Typing Tensor Calculus in 2-Categories (I)

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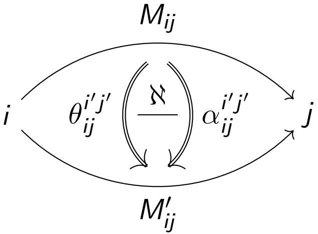
Outline

Big Picture



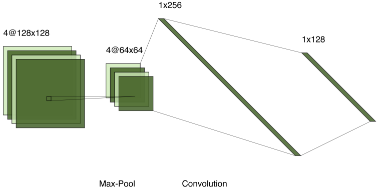
$$\aleph^{(iji'j')_\alpha}_{(iji'j')_\theta} : \theta_{ij}^{i'j'} \Rrightarrow \alpha_{ij}^{i'j'}$$

$\cdots$

Figure: n-morphisms are tensors of rank 2^n.

Motivation

- Tensor calculus appears in many domains (physics, ML, etc.)



- Traditional notation is index-heavy; one needs to keep track of indices for different calculations.

$$T \times P = \sum_{j_2, k_2} T^{j_1 j_2 \dots j_m}_{i_i i_2 \dots i_n} P^{l_1 l_2 \dots l_q}_{k_1 k_2 \dots k_w}$$

- Categorical typing of tensors offers abstraction and index-free typing; (multiplication=composing two arrows).

Typed vs. Untyped Systems

- In **typed systems**, operations are only allowed when they make sense type-wise. For example, you can't add a number to a logical proposition.
- In **untyped systems**, everything is more flexible (and risky). You can try to do anything, but it might lead to contradictions or undefined behaviour.

JavaScript example

let result = "The answer is " + 42;

Output: "The answer is 42"

Category of Matrices is the suitable primary category for **typing** elements of linear algebra.

- **Objects:** Natural numbers, 0, 1, 2, ...
- **Morphisms:** Matrices of a finite fields, $Mat_{\mathbb{F}}$.

$$M_{q\times p} = \begin{bmatrix} m_{11} & \dots & m_{1p} \\ \dots & \dots & \dots \\ m_{q1} & \dots & m_{qp} \end{bmatrix}, M : p \longrightarrow q$$

- **Composition:** Matrix multiplication.

$$p \xrightarrow{M_{q\times p}\in hom(p,q)} q, \qquad p \xrightarrow{M_{q\times p}\in hom(p,q)} q \xrightarrow{N_{q\times w}\in hom(q,w)} w$$

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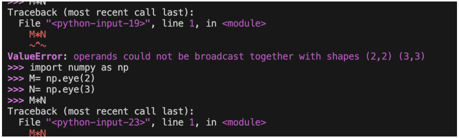
- **type Check:**

$$M_{q\times p} * M_{z\times w}, \qquad \text{if} \;\; p = z$$



Vectorisation

Vectorisation is a common practice in many algorithms including Machine Learning; for example, if you are working with Neural Networks, you need to transfer the last layer to a vector. Also in the hardware level, some components work better with vectors than matrices.

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix}, \qquad \mathbf{vec} f = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \end{bmatrix}$$

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The vectorization map, **currying**?

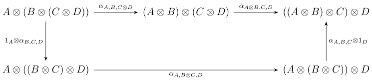
$$\mathbf{vec} :: (3 \xleftarrow{f} 2 \times 1) \leftarrow (3 \times 2 \xleftarrow{\mathbf{vec} f} 1)$$

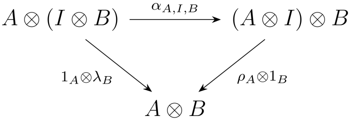
Generalisation to semiadditive monoidal categories

Definition

A **monoidal category** $\mathcal{C}$ is a category with a bifunctor, $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that for every three objects, there exists a (natural) isomorphism $a_{A,B,C}$ such that it satisfies the pentagonal and tiangle equations.

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$





Biproducts in Categories (strong version)

Definition

An object is a **zero object** if it is both initial and terminal objects. Morphisms that factorize through the zero object are **zero morphisms**.

Definition

Algebraic Definition: In a category whose hom-sets are commutative monoids, a *biproduct* of a pair of objects (A, B) is a tuple $(P = A \oplus B, p_A : P \longrightarrow A, p_B : P \longrightarrow B, i_A : A \longrightarrow P, i_B : B \longrightarrow P)$ where 0 are zero morphisms, such that:

$$p_A i_A = id_A, \quad p_B i_B = id_B, \quad p_A i_B = 0_{A,B}, \quad p_B i_A = 0_{B,A},$$
$$i_A p_A + i_B p_B = id_{A \oplus B}$$

Definition

In a locally pointed category, the **canonical morphism** between a coproduct of a pair of objects A_1, A_2, i.e. $A_1 \sqcup A_2$ and a product $A_1 \times A_2$ is morphim r such that it satisfies $p_k r i_j = \delta_{k,j}$.

$$A_j \xrightarrow{i_j} A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2 \xrightarrow{p_k} A_k, \qquad \text{if} \quad i, j \in \{1, 2\}$$

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Definition

In a category with zero morphisms, a pair of objects A and B has a **biproduct** if the canonical morphism r is an isomorphism.

Theorem

The limit-form and algebraic form of biproducts are equivalent.

Proof.
Algebraic $\Rightarrow$Limit-form: One needs to show that (P, p_A, p_B) is a product and (P, i_A, i_B) is a coproduct. Meaning, they satisfy the universality condition.
Algebraic $\Leftarrow$Limit-form: One needs to show the canonical morphism is an identity morphism.

$$A_j \xrightarrow{i_j} A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2 \xrightarrow{p_k} A_k, \qquad \text{if} \quad i, j \in \{1, 2\}$$

Semiadditive Categories

Definition

Semiadditive category is a category with finite biproducts.

Monoidal Semiadditive Categories

Definition

Semiadditive category is a category with finite biproducts.

Definition

A **monoidal semiadditive category** is a category with finite biproducts and a monoidal product such that the monoidal product distributes over biproducts.

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$$

Typing Matrices and Vectorisation

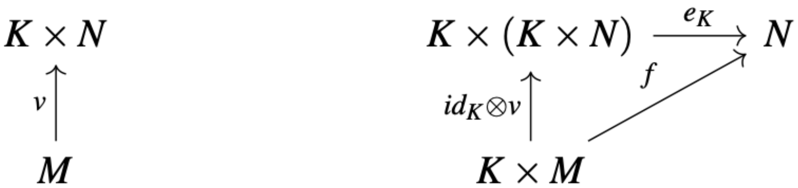
In a monoidal semiadditive category, we then have the same structures as Category of $Mat_{\mathbb{F}}$. The vectorisation is also more general, as one can potentially transfer matrices of different shapes to each other.

$$f : A_1 \oplus A_2 \longrightarrow B_1 \oplus B_2 \oplus B_3, \qquad f_{ij} : A_i \longrightarrow B_j$$

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix}, \qquad \mathbf{vec} f = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \end{bmatrix}$$

Typing Matrices and Vectorisation

$$v = \mathbf{vec}_K f \Leftrightarrow f = e_K(id_K \otimes f)$$



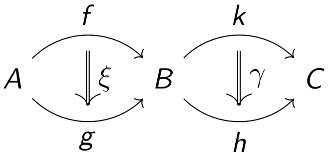
K is called the thinning factor!

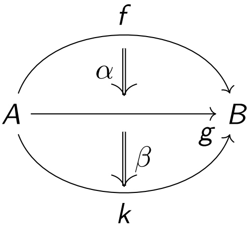
Other operations in terms of vectorisation: Having a matrix $N \xleftarrow{f} K \times 1$, we start by first de-vectorization, $1 \xleftarrow{g=\mathbf{unvec}f} N \times K$, then vectorization, $N \times K \xleftarrow{h=\mathbf{vec}g} 1$ and finally, $K \xleftarrow{\mathbf{unvec}h} N$. Hence,

$$f^T = \mathbf{unvec}(\mathbf{vec}(\mathbf{unvec}f)))$$

2-Categories

- Objects, 1-morphisms $A \xrightarrow{f} B$, 2-morphisms $\gamma : f \Rightarrow g$.
- Composition: horizontal $\gamma \circ \xi$ and vertical $\beta \odot \alpha$.

$\gamma \circ \xi =$ 

$\beta \odot \alpha =$ 

- Compositions are strict.

- We follow the same procedure, first defining the algebraic form of biproducts and then the limit-form definition.
- To define limit in 2-categories, let us first define products and coproducts. One can consider four possible limits in 2-categories: **strict, weak, lax, and oplax**. We define the weak version here.

Weak Products in 2-Categories

Definition

In a 2-category, a *weak 2-product* of a pair of objects A, B is an object $A \times B$ equipped with 1-morphism projections $(p_A : A \times B \longrightarrow A, p_B : A \times B \longrightarrow B)$ such that:

- for every cone $(X, f : X \longrightarrow A, g : X \longrightarrow B)$, there exist a 1-morphism $b : X \longrightarrow A \times B$ and 2-isomorphisms $\{\xi\}$ such that $(\xi_A : p_A b \Longrightarrow f, \xi_B : p_B b \Longrightarrow g)$ (the red cone in Figure 4).
- Moreover, for any other cone $(X, f' : X \longrightarrow A, g' : X \longrightarrow B)$ with a corresponding 1-morphism $b' : X \longrightarrow A \times B$ and 2-isomorphisms $(\xi'_A : p_A b' \Longrightarrow f', \xi'_B : p_B b' \Longrightarrow g')$ (the blue cone in Figure 4) and given 2-morphisms $(\Sigma_A : f \Longrightarrow f', \Sigma_B : g \Longrightarrow g')$,

there exists a unique 2-morphism $\gamma : b \Longrightarrow b'$ which satisfies the following condition:

$$(p_A \gamma) = ({\xi'}_A)^{-1} \odot \Sigma_A \odot (\xi_A) \tag{1}$$

Weak Products in 2-Categories

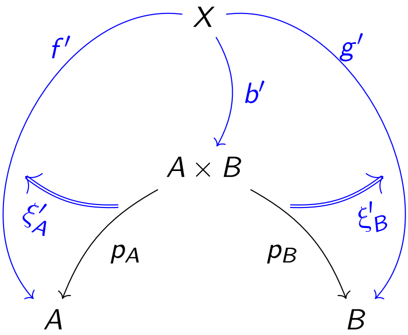


Figure: Weak 2-product in 2-categories.

Weak Products in 2-Categories

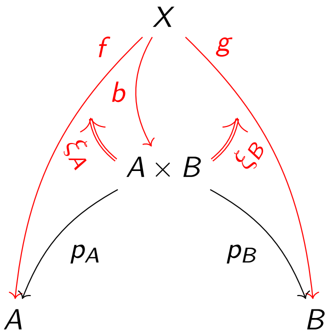


Figure: Weak 2-product in 2-categories.

Weak Products in 2-Categories

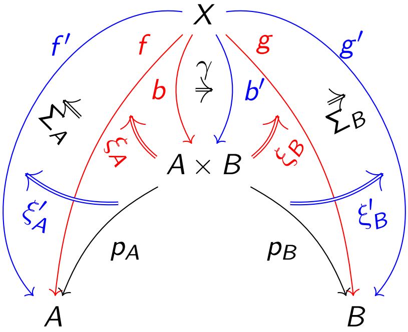


Figure: Weak 2-product in 2-categories.

Some Definition

Definition

A *locally semiadditive 2-category* is a 2-category whose hom-categories are semiadditive(have finite biproducts defined in categories).

Definition

A *locally semiadditive and compositionally distributive 2-category* is a locally semiadditive 2-category whose 2-morphisms distribute over addition of 2-morphisms. That is Equations 2 and 3 hold in a compositionally distributive 2-category:

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta \tag{2}$$

$$\alpha \odot (\beta + \gamma) = \alpha \odot \beta + \alpha \odot \gamma \tag{3}$$

Biproduct in 2-Categories

Definition

In a locally semiadditive and compositionally distributive 2-category, a *weak 2-biproduct* of a pair of objects (A, B) is a tuple

$$(A \boxplus B, p_A, p_B, i_A, i_B, \theta_A, \theta_B, \theta_{AB}, \theta_{BA}, \theta_P)$$

such that:

- *1-Morphism projections and injections:*

$$p_A : A \boxplus B \longrightarrow A, \qquad p_B : A \boxplus B \longrightarrow B$$
$$i_A : A \longrightarrow A \boxplus B, \qquad i_B : B \longrightarrow A \boxplus B$$

- *Weakening 2-isomorphisms:*

$$\theta_A : p_A i_A \Rightarrow id_A, \theta_B : p_B i_B \Rightarrow id_B, \theta_{BA} : p_B i_A \Rightarrow 0_{B,A}, \theta_{AB} : p_A i_B \Rightarrow 0_{A,B}$$
$$\theta_P : i_A p_A \oplus i_B p_B \Rightarrow id_{A \boxplus B}$$

- *Conditions for 2-biproducts:*

$$p_A \theta_P i_A = \begin{bmatrix} (p_A i_A)\theta_A & 0 \\ 0 & 0 \end{bmatrix}, \qquad p_B \theta_P i_B = \begin{bmatrix} 0 & 0 \\ 0 & (p_B i_B)\theta_B \end{bmatrix} \tag{4}$$

Definition

In a locally semiadditive and compositionally distributive 2-category, a *canonical 1-morphism* between a 2-coproduct of a pair of objects A, B, i.e. $A \sqcup B$ and a 2-product $A \times B$ is a 1-morphism which satisfies $\theta_{k,j} : p_k r i_j \Rightarrow \delta_{k,j} id_j$, if $\theta_{k,j}$ are 2-isomorphisms.

$$A_j \xrightarrow{i_j} A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2 \xrightarrow{p_k} A_k, \quad \text{if} \quad j, k \in \{1, 2\}$$

Limit-Form Definition

Definition

In a locally semiadditive and compositionally distributive 2-category, a pair of objects A and B has a *weak 2-biproduct* if the canonical 1-morphism r is an equivalence and satisfies the following conditions:

$$A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2, \qquad A_1 \sqcup A_2 \xleftarrow{r'} A_1 \times A_2$$

$$\xi_{A\times B} : rr' \Rightarrow id_{A\times B}, \qquad \xi_{A\sqcup B} : id_{A\sqcup B} \Rightarrow r'r$$

$$(r'\xi_{A\times B}) \odot (\xi_{A\sqcup B} r') = 1_{r'}, \qquad (\xi_{A\times B} r) \odot (r\xi_{A\sqcup B}) = 1_r$$

Biproducts in 2-Categories

Theorem

In a locally semiadditive and compositionally distributive 2-category, the following conditions for a pair of objects A and B are equivalent:

1. *the weak 2-product (P, p_A, p_B) of A, B exists.*
2. *the weak 2-coproduct (P, i_A, i_B) of A, B exists.*
3. *the weak 2-biproduct $(P, p_A, p_B, i_A, i_B, \theta_A, \theta_B, \theta_{AB}, \theta_{BA}, \theta_P)$ of A, B exists.*

Biproducts in 2-Categories

Proof. $1 \implies 3$: Assuming a pair of objects A, B has a weak 2-product P, we want to show P is also the weak 2-biproduct. Check the universal property of weak 2-products for $(A, {}_A : A \longrightarrow A, 0_{B,A} : A \longrightarrow B, i_A : A \longrightarrow P)$ and $(B, {}_B : B \longrightarrow B, 0_{A,B} : B \longrightarrow A, i_B : B \longrightarrow P)$. From the definition of 2-products, we know that there exist two 2-morphisms $(\gamma_A : p_A i_A \Longrightarrow_A, \gamma_B : p_B i_B \Longrightarrow_B)$. We let $\theta_A := \gamma_A$ and $\theta_B := \gamma_B$. To find θ_p, we check the universality condition for P and two 2-cones:

1. $(P, p_A I : P \longrightarrow A, p_B I : P \longrightarrow B, I : P \longrightarrow P), \qquad I := i_A p_A \oplus i_B p_B$
2. $(P, p_A : P \longrightarrow A, p_B : P \longrightarrow B, {}_P : P \longrightarrow P)$

2-morphisms between these two cones are as below:

$$\Sigma_A = \begin{bmatrix} \gamma_A p_A & 0 \end{bmatrix} : p_A i_A p_A \oplus p_A i_B p_B \Rightarrow p_A,$$
$$\Sigma_B = \begin{bmatrix} 0 & \gamma_B p_B \end{bmatrix} : p_B i_A p_A \oplus p_B i_B p_B \Rightarrow p_B$$

Therefore, $\theta_P = \begin{bmatrix} i_A \Sigma_A \\ i_B \Sigma_B \end{bmatrix} = \begin{bmatrix} i_A \gamma_A p_A & 0 \\ 0 & i_B \gamma_B p_B \end{bmatrix}$.

Proof Continues

We need to now show it satisfies the condition expressed earlier in the definition. Since this is the horizontal composition, projection and injection sandwich θ_A and θ_B.

$$p_A \theta_P i_A = \begin{bmatrix} (p_A i_A)\theta_A(p_A i_A) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (p_A i_A)\theta_A & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition

A (weak)semiadditive 2-category is a locally semiadditive and compositionally distributive 2-categories whose objects have finite 2-biproducts.

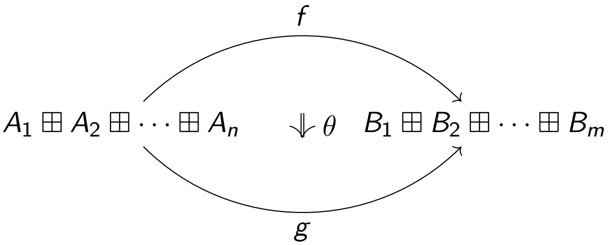
Definition

A monoidal semiadditive 2-category is a semiadditive 2-category with a monoidal product where the monoidal product and 2-biproduct are compatible; i.e. the monoidal product, $\bigotimes$, distributes over biproduct $\boxplus$.

Projections and Injections at 2-Levels

- **Semiadditive 2-Categories**: 2-Biproducts $(A \boxplus B)$ at this level have 1-morphisms projections and injections indexed by objects $(p_A : A \boxplus B \longrightarrow A, i_A : A \longrightarrow A \boxplus B)$.
- **Semiadditive Hom-categories**: biproducts $(f \oplus g)$ at this level have 2-morphisms projections and injections indexed by 1-morphisms, $(\pi_f : f \oplus g \longrightarrow f, \nu_f : f \longrightarrow f \oplus g)$.

$$A_1 \boxplus A_2 \boxplus \cdots \boxplus A_n \;\underset{g}{\overset{f}{\rightrightarrows}}\; B_1 \boxplus B_2 \boxplus \cdots \boxplus B_m \qquad \Downarrow \theta \tag{5}$$



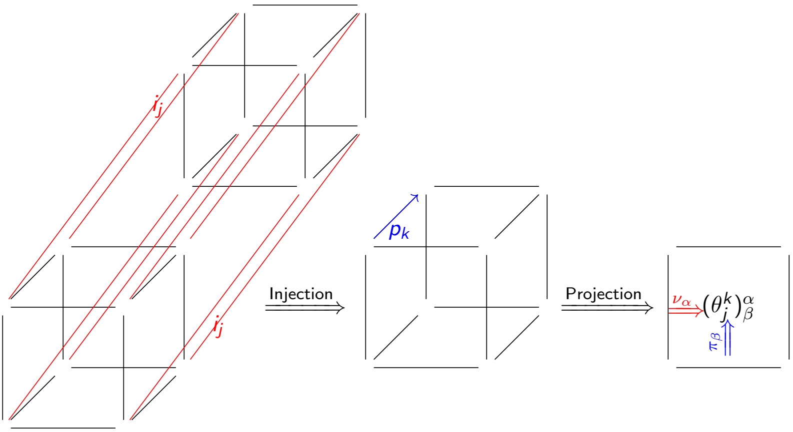
(6)

$$f_{jk} : A_j \longrightarrow B_k$$

$$f_{jk} = \bigoplus_{i=0}^{p} h_i \qquad\qquad g_{jk} = \bigoplus_{l=0}^{z} g_l \tag{7}$$

$$\theta_{jk} = \sum_{\alpha=0}^{p} \sum_{\beta=0}^{z} (\theta_{jk})^{\alpha\beta} (\nu_\alpha \otimes \pi_\beta) \tag{8}$$

Tensors in 2-Categories



Vectorisation in 2-Categories

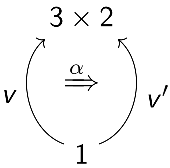
- External **currying** for 1-morphisms and 2-morphisms.
- Internal **currying** (in Hom-categories) for 2-morphisms.

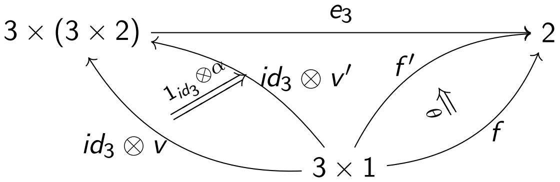
$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix}, f' = \begin{bmatrix} f'_{11} & f'_{12} \\ f'_{21} & f'_{22} \\ f'_{31} & f'_{32} \end{bmatrix}, \theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \\ \theta_{31} & \theta_{32} \end{bmatrix}$$

Currying

$$v = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \end{bmatrix}, v' = \begin{bmatrix} f'_{11} \\ f'_{12} \\ f'_{21} \\ f'_{22} \\ f'_{31} \\ f'_{32} \end{bmatrix}, \alpha = \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{21} \\ \theta_{22} \\ \theta_{31} \\ \theta_{32} \end{bmatrix} \tag{9}$$

External Currying





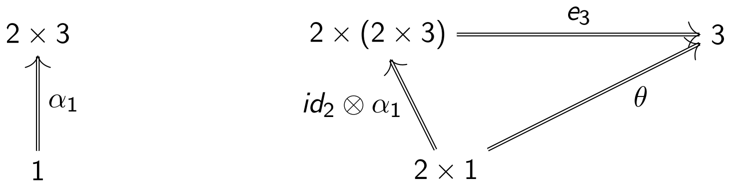
(10)

$$e_3(id_3 \otimes v) = f, e_3(id_3 \otimes v') = f', 1_{e_3} \circ (id_3 \otimes \alpha) = \theta \qquad (11)$$

$$v = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \end{bmatrix}, v' = \begin{bmatrix} f'_{11} \\ f'_{12} \\ f'_{21} \\ f'_{22} \\ f'_{31} \\ f'_{32} \end{bmatrix}, \alpha = \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{21} \\ \theta_{22} \\ \theta_{31} \\ \theta_{32} \end{bmatrix} \tag{12}$$

The consider,

$$\theta_{11} = \begin{bmatrix} \theta_{11}^1 & \theta_{11}^2 & \theta_{11}^3 \\ \theta_{11}^4 & \theta_{11}^5 & \theta_{11}^6 \end{bmatrix}$$


2×3
α_1
1
$2 \times (2 \times 3)$
e_3
3
$id_2 \otimes \alpha_1$
θ
2×1

(13)

$$v = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \end{bmatrix}, v' = \begin{bmatrix} f'_{11} \\ f'_{12} \\ f'_{21} \\ f'_{22} \\ f'_{31} \\ f'_{32} \end{bmatrix}, \alpha = \begin{bmatrix} \theta^1_{11} \\ \theta^2_{11} \\ \theta^3_{11} \\ \theta^4_{11} \\ \theta^5_{11} \\ \theta^6_{11} \\ \theta_{12} \\ \theta_{21} \\ \theta_{22} \\ \theta_{31} \\ \theta_{32} \end{bmatrix} \tag{14}$$

The consider,

$$\theta_{11} = \begin{bmatrix} \theta^1_{11} & \theta^2_{11} & \theta^3_{11} \\ \theta^4_{11} & \theta^5_{11} & \theta^6_{11} \end{bmatrix}$$

Future works

- Cartesian close bicategories.
- Curry-Howard in Cartesian close bicategories.
- Potential for scalable computation and formal proofs.

References

F. R. Ahmadi, "Typing Tensor Calculus in 2-Categories (I)," arXiv:1908.01212.
https://arxiv.org/abs/1908.01212