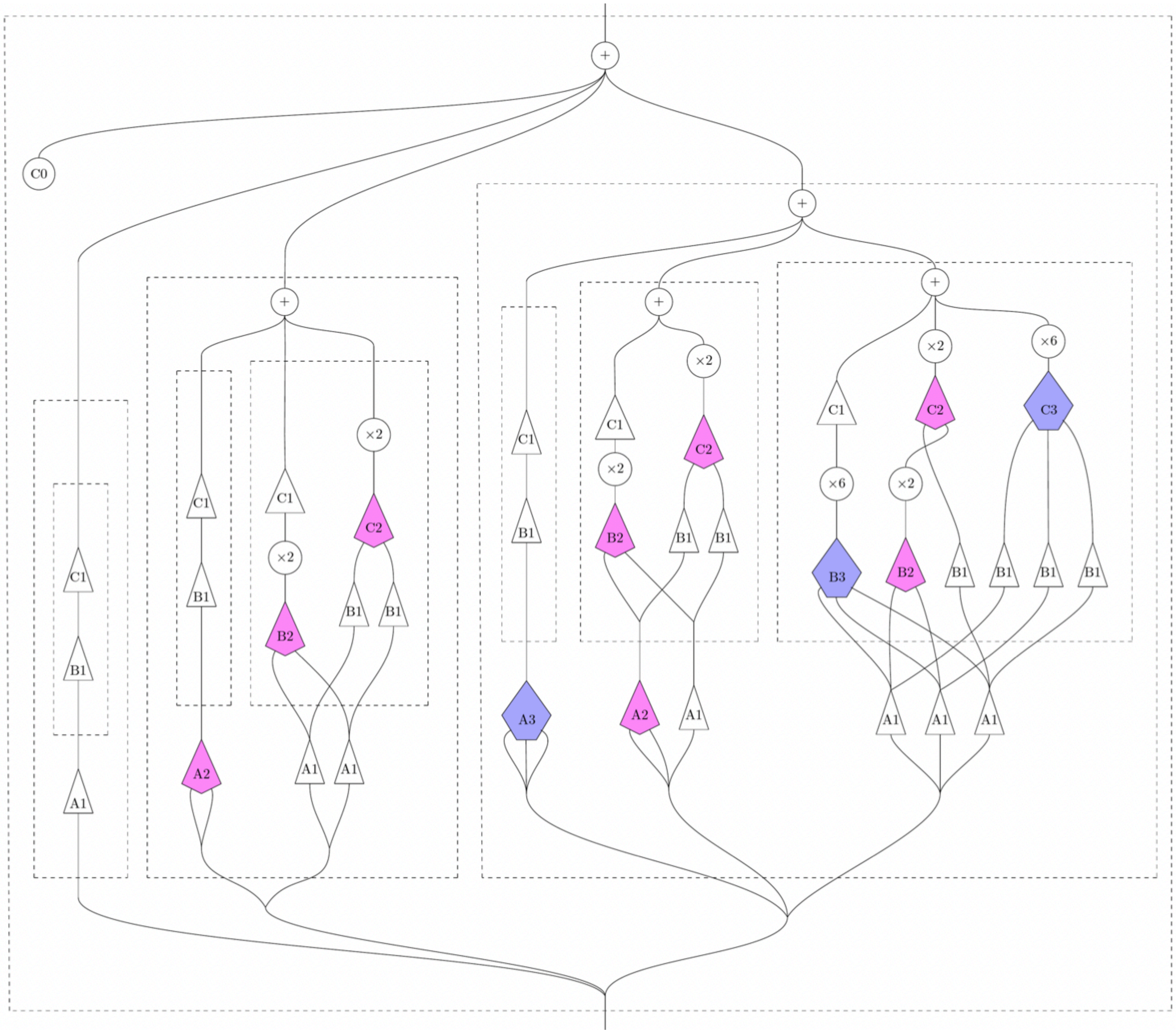

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Part 4: Time-Frequency Analysis

Time-frequency concerns the joint localization of signals. Used to analyze non-stationary signals, whose spectra are time-varying. Connections to QM and symplectic geometry.

See, e.g., *Explorations in time-frequency analysis* by Patrick Flandrin

Core object: the *Wigner-Ville distribution*

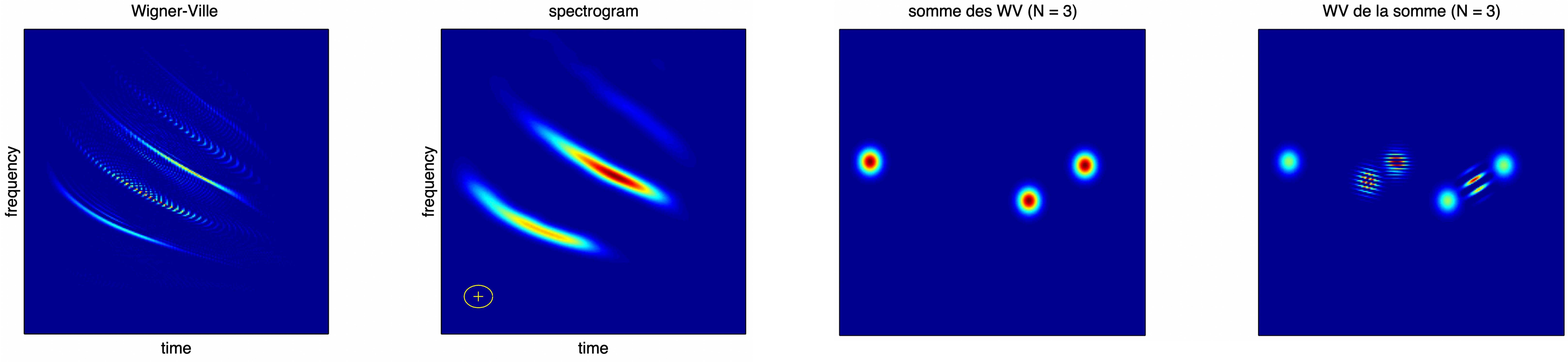
$$W_x(s, f) = \int x(s + \frac{\tau}{2}) x^*(s - \frac{\tau}{2}) e^{-i2\pi f \tau} d\tau$$

$$x(t) = A(t)e^{i\varphi(t)}$$

$$\varphi(t) = \sum_{p=0}^{P} a_p t^p$$

$$f_{\text{inst}}(t) = \varphi'$$

- perfectly localizes *linearly frequency-modulated signals* (quadratic-phase chirps) along their instantaneous frequency.[1]



[2]

1 x here should be analytic: $x(t) = s(t) + iH(s)(t)$

2 *Data driven time-frequency analysis*, Patrick Flandrin

A broad class of time-frequency distributions, *Cohen's class*, can be represented by Volterra series.[1]

But we need multivariate, parameterized Volterra series.

[1] *Volterra series representation of time-frequency distributions*, Powers and Nam

Multivariate and parameterized Volterra series

Second-order, double (or bivariate) Volterra series:

$$y(t) = H_2[x_u, x_v] = \int \int h_2(u, v) \cdot x_a(t-u) x_b(t-v) du\, dv$$

Volterra series with parameterized kernel function:

$$V(s)(t, \theta) = y(t, \theta) = \int_{\boldsymbol{\tau}_2 \in \mathbb{R}^2} v_{2,\theta}(\tau_1, \tau_2) \cdot x_{\tau_1}(t - \tau_1) x_{\tau_2}(t - \tau_2) d\tau_1\, d\tau_2$$

Volterra series form of the Wigner-Ville distribution

The WVD as a Volterra series is

$$W(t,f) = \int\int 2e^{-2\pi i f(u-v)}\delta(u+v)\cdot s^*(t-u)\cdot s(t-v)dudv$$

$$= \int\int \delta(f+\frac{1}{2}f_1-\frac{1}{2}f_2)\cdot e^{2\pi i(f_1+f_2)t}\cdot S^*(-f_1)S(f_2)df_1df_2.$$

The parameterized kernel function is

$$v_{2,\theta}(\tau_1,\tau_2) = \delta(\tau_1+\tau_2)e^{-2\pi i\theta(\tau_1+\tau_2)}$$

Conclusions

Volterra series model nonlinear systems; generalize LTI to the nonlinear regime;

- are functorial over $S'(\mathbb{R})$

Morphisms of VS model how nonlinear systems change

- and are natural (under certain restrictions)

VS and their morphisms assemble into a category, *Volt*

- whose monoidal products model ways of interconnecting VS

Core time-frequency transforms can be represented within *Volt*

Extensions and generalizations

- graph Volterra series[0]; topological signal processing[1, 2]
- explore categorical structure of *Volt*; connections to Poly[3]?
- study key transforms within *Volt*
- nonlinear system identification[4]; system decomposition

[0] *Topological Volterra Filters*, Leus et al.

[1] *Topological Signal Processing*, Michael Robinson

[2] *Topology in Sound Synthesis and Digital Signal Processing--DAFx2022 Lecture Notes*, Georg Essl

[3] *Polynomial Functors: A Mathematical Theory of Interaction*, Niu and Spivak

[4] *Volterra Neural Networks*, Krim et al.

1

Thanks for listening.

2

reference: https://arxiv.org/abs/2308.07229v4

questions, feedback:
jaraujosimon@protonmail.com

1 Hélène Vogelsinger, Illumination

2 Jimi Hendrix, photo origin unknown

Symmetric kernel functions

Volterra series kernel functions are assumed symmetric; they can be symmetrized via

$$\hat{v}_j^{\mathrm{sym}}(\Omega_j) = \frac{1}{n^*(\Omega_j)} \sum_{\sigma \in S_j} \hat{v}_j(\sigma(\Omega_j))$$

$$n^*(\Omega_j) = \binom{j}{n_\iota(\omega_1), \ldots, n_\iota(\omega_j)} = \frac{j!}{n_\iota(\omega_1)! \cdot \cdots \cdot n_\iota(\omega_j)!}$$

$$n_\iota(\omega_i) = \left|\iota^{-1}(\omega_i)\right|.$$

Multivariate Volterra series

$$V : S(\mathbb{R})^B \to S(\mathbb{R})^A$$

$$V^{(a)}[U](t) = y^{(a)}(t) = \sum_{j}^{\infty} y_j^{(a)}(t)$$

$$y_j^{(a)}(t) = \sum_{\tilde{f} \in U^j / S_j} \binom{j}{n_f(u_1), \ldots, n_f(u_B)} \int_{\boldsymbol{\tau}_j \in \mathbb{R}^j} v_{j,a}^{\mathrm{sym},\tilde{f}}(\boldsymbol{\tau}_j) \prod_{i=1}^{j} u_{f(i)}(t - \tau_i) d\tau_i$$

$$v_{j,a}^{\mathrm{sym},\tilde{f}}(\boldsymbol{\tau}_j) = n^*(\boldsymbol{\tau}_j) \sum_{\sigma \in S_j} v_{j,a}^{f}(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(j)})$$

An important combinatorial identity:

$$\sum_{p \in \binom{j+k-1}{j}} \binom{j}{\alpha_1^p, \alpha_2^p, \ldots, \alpha_k^p} = k^j$$

Projection-Slice theorem and Radon transform

$$F_1 P_1 = S_1 F_2$$

P_1 - Proj.

S_1 - Slice

$$\hat{y}_j(\omega) = \int_{\mathbf{\Omega}_j \in \mathbb{R}^j \,|\, \Sigma \mathbf{\Omega}_j = \omega} \hat{v}_j(\mathbf{\Omega}_j) \prod_{q=1}^{j} \hat{s}(\omega_q) d\omega_q$$

Integrating over *hyperplanes* in the frequency domain.

Non-commutativity of time- and frequency-shifts:

$$M_\nu T_\tau = e^{-2\pi i \tau \nu} T_\tau M_\nu$$