A topos-theoretic framework for reconstruction theorems in model theory

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Logical motivation

Theorem (Ahlbrandt-Ziegler [AZ86], Coquand)

Let M, N be countable, ω-categorical structures.

- A theory is ω-categorical if any pair of countable models are isomorphic.

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Let M, N be countable, ω-categorical structures.

There is a homeomorphism of topological groups

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(N),$$

if and only if M and N are *bi-interpretable*.

- A structure is interpretable in another if it can be obtained as a definable quotient of definable subsets.

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Theorem (Ben Yaacov [BY22])

For any pair of classical theories $\mathbb{T}_1, \mathbb{T}_2$, there are *topological groupoids* $\mathbf{G}(\mathbb{T}_1)$ and $\mathbf{G}(\mathbb{T}_2)$ such that there is a homeomorphism

$$\mathbf{G}(\mathbb{T}_1) \cong \mathbf{G}(\mathbb{T}_2)$$

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if and only if $\mathbb{T}_1$ and $\mathbb{T}_2$ are *bi-interpretable*.

However, the groupoid $\mathbf{G}(\mathbb{T})$ is *not* a groupoid of models.

- ▷ How can we generalise this correspondence?
  - If M and N are not countable models, we must weaken the *homeomorphism* condition.
  - If $\mathbb{T}_1, \mathbb{T}_2$ are not atomic, we must use *topological groupoids* of models.

Theorem template

The theorems we seek to generalise involve comparing two species of data.

topological and algebraic data	**logical data**

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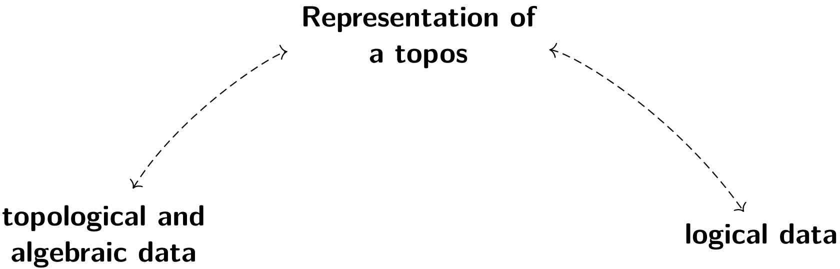


Both species of data generate a topos where they can be compared:

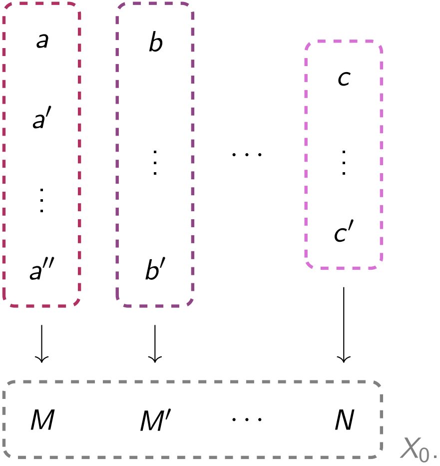
(i) each topological category generates a *topos of sheaves*,

(ii) and each (geometric) theory has a *classifying topos*.

This will form a template for our reconstruction theorems.

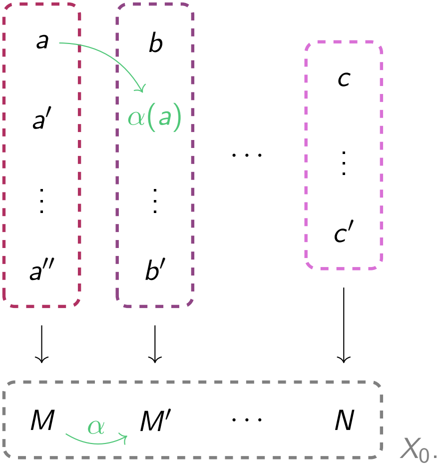
Equivariant sheaves on a topological category

Given a category $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$,



Equivariant sheaves on a topological category

Given a category $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$, equipped with an X_1-action $\beta\colon Y \times_{X_0} X_1 \to Y$,



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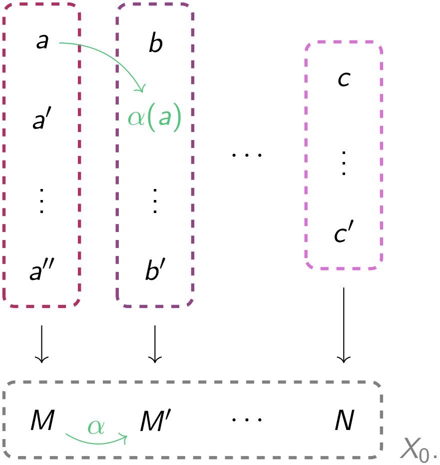

If $\mathbb{X}$ is endowed with topologies making it a *topological category*, a bundle is a *sheaf* if

(i) $q\colon Y \to X_0$ is a local homeomorphism,

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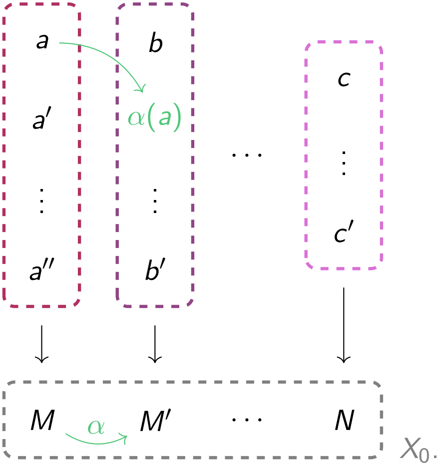

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A *morphism* of sheaves is a continuous map $f\colon Y \to Y'$ such that the following commute:

$$\begin{array}{ccc} Y \times_{X_0} X_1 & \xrightarrow{f \times_{X_0} \mathrm{id}_{X_1}} & Y' \times_{X_0} X_1 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ Y & \xrightarrow{\quad f \quad} & Y', \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & Y' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & X_0. & \end{array}$$

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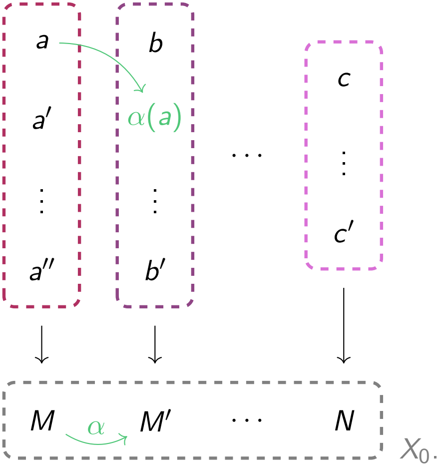

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Definition

The category of sheaves and their morphisms define a topos $\mathbf{Sh}(\mathbb{X})$.

Basic examples

(i) For every space X,

$$X \xrightarrow{\mathrm{id}_X} X \;\substack{\xrightarrow{\mathrm{id}_X} \\ \xleftarrow{\mathrm{id}_X} \\ \xrightarrow{\mathrm{id}_X}}\; X,$$

is a topological category, whose topos of equivariant sheaves is the usual topos $\mathbf{Sh}(X)$ of sheaves on X.

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(ii) If G is a topological group,

$$G \times G \xrightarrow{m} G \overset{\xrightarrow{!}}{\underset{\xrightarrow{!}}{\xleftarrow{e}}} \{*\},$$

is a topological category, whose topos of equivariant sheaves is the topos $\mathbf{B}G$ of continuous actions $G \times X \to X$ on discrete sets.

Classifying topos theory

Toposes also admit a logical description via the notion of a *classifying topos*.

Definition

Let $\mathbb{T}$ be a theory. A *classifying topos* $\mathcal{E}_{\mathbb{T}}$ for $\mathbb{T}$ is a topos that satisfies

$$\mathbb{T}\text{-}\mathbf{Mod}(\mathbf{Sets}) \simeq \mathbf{Geom}(\mathbf{Sets}, \mathcal{E}_{\mathbb{T}}).$$

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Example

Let $\mathbb{T}$ be a (classical) propositional theory, and let $X_{\mathbb{T}}$ be the associated Stone space.

Then $\mathbf{Sh}(X_{\mathbb{T}})$, the topos of sheaves on the space $X_{\mathbb{T}}$, classifies $\mathbb{T}$.

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Two theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are *Morita equivalent* if there is a natural equivalence

$$\mathbb{T}_1\text{-}\mathbf{Mod}(\mathcal{F}) \simeq \mathbb{T}_2\text{-}\mathbf{Mod}(\mathcal{F}),$$

or equivalently if $\mathcal{E}_{\mathbb{T}_1} \simeq \mathcal{E}_{\mathbb{T}_2}$.

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(e.g. see Kamsma [Ka23]).

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Proposition (McEldowney [Mc20])

If $\mathbb{T}_1, \mathbb{T}_2$ both prove that $\exists x, y\ x \neq y$, then

$\mathbb{T}_1, \mathbb{T}_2$ are bi-interpretable $\iff$ $\mathbb{T}_1, \mathbb{T}_2$ are Morita equivalent.

A topos-theoretic template

Using this framework, we can recover Ahlbrandt-Ziegler type results.

By Caramello's *Topological Galois theory* [Ca16],

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{B}\mathrm{Aut}(M)$$

$\mathbb{T}$ a complete,
ω-categorical theory

$M \vDash \mathbb{T}$
a countable model.

Hence, for complete, ω-categorical theories $\mathbb{T}_1, \mathbb{T}_2$ and countable models $M \vDash \mathbb{T}_1$, $N \vDash \mathbb{T}_2$, there is a chain of equivalences

$$\begin{aligned} \mathbb{T}_1, \mathbb{T}_2 \text{ are Morita equivalent} &\iff \mathcal{E}_{\mathbb{T}_1} \simeq \mathcal{E}_{\mathbb{T}_2}, \\ &\iff \mathbf{B}\mathrm{Aut}(M) \simeq \mathbf{B}\mathrm{Aut}(N) \end{aligned}$$

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And so we recover the classical Ahlbrandt-Ziegler result.

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Using this framework, we can recover Ahlbrandt-Ziegler type results.

We have that

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(X_{\mathbb{T}})$$

$\mathbb{T}$ a classical propositional

$X_{\mathbb{T}}$ the Stone space of models for $\mathbb{T}$.

Hence, for classical, propositional theories $\mathbb{T}_1, \mathbb{T}_2$, there is a chain of equivalences

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This is a reformulation of Stone duality.

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- Each topological groupoid $\mathbb{X}$ yields a topos $\mathbf{Sh}(\mathbb{X})$.
- For certain *representing* groupoids of models, $\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{X})$.

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$\mathbb{T}$ a geometric theory

$\mathbb{X}$ a *representing* groupoid of models.

For theories $\mathbb{T}_1, \mathbb{T}_2$ with representing groupoids $\mathbb{X}$ and $\mathbb{Y}$,

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Overview

A. We characterise which groupoids of models are *representing*.

B. We establish a *bi-equivalence* between topoi with enough points and a *localisation* of topological groupoids.

Hence, we deduce when two topological groupoids are *Morita equivalent*.

Representing groupoids overview

Theorem A (W.)

A groupoid of models represents a geometric theory if and only if

(i) it is *conservative*,

(ii) and it *eliminates parameters*.

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Overview of Part A

1. Define elimination of parameters.
2. Technically restate the classification theorem.

Indexed structures

Let M be a structure over a signature Σ.

Given a set $\mathfrak{K}$ of *parameters*, a $\mathfrak{K}$-*indexing* of M consists of:

(i) a subset $\mathfrak{K}' \subseteq \mathfrak{K}$,

(ii) and an *expansion* of M to the signature $\Sigma \cup \{ c_m \mid m \in \mathfrak{K}' \}$ such that M satisfies

$$\top \vdash_x \bigvee_{m \in \mathfrak{K}} x = c_m,$$

i.e. every $n \in M$ is the interpretation of some parameter $m \in \mathfrak{K}$.

Equivalently, this is a choice of partial surjection $\mathfrak{K} \rightharpoonup\!\!\!\!\rightharpoonup M$.

Definables

Let M be a model of $\mathbb{T}$ with an indexing $\mathfrak{K} \twoheadrightarrow M$.

(i) A *definable subset* is a subset of the form

$$[\![\vec{x} : \varphi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \varphi(\vec{n}) \} \subseteq M^n$$

for some formula $\{ \vec{x} : \varphi \}$.

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(ii) A *definable subset with parameters* is a subset of the form

$$[\![\vec{x}, \vec{m} : \psi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \psi(\vec{n}, \vec{m}) \} \subseteq M^n$$

for some formula $\{ \vec{x}, \vec{y} : \psi \}$ and a tuple of parameters $\vec{m} \subseteq \mathfrak{K}$.

Definables

For a groupoid $\mathbb{X}$ of $\mathbb{T}$-models, a $\mathfrak{K}$-indexing of $\mathbb{X}$ is a choice of $\mathfrak{K}$-indexing $\mathfrak{K} \twoheadrightarrow M$ for each $M \in \mathbb{X}$.

(i) A *definable* or *definable without parameters* is a subset of the form

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} = \{ \langle \vec{n}, M \rangle \mid \vec{n} \subseteq M \in X_0,\ M \vDash \varphi(\vec{n}) \} \subseteq \coprod_{M \in X_0} M^n$$

for some formula $\{ \vec{x} : \varphi \}$.

(ii) A *definable with parameters* is a subset of the form

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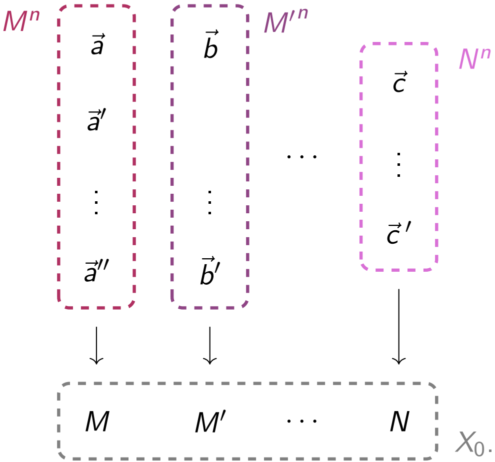
Interpreting definables and elimination of parameters

For each n, there is a bundle



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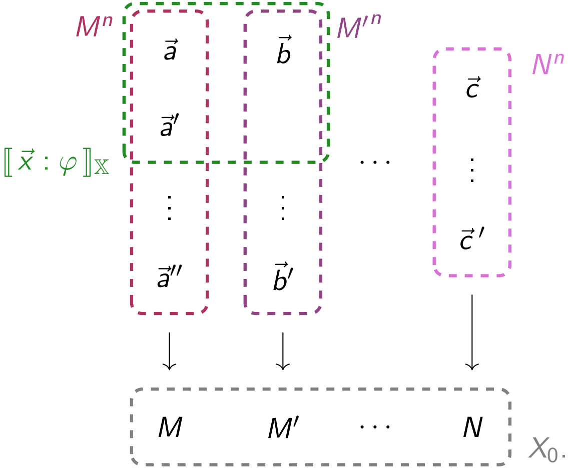
Each definable defines a subset



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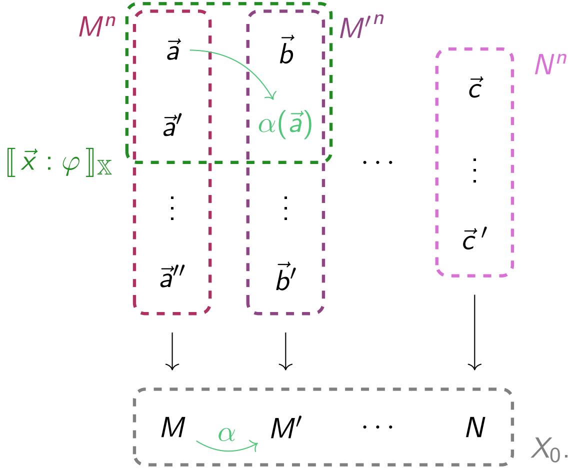
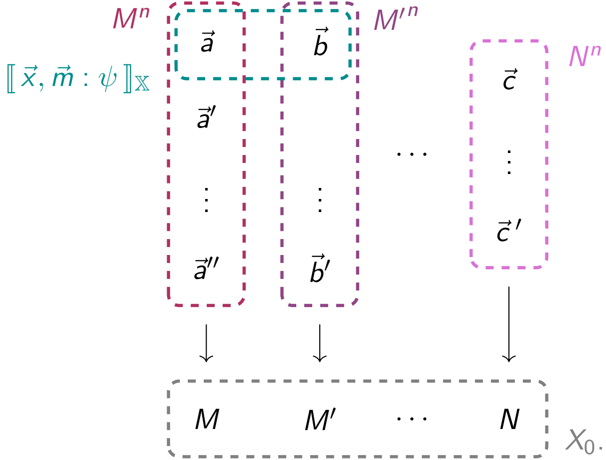


Note that $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ is *stable* under the X_1-action.

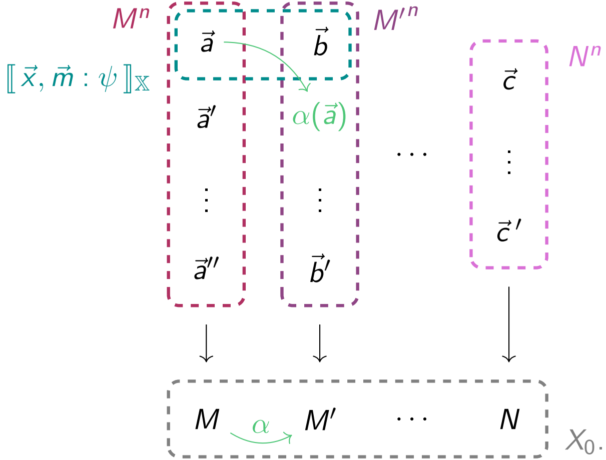
Interpreting definables and elimination of parameters

Each definable with parameters also defines a subset



Interpreting definables and elimination of parameters

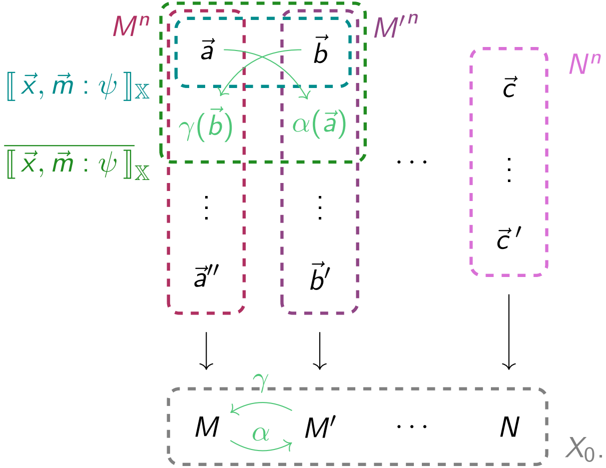
Each definable with parameters also defines a subset



However, $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ is not stable under the X_1-action.

Interpreting definables and elimination of parameters

We can consider the closure of $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ under the X_1-action



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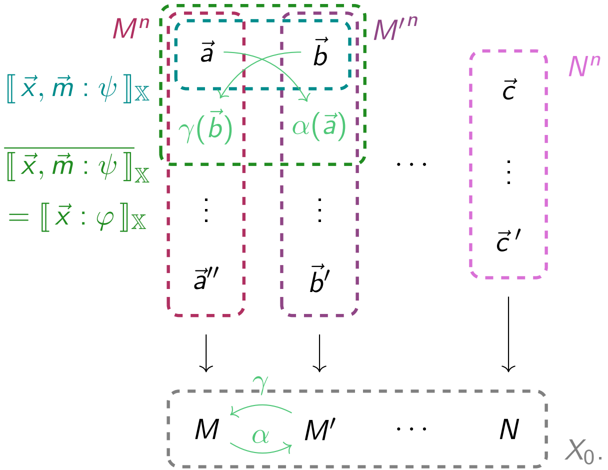

In this example, $\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$.

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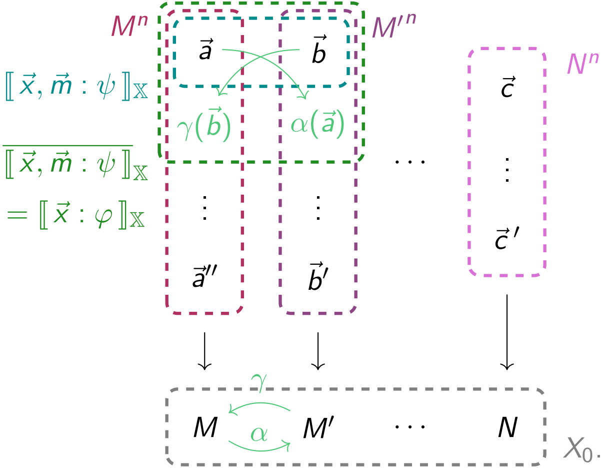

Definition

Given a groupoid $\mathbb{X}$ of $\mathbb{T}$-models and an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$,

$\mathbb{X}$ *eliminates parameters* if, for every ψ and $\vec{m}$, there exists some *geometric* formula φ such that

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

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Classification result

Theorem (W.)

Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a small groupoid of $\mathbb{T}$-models.

We can endow $\mathbb{X}$ with the structure of an **open** topological groupoid for which

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if

(i) X_0 is a conservative set –

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} = [\![\vec{x} : \chi]\!]_{\mathbb{X}} \implies \varphi \equiv^{\mathbb{T}}_{\vec{x}} \chi,$$

(ii) there is an indexing of $\mathbb{X}$ by parameters $\mathfrak{K}$ for which $\mathbb{X}$ eliminates parameters –

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

Examples

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(i) The groupoid of all $\mathfrak{K}$-indexed models eliminates parameters.

(ii) The groupoid of all $\mathfrak{K}$-*enumerated* –

every element is indexed by infinitely many parameters

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(iii) If $\mathbb{T}$ is an atomic theory, then

$$\mathrm{Aut}(M) \text{ eliminates parameters } \iff M \text{ is } \textit{ultrahomogeneous},$$

i.e. every finite partial isomorphism of M extends to a total isomorphism,

$$\begin{array}{ccc} \vec{n} & \xrightarrow{\sim} & \vec{n}' \\ \downarrow & & \downarrow \\ M & \overset{\sim}{\dashrightarrow} & M. \end{array}$$

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Thus, if $\mathbb{T}$ is a complete, atomic (i.e. ω-categorical) theory, and $M \vDash \mathbb{T}$ is countable, then

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{B}\mathrm{Aut}(M).$$

A biequivalence for topoi with enough points

Now we know when $\mathcal{E} \simeq \mathbf{Sh}(\mathbb{X})$, but when are two topological groupoids $\mathbb{X}, \mathbb{Y}$ *Morita equivalent*?

That is, when do we have $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y})$?

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- $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ is the bicategory of toposes with enough points, geometric morphisms, and natural isomorphisms,
- **LogGrpd** is a full 2-subcategory of **TopGrpd**,
- $\mathfrak{W}$ is a *bi-calculus of fractions* on **LogGrpd**, i.e. a class of weak equivalences.

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Overview of Part B

1. We first compare our localisation on the *left* with Moerdijk's localisation on the *right*.
2. We define the bicategory of *logical groupoids* and the *weak equivalences*.
3. Finally, we restate the biequivalence.

   Thereby, we deduce a groupoidal characterisation of Morita equivalence for theories.

Homomorphisms of topological groupoids

A *homomorphism* of topological groupoids is a continuous functor

$$f : \mathbb{X} \to \mathbb{Y},$$

i.e. a pair of continuous maps $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ such that

$$\begin{array}{ccc}
X_1 \times_{X_0} X_1 & \dashrightarrow & Y_1 \times_{Y_0} Y_1 \\
\downarrow m & & \downarrow m' \\
X_1 & \xrightarrow{f_1} & Y_1 \\
s \downarrow \uparrow \downarrow t & & s' \downarrow \uparrow \downarrow t' \\
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commutes.

A *transformation* $f \overset{a}{\Rightarrow} g$ between homomorphisms is a natural transformation where the map

$$a\colon X_0 \to Y_1,$$
$$x \mapsto f_0(x) \xrightarrow{a_x} g_0(x)$$

is continuous.

Together, this forms the datum of a *bicategory* **TopGrpd**, and there is a bifunctor

$$\mathbf{Sh}\colon \mathbf{TopGrpd} \to \mathbf{Topos}^{\text{iso}}_{w.e.p.} \subseteq \mathbf{Topos}.$$

Moerdijk's equivalence

Theorem (Moerdijk [Mo88], Pronk [Pr96])

There is a biequivalence

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where $\mathbf{ECG} \subseteq \mathbf{LocGrpd}$.

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Proposition

For any 2-subcategory $\mathcal{C} \subseteq \mathbf{TopGrpd}$, and any **right** bicalculus W^{-1} on $\mathcal{C}$,

$$\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \not\simeq \mathcal{C}[W^{-1}].$$

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Proof: Suppose that $\mathbf{Topos}^{\text{iso}}_{w.e.p.} \not\simeq \mathcal{C}[W^{-1}]$.

Let $\mathcal{E}$ be a topos with a large (and jointly conservative) set of points, e.g. the classifying topos for groups.

Let $\mathbb{X} \in \mathcal{C}$ be a representing groupoid for $\mathcal{E}$. By assumption there is a point $p \colon \mathbf{Sets} \to \mathcal{E}$ that is not included in X_0.

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Now p is induced by a homomorphism $f \in \mathcal{C}$, i.e.

$$\mathbf{Sets} \simeq \mathbf{Sh}(\mathbb{Y}) \xrightarrow{\mathbf{Sh}(f)=p} \mathbf{Sh}(\mathbb{X}),$$

by the hypothesis $\mathbf{Topos}^{\text{iso}}_{w.e.p.} \not\simeq \mathcal{C}[W^{-1}]$.

But any section of $\mathbf{Sh}(Y_0) \to \mathbf{Sets}$ yields a diagram

$$\begin{array}{ccc} \mathbf{Sh}(Y_0) & \xrightarrow{\mathbf{Sh}(f_0)} & \mathbf{Sh}(X_0) \\ \downarrow & & \downarrow \\ \mathbf{Sets} \simeq \mathbf{Sh}(\mathbb{Y}) & \xrightarrow{\mathbf{Sh}(f)=p} & \mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}, \end{array}$$

a contradiction. Hence, Y_0 is empty, and so $\mathbf{Sets} \simeq \mathbf{0}$.

Logical groupoids

Definition

An open topological groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ is said to be *logical* if X_0, X_1 are sober, and $\mathbb{X}$ is *étale complete* in the sense that:

(i) for any pair $x, y \in X_0$, a natural isomorphism

$$\begin{array}{ccc} \mathbf{Sets} & \xrightarrow{\ x\ } & \mathbf{Sh}(X_0) \\ {\scriptstyle y}\downarrow & \overset{\alpha}{\underset{\sim}{\swarrow}} & \downarrow{\scriptstyle u} \\ \mathbf{Sh}(X_0) & \xrightarrow[\ u\ ]{} & \mathbf{Sh}(\mathbb{X}), \end{array}$$

is instantiated by an point $\alpha \in X_1$,

(ii) the topology on X_1 is the coarsest such topology determined by $\mathbf{Sh}(\mathbb{X})$,

i.e. given another topology T on the points of X_1 yielding a topological groupoid $\mathbb{X}'$, if $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{X}')$ then T contains the topology on X_1.

We denote by **LogGrpd** $\subseteq$ **TopGrpd** the full 2-subcategory of logical groupoids.

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Remark

A topological groupoid $\mathbb{X}$ is a logical groupoid if and only if it is a representing groupoid of models, with *all possible* isomorphisms, for a theory classified by $\mathbf{Sh}(\mathbb{X})$.

Weak equivalences of logical groupoids

Definition

A homomorphism of logical groupoids

$$\psi \colon \mathbb{Y} \to \mathbb{W}$$

is a *weak equivalence* if

(i) ψ is full inclusion $\mathbb{Y} \hookrightarrow \mathbb{W}$,

(ii) Let $\mathbb{T}$ be a theory classified by **Sh**$(\mathbb{W})$.

When $\mathbb{W}$ is viewed as a representing groupoid of indexed $\mathbb{T}$-models, the subgroupoid $\mathbb{Y}$ is still conservative and eliminates parameters.

Denote the class of weak equivalences by $\mathfrak{W}$.

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Corollary

If $\psi\colon \mathbb{Y} \hookrightarrow \mathbb{W}$ is a weak equivalence, then

$$\mathbf{Sh}(\psi)\colon \mathbf{Sh}(\mathbb{Y}) \xrightarrow{\sim} \mathbf{Sh}(\mathbb{W})$$

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Proposition

Every geometric morphism

$$f \colon \mathbf{Sh}(\mathbb{X}) \to \mathbf{Sh}(\mathbb{Y})$$

is induced by a cospan

$$\begin{array}{ccc} & & \mathbb{Y} \\ & & \big\downarrow{\scriptstyle \psi} \\ \mathbb{X} & \xrightarrow{\varphi} & \mathbb{W} \end{array}$$

where ψ is a weak equivalence.

A bi-equivalence for toposes

Theorem (W.)

There is a bi-equivalence

$$[\mathfrak{W}^{-1}]\mathbf{LogGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}.$$

- An object of $[\mathfrak{W}^{-1}]\mathbf{LogGrpd}$ is a logical groupoid.
- An arrow $\mathbb{X} \xrightarrow{(\varphi,\psi)} \mathbb{Y} \in [\mathfrak{W}^{-1}]\mathbf{LogGrpd}$ is a cospan

$$\begin{array}{ccc} & & \mathbb{Y} \\ & & \big\downarrow{\scriptstyle \psi} \\ \mathbb{X} & \xrightarrow{\varphi} & \mathbb{W} \end{array}$$

  where ψ is a weak equivalence.
- A 2-cell $(\varphi, \psi) \Rightarrow (\varphi', \psi')$ is a transformation $\varphi \overset{a}{\Rightarrow} \varphi'$.

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Corollary

Two logical groupoids $\mathbb{X}, \mathbb{Y}$ are *Morita equivalent*, i.e. $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y})$, if and only if there is a co-span of weak equivalences

$$\mathbb{X} \hookrightarrow \mathbb{W} \hookleftarrow \mathbb{Y}$$

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Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be theories with representing groupoids $\mathbb{X}$ and $\mathbb{Y}$ respectively.

Then $\mathbb{T}_1$ and $\mathbb{T}_2$ are Morita equivalent if and only if there is a co-span of weak equivalences

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Future directions

(i) What *Morita-invariant* properties of topological groupoids are preserved by weak equivalences?

(ii) What is an entirely topological description of weak equivalences?

References

Based on PhD thesis, available here: `jlwrigley.github.io/`

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Thank you for listening!