The Markov category of a random graph

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Probabilistic programming interfaces for random graphs, 2024

Probabilistic Programming Interfaces for Random Graphs: Markov Categories, Graphons, and Nominal Sets

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We study semantic models of probabilistic programming languages over graphs, and establish a connection to graphons from graph theory and combinatorics. We show that every well-behaved equational theory for our graph probabilistic programming language corresponds to a graphon, and conversely, every graphon arises in this way.

We provide three constructions for showing that every graphon arises from an equational theory. The first is an abstract construction, using Markov categories and monoidal indeterminates. The second and third are more concrete. The second is in terms of traditional measure theoretic probability, which covers 'black-and-white' graphons. The third is in terms of probability monads on the nominal sets of Gabbay and Pitts. Specifically, we use a variation of nominal sets induced by the theory of graphs, which covers Erdős-Rényi graphons. In this way, we build new models of graph probabilistic programming from graphons.

CCS Concepts: • **Theory of computation** → **Semantics and reasoning**; **Probabilistic computation**.

Additional Key Words and Phrases: probability monads, exchangeable processes, graphons, nominal sets, Markov categories, probabilistic programming

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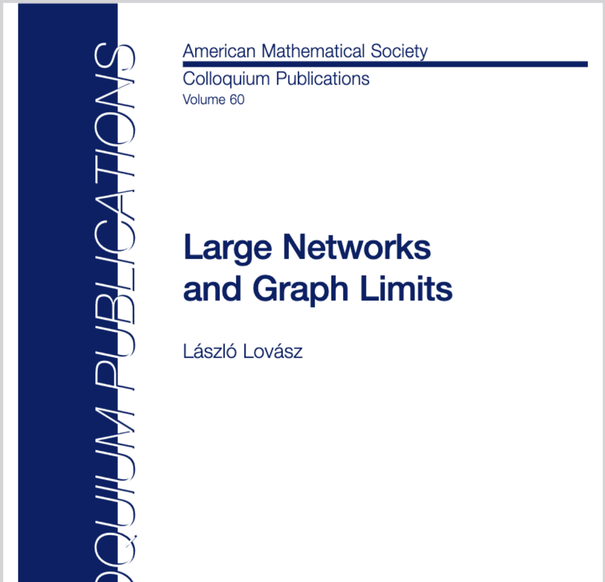
Large networks and graph limits, Lovász, 2012



Outline

Graphs

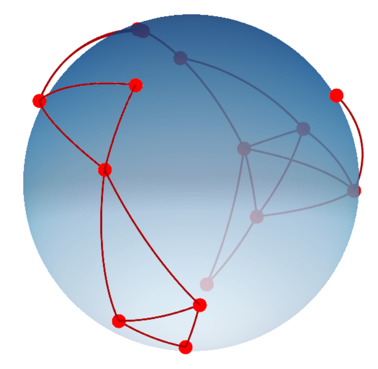
A simple graph $G = (V, E)$ is

- a set $V = V(G)$ of nodes,
- a symmetric, irreflexive binary relation $E = E(G) \subseteq V \times V$ (edges).

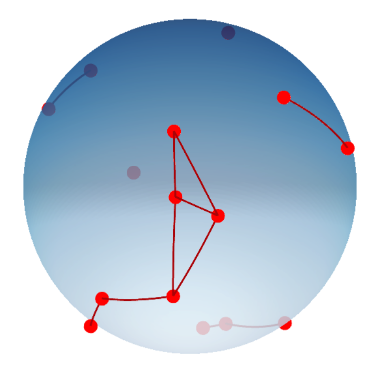
Write $\mathcal{G}_n \in \mathsf{Set}$ for the set of simple graphs G with $V(G) = [n] = \{1, \ldots, n\}$. Thus $|\mathcal{G}_n| = 2^{\binom{n}{2}}$

A large graph: proximity on a sphere

For a fixed θ, let $x, y \in S^2$ be connected if the angle between them is $< \theta$.



$\theta = \pi/3$



$\theta = \pi/6$

Graphons — a model of large graphs

A graphon* consists of

- a probability space $(\Omega, \mathcal{F}, \mu)$,
- a symmetric measurable function $W : \Omega^2 \to [0, 1]$.

- E.g. $\Omega = S^2$, μ = uniform distribution, $W(x, y) = [\![x \cdot y > \cos\theta]\!]$.
- E.g. Any finite graph G: let $\Omega = V(G)$, μ = uniform distribution, $W(u, v) = [\![uv \in E(G)]\!]$.
- Models dense rather than sparse graphs.
- Equivalence of graphons is non-trivial.

*Usually restricted to $(\Omega, \mathcal{F}, \mu) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$.

Graphons — subgraph sampling

A graphon consists of

- a probability space (Ω, μ),
- a symmetric measurable function $W : \Omega^2 \to [0, 1]$.

- Induce a distribution on graphs $\mathcal{G}_n$ as follows.
  - Sample a vector $\vec{x} \in \Omega^n$ using $\mu^{\otimes n}$.
  - For each $1 \leq i < j \leq n$, let there be an edge $ij \in E(G)$ with probability $W(x_i, x_j)$.
- I.e.,

$$\mathbb{P}_{W,n}(G) = \int_{x_1 \in \Omega} \cdots \int_{x_n \in \Omega} \prod_{ij \in E(G)} W(x_i, x_j) \times \prod_{ij \notin E(G)} (1 - W(x_i, x_j)) \mu(dx_n) \ldots \mu(dx_1).$$

Random graph models

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

- exchangeable: each p_n is invariant under permutations of $\{1, \ldots, n\}$,
- consistent: $p_n(G) = \sum p_{n+1}(G')$ where G' varies over the 2^n one-node extensions of G,
- local: $p_h(G_1)p_k(G_2) = \sum p_{h+k}(G')$ where G' varies over all 2^{hk} graphs on $\{1, \ldots, h+k\}$ that restrict to G_1 on $\{1, \ldots, h\}$ and G_2 on $\{h+1, \ldots, h+k\}$.

Basic result: a graphon gives rise to a random graph model, and conversely every random graph model arises from a graphon.

Aside: countable random graph models and graph limits

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

- exchangeable,
- consistent,
- local.

- We could also use a graphon to sample a graph on vertex set $\mathbb{N}$.
- Can also consider graphons as 'graph limits':

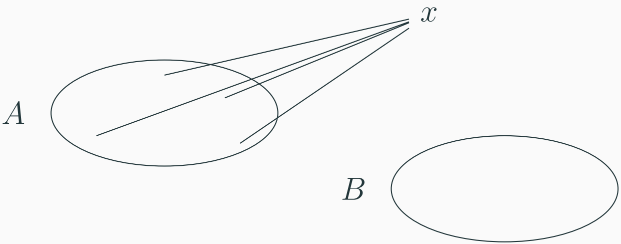
$$\mathsf{FinGraph} \to [0,1]^{\mathsf{FinGraph}}$$

$$G \mapsto \lambda H. \frac{|\operatorname{hom}(H,G)|}{|V(G)|^{|V(H)|}}.$$

The Rado graph

The Rado graph is the unique countable graph with the extension property:

> For all finite disjoint $A, B \subseteq V(G)$, $\exists x \in V(G) \setminus (A \cup B)$ with $\forall y \in A. xy \in E(G)$ and $\forall z \in B. xz \notin E(G)$.



The Erdős-Rényi model

For $0 < \alpha < 1$, let $\Omega = \{*\}$ and $W_\alpha(*,*) = \alpha$.

Theorem (Erdős-Rényi)

With probability 1, the countable graph sampled from W_α is isomorphic to the Rado graph.

- N.B. The W_α are different random graph models:

$$\mathbb{P}_{W_\alpha,n}(G) = \alpha^{|E(G)|} \cdot (1-\alpha)^{\binom{n}{2}-|E(G)|}.$$

- Impossible to present with a black-and-white ($\{0,1\}$-valued) graphon.

Summary

A graphon consists of

- a probability space $(\Omega, \mathcal{F}, \mu)$,
- a symmetric measurable function $W : \Omega^2 \to [0,1]$.

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

▶ exchangeable, ▶ consistent, ▶ local.

- We have a model of 'large graphs with a measure on the space of vertices'.
- Next: we give a more categorical presentation of random graph model.
- Then: identify random graph models with black-and-white graphons internal to Markov categories, i.e. with equational semantics for programming language with a 'graph interface'.

Outline

Recap of finite distributions monad

- Let Fin = a skeleton of the category of finite sets and functions.
- Finite distributions form a functor $D : \mathsf{Fin} \to \mathsf{Set}$ where

$$D(n) = \left\{ \alpha \in [0,1]^n \mid \sum \alpha(i) = 1 \right\}.$$

- (Abstract) clone with

$$\eta_n : n \to D(n) \qquad \eta_n(i) = \delta_i = \lambda j. [\![i = j]\!]$$

$$(\gg\!=) : D(m) \times D(n)^m \to D(n) \qquad \alpha \gg\!= (\beta_i)_i = \textstyle\sum_i \alpha(i) \cdot \beta_i$$

- clones ≃ finitary monads (left Kan extend to an endofunctor on Set).

Commutative clones

- A clone T is commutative if for all $\alpha \in T(m)$, $\beta \in T(n)$

$$\alpha \gg= (\lambda i.\beta \gg= (\lambda j.\eta(i,j))) = \beta \gg= (\lambda j.\alpha \gg= (\lambda i.\eta(i,j)))$$

  in $T(m \times n)$.

- Define $\alpha \star \beta$ to be the common value of the two sides. Then

$$(\star) : T(m) \times T(n) \to T(m \times n)$$

  with $\eta_1 : 1 \to T(1)$ makes T a symmetric monoidal functor.

- For finite distributions, actually $1 \cong D(1)$ and

$$\alpha \star \beta = \lambda(i,j).\alpha(i)\beta(j)$$

The category of finite kernels

Let FinStoch = Fin_D be the Kleisli category of D.

This is a symmetric monoidal category with a terminal unit, with an identity-on-objects symmetric monoidal functor Fin → FinStoch.

- Maps $m \to n$ in FinStoch are matrices $A : [m] \times [n] \to [0,1]$ with $\sum_j A_{ij} = 1$ for all $i \in [m]$.
- Composition of $A : l \to m$ with $B : m \to n$ is given by
$$(B \circ A)_{ik} = \sum_{j \in [m]} A_{ij} B_{jk}.$$
- The tensor of $A : m \to n$ with $B : m' \to n$ is given by
$$(A \otimes B)_{(i,i')(j,j')} = A_{ij} B_{i'j'}.$$

Random graph models

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

- exchangeable: [...]
- consistent: [...]
- local: [...]

In particular, it is a sequence $(p_n : 1 \to \mathcal{G}_n)$ of maps in FinStoch.

The diagram of finite graphs

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

- exchangeable: each p_n is invariant under permutations of $\{1, \ldots, n\}$,
- consistent: $p_n(G) = \sum p_{n+1}(G')$ where G' varies over the 2^n one-node extensions of G,
- local: [...]

- Let Inj $\hookrightarrow$ Fin be the wide subcategory of injections.
- $\mathcal{G}_{(-)}$ extends to a functor $\mathsf{Inj}^{\mathrm{op}} \to \mathsf{Fin} \to \mathsf{FinStoch}$.
- A cone $p : 1 \Rightarrow \mathcal{G}_{(-)}$ in FinStoch is an exchangeable, consistent sequence.

locality

A random graph model is $(p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})$ which is:

- exchangeable: [...]
- consistent: [...]
- local: $p_h(G_1)p_k(G_2) = \sum p_{h+k}(G')$ where G' varies over all 2^{hk} graphs on $\{1, \ldots, h+k\}$ that restrict to G_1 on $\{1, \ldots, h\}$ and G_2 on $\{h+1, \ldots, h+k\}$.

- $+$ on Fin restricts to monoidal structure on Inj.
- $\mathcal{G}_{(-)} : \mathsf{Inj}^{\mathrm{op}} \to \mathsf{Fin}$ is oplax monoidal:

$$(\mathcal{G}_{\iota_1}, \mathcal{G}_{\iota_2}) : \mathcal{G}_{m+n} \to \mathcal{G}_m \times \mathcal{G}_n$$

- A cone $p : 1 \Rightarrow \mathcal{G}_{(-)}$ in FinStoch is monoidal iff (p_n) is local.

Theory of graph-sampling

Cones are equivalently:

$$\lim_{n \in \mathsf{Inj}^{\mathrm{op}}} D(\mathcal{G}_n) \cong \lim_{n \in \mathsf{Inj}^{\mathrm{op}}} [\mathsf{Fin}, \mathsf{Set}](\mathsf{Fin}(\mathcal{G}_n, -), D)$$
$$\cong [\mathsf{Fin}, \mathsf{Set}](\mathrm{colim}_{n \in \mathsf{Inj}^{\mathrm{op}}} \mathsf{Fin}(\mathcal{G}_n, -), D)$$

$$\boxed{\text{Define } \mathcal{G} := \mathrm{colim}_{n \in \mathsf{Inj}^{\mathrm{op}}} \mathsf{Fin}(\mathcal{G}_n, -) : \mathsf{Fin} \to \mathsf{Set}.}$$

$\mathcal{G}$ is a commutative clone

$$\mathcal{G}(m) \times \mathcal{G}(n)^m \cong \mathrm{colim}_h \mathsf{Fin}(\mathcal{G}_h, m) \times (\mathrm{colim}_k \mathsf{Fin}(\mathcal{G}_k, n))^m$$
$$\to \mathrm{colim}_{h,k} \mathsf{Fin}(\mathcal{G}_{h+k}, m \times n^m)$$
$$\to \mathcal{G}(n).$$

Monoidal cone $\iff$ monoidal natural transformation $\mathcal{G} \to D$

Morphisms of clones

- If T is a commutative clone with $T(1) \cong 1$ ("affine"), then $\forall \alpha \in T(m), \beta_i \in T(n)$

$$\alpha \gg= (\beta_i)_i = T(\mathrm{ev})(\alpha \star \beta_1 \star \ldots \star \beta_m)$$

where $\mathrm{ev} : m \times n^m \to n$ is the evaluation map, and $\eta_n : n \to T(n)$ is

$$n \cong 1 + \ldots + 1 \cong T(1) + \ldots + T(1) \to T(n).$$

- Therefore, a monoidal natural transformation $\mathcal{G} \to D$ is the same as a morphism of clones $\mathcal{G} \to D$ (i.e. a morphism of monads).

Summary

- A 'theory of probability over finite sets' is a commutative, affine clone.

$\Longrightarrow$ The Kleisli category is symmetric monoidal with terminal unit.

- Random graph models are equivalent to theory morphisms $\mathcal{G} \to D$.
- We would like to characterize random graph models in terms of programs that interface with the sets of vertices and their edge relation.
- Next: we will recap Markov categories.

Outline

Markov categories [Fritz 2020], …

Definition

A Markov category is a symmetric monoidal category $\mathbb{C}$, where

- every object X is equipped with the structure of a commutative comonoid $\mathsf{copy}_X : X \to X \otimes X$, $\mathsf{disc}_X : X \to I$,
- the $\mathsf{copy}_X, \mathsf{disc}_X$ maps are compatible with the monoidal structure,
- disc_X is natural (which implies I is a terminal object).

- Basic fact: there is a subcategory $\mathbb{C}_{\mathrm{det}}$ of deterministic maps: those $f : X \to Y$ such that $\mathsf{copy}_Y \circ f = (f \otimes f) \circ \mathsf{copy}_X$.
- $\mathbb{C}_{\mathrm{det}}$ is cartesian (has chosen finite products).
- $\mathbb{C}_{\mathrm{det}} \hookrightarrow \mathbb{C}$ is symmetric and strict monoidal.

Probability theory over a cartesian category

Let $\mathbb{V}$ be a cartesian category.

A "probability theory over"* $\mathbb{V}$ is an identity-on-objects symmetric strict monoidal functor $i : \mathbb{V} \to \mathbb{C}$ into a symmetric monoidal category whose unit is a terminal object.

- For $X \in \mathbb{V}$, the images of $X \to X \times X$ and $X \to 1$ under i make X a commutative comonoid in $\mathbb{C}$.
- $\mathbb{C}$ becomes a Markov category.
- Get a factorization $\mathbb{V} \to \mathbb{C}_{\mathrm{det}} \hookrightarrow \mathbb{C}$.

* $=$ commutative Freyd category over $\mathbb{V}$ with terminal unit.

Markov categories and monads

- Basic example: $\mathbb{V} \to \mathbb{V}_T$, for T a commutative affine monad on $\mathbb{V}$.

Conversely, let $i : \mathbb{V} \to \mathbb{C}$ be a 'probability theory'.

The adjunction

$$\begin{tikzcd}[\mathbb{V}^{\mathrm{op}}, \mathsf{Set}] \arrow[rr, bend left, "i_!"] \arrow[rr, phantom, "\perp"] && [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}] \arrow[ll, bend left, "i^*"]\end{tikzcd}$$

induces a commutative affine monad on T on the cartesian category $[\mathbb{V}^{\mathrm{op}}, \mathsf{Set}]$, such that the bijective-on-objects factorization of

$$\mathbb{V} \to [\mathbb{V}^{\mathrm{op}}, \mathsf{Set}] \to [\mathbb{V}^{\mathrm{op}}, \mathsf{Set}]_T$$

is isomorphic to $i : \mathbb{V} \to \mathbb{C}$.

Summary of Markov categories

Roughly, a Markov category is

- a symmetric monoidal category,
- whose unit is terminal,
- which contains a wide cartesian subcategory.

- We 'forget' exactly which cartesian subcategory. Maximal possibility is $\mathbb{C}_{\mathrm{det}}$.
- Sufficient to consider commutative affine monads on cartesian categories.
- E.g.
  - Finite distributions Fin → FinStoch.
  - Meas → Stoch = Meas_P, the Kleisli category of the Giry monad P on measurable spaces.

Coproducts in a Markov category

Consider a coproduct diagram in a Markov category $\mathbb{C}$:

$$A \xrightarrow{\iota_A} A + B \xleftarrow{\iota_B} B$$

The following are equivalent.

- ι_A, ι_B induce $\mathbb{C}_{\mathrm{det}}(A, -) \times \mathbb{C}_{\mathrm{det}}(B, -) \cong \mathbb{C}_{\mathrm{det}}(A + B, -)$,
- ι_A, ι_B are deterministic,
- "+ is chosen compatibly with the comonoid structures", i.e.

$$\mathsf{copy}_{A+B} = [(\iota_A \otimes \iota_A) \circ \mathsf{copy}_A, (\iota_B \otimes \iota_B) \circ \mathsf{copy}_B].$$

Distributivity

Definition

A distributive symmetric monoidal category is a symmetric monoidal category $\mathbb{C}$, where

- $\mathbb{C}$ has chosen finite coproducts,
- The canonical maps

$$X \otimes Z + Y \otimes Z \to (X + Y) \otimes Z \qquad\qquad 0 \to 0 \otimes Z$$

  are isomorphisms.

A distributive category is a *cartesian* distributive symmetric monoidal category.

Distributive Markov categories

Definition

A distributive Markov category is a Markov category $\mathbb{C}$, which is

- distributive as a symmetric monoidal category, and moreover
- the chosen coproduct inclusions are deterministic.

Probability theory over a distributive category

Let $\mathbb{V}$ be a distributive category.

A "(distributive) probability theory over"* $\mathbb{V}$ is an identity-on-objects symmetric strict monoidal functor $i : \mathbb{V} \to \mathbb{C}$ into a symmetric monoidal category whose unit is a terminal object such that i preserves finite coproducts.

- $\mathbb{C}$ becomes a distributive Markov category.
- Get a factorization $\mathbb{V} \to \mathbb{C}_{\mathrm{det}} \hookrightarrow \mathbb{C}$.
- Basic example: $\mathbb{V} \to \mathbb{V}_T$, for T a commutative affine monad on a distributive $\mathbb{V}$.

* = commutative distributive Freyd category over $\mathbb{V}$ with terminal unit.

Distributive Markov categories and monads

Conversely, let $i : \mathbb{V} \to \mathbb{C}$ be a 'distributive probability theory'.

There is an adjunction

$$\begin{tikzcd} \mathrm{FP}(\mathbb{V}^{\mathrm{op}}, \mathsf{Set}) \arrow[rr, bend left, "i_!"] \arrow[rr, phantom, "\perp"] & & \mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set}) \arrow[ll, bend left, "i^*"] \end{tikzcd}$$

which induces a commutative affine monad on T on the distributive category $\mathrm{FP}(\mathbb{V}^{\mathrm{op}}, \mathsf{Set})$, such that the bijective-on-objects factorization of

$$\mathbb{V} \to \mathrm{FP}(\mathbb{V}^{\mathrm{op}}, \mathsf{Set}) \to \mathrm{FP}(\mathbb{V}^{\mathrm{op}}, \mathsf{Set})_T$$

is isomorphic to $i : \mathbb{V} \to \mathbb{C}$.

$\implies$ If $\mathbb{V} = \mathsf{Fin}$ then precisely a commutative affine clone.

Summary of distributive Markov categories

Roughly, a distributive Markov category is

- a distributive symmetric monoidal category,
- whose unit is terminal,
- which contains a wide distributive subcategory.

- Sufficient to consider commutative affine monads on distributive categories.
- E.g.
  - Finite distributions Fin → FinStoch.
  - Kleisli of Giry monad Meas → Stoch.
- Over Fin, equivalent to finitary commutative affine clones.

Numerals of a distributive Markov category

The numerals

$$0, \qquad 1, \qquad 2 := 1 + 1, \qquad 3 := 2 + 1, \qquad \ldots$$

are the image of the canonical distributive symmetric monoidal functor Fin $\to \mathbb{C}$.

Definition

For a distributive Markov category $\mathbb{C}$, write $\mathbb{C}_{\mathbb{N}}$ for the induced clone.

I.e.

$$\mathbb{C}_{\mathbb{N}}(m) := \mathbb{C}(1, \underbrace{1 + \ldots + 1}_{m}).$$

E.g. $\mathsf{FinStoch}_{\mathbb{N}} \cong \mathsf{Stoch}_{\mathbb{N}} \cong D$.

Bernoulli bases

Definition

A Bernoulli base for a distributive Markov category $\mathbb{C}$ is an injective monoidal natural transformation (equivalently, clone morphism),

$$\mathbb{C}_{\mathbb{N}} \hookrightarrow D.$$

Concretely, each n-measurement of $\mathbb{C}$ can be identified with classical probability distribution:

$$\Phi_n : \mathbb{C}(1, n) \hookrightarrow D(n).$$

Since $\mathsf{FinStoch}_{\mathbb{N}} \cong \mathsf{Stoch}_{\mathbb{N}} \cong D$, FinStoch and Stoch each have a unique Bernoulli base.

Outline

The graph programming interface

Let $\mathbb{C}$ be a distributive Markov category.

Definition

A graph interface in $\mathbb{C}$ consists of

$$V \in \mathbb{C} \qquad \mathsf{edge} : V \otimes V \to 2 \qquad \mathsf{new} : 1 \to V$$

such that

- edge is deterministic: $(\mathsf{edge} \otimes \mathsf{edge}) \circ \mathsf{copy}_{V \otimes V} = \mathsf{copy}_2 \circ \mathsf{edge}$
- edge is irreflexive and symmetric:

$$\mathsf{edge} \circ \mathsf{copy}_V = \iota_0 \circ \mathsf{disc}_V : V \to 2 \qquad \mathsf{edge} \circ \mathsf{swap} = \mathsf{edge} : V \otimes V \to 2.$$

Random graph models from a graph programming interface

Let $(V, \mathsf{edge}, \mathsf{new})$ be a graph interface in a distributive Markov category $\mathbb{C}$.

As before,

$$\mathsf{Inj}^{\mathrm{op}} \xrightarrow{\mathcal{G}_{(-)}} \mathsf{Fin} \to \mathbb{C}$$

is an oplax monoidal functor.

Define a monoidal cone over $\mathcal{G}_{(-)}$ where p_n is

$$1 \cong 1^{\otimes n} \xrightarrow{\mathsf{new}^{\otimes n}} V^{\otimes n} \to (V^{\otimes 2})^{\otimes \binom{n}{2}} \xrightarrow{(\mathsf{edge}(\pi_i, \pi_j) \mid i<j)} 2^{\otimes \binom{n}{2}} \cong \mathcal{G}_n.$$

Random graph models from a graph programming interface

Define a monoidal cone over $\mathcal{G}_{(-)}$ where p_n is

$$1 \cong 1^{\otimes n} \xrightarrow{\mathsf{new}^{\otimes n}} V^{\otimes n} \to (V^{\otimes 2})^{\otimes \binom{n}{2}} \xrightarrow{(\mathsf{edge}(\pi_i,\pi_j) \mid i<j)} 2^{\otimes \binom{n}{2}} \cong \mathcal{G}_n.$$

As before, a monoidal cone is the same as a monoidal transformation $\mathcal{G} \to \mathbb{C}_{\mathbb{N}}$.

Suppose that $\mathbb{C}$ has a Bernoulli base $\Phi : \mathbb{C}_{\mathbb{N}} \to D$.

Then we get a random graph model by composition.

$$\mathcal{G} \to \mathbb{C}_{\mathbb{N}} \xrightarrow{\Phi} D$$

Black-and-white graphons

A graphon consists of

- a probability space $(\Omega, \mathcal{F}, \mu)$,
- a symmetric measurable function let $W : \Omega^2 \to [0,1]$.

- A graphon is black-and-white (or random-free) if $W : \Omega^2 \to \{0,1\}$.
- Gives a graph interface in Stoch:

$$V = \Omega \qquad \text{edge} = W \qquad \text{new} = \mu$$

- Bernoulli-based: $\mathsf{Stoch}_{\mathbb{N}} \cong \mathsf{FinStoch}$.
- We recover the same random graph model.

What about general graphons?

Programming with an edge relation

Let $\mathbb{G}$ be the category of finite simple graphs and functions that preserve and reflect the edge relation.

Then $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})$ has

- as objects, sequence $(G_1, \ldots, G_n)$ of finite simple graphs;
- as morphisms $(G_1, \ldots, G_m) \to (H_1, \ldots, H_n)$, functions $f : m \to n$ with $\mathbb{G}$-morphisms $f_i : H_{f(i)} \to G_i$ for $1 \leq i \leq m$.

Proposition

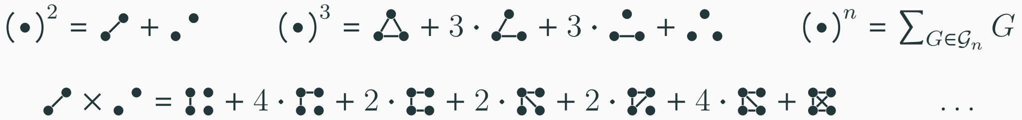
$\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})$ is a distributive category modelling the (V, edge) part of the graph interface.

Programming with an edge relation

Proposition

$\mathrm{Fam}(\mathbb{G}^{\mathrm{op}})$ is a distributive category modelling the (V, edge) part of the graph interface.

Let $V = \bullet$ the one-vertex graph. Cartesian products:



The edge map is + $\to 1 + 1 = 2$.

Sampling a fresh vertex

We want to add a new global element $1 \to V$.

Define $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}}) \to \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu]$ by

$$\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu](G, \vec{H}) \coloneqq \operatorname{colim}_{k \in \mathsf{Inj}} \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})((\bullet)^k \times G, \vec{H})$$

and extend to $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu](\vec{G}, \vec{H})$ by distributivity.

This is indeed a distributive probability theory over $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})$.

Define $\mathsf{new} : 1 \to (\bullet)$ by $[1, \pi_1 : \bullet \times 1 \to \bullet]$.

See [Hermida & Tennent 2012] for the *non-distributive* case.

Cf. the 'para construction'..

Sampling a fresh vertex

Define $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}}) \to \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu]$ by

$$\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu](G, \vec{H}) \coloneqq \operatorname{colim}_{k \in \mathsf{Inj}} \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})((\bullet)^k \times G, \vec{H})$$

and extend to $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu](\vec{G}, \vec{H})$ by distributivity.

The numerals of this distributive Markov category are familiar:

$$\begin{aligned} \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu]_{\mathbb{N}}(n) &\cong \operatorname{colim}_{k \in \mathsf{Inj}} \mathsf{Fam}(\mathbb{G}^{\mathrm{op}})((\bullet)^k, n) \\ &\cong \operatorname{colim}_{k \in \mathsf{Inj}} \mathsf{Set}(\mathcal{G}_k, n) \\ &= \mathcal{G}(n) \end{aligned}$$

Quotients of distributive Markov categories

Theorem

Suppose

- $\mathbb{C}$ is a distributive Markov category,
- $\Psi : \mathbb{C}_{\mathbb{N}} \to D$ is a monoidal natural transformation,
- $\forall X \in \mathbb{C}$, either $X \cong 0$ or $\mathbb{C}(1, X) \neq \varnothing$.

Then there is a distributive Markov category $\mathbb{C}_{/\psi}$ and an identity-on-objects strict distributive Markov functor $\mathbb{C} \to \mathbb{C}_{/\Psi}$ inducing a factorization

$$\mathbb{C}_{\mathbb{N}} \twoheadrightarrow (\mathbb{C}_{/\Psi})_{\mathbb{N}} \rightarrowtail D.$$

Idea: for $f, g \in \mathbb{C}(X, Y)$, say $f \sim g$ iff

$$\forall Z, n, 1 \xrightarrow{h} X \otimes Z, Y \otimes Z \xrightarrow{k} n. \Psi(k \circ (f \otimes Z) \circ h) = \Psi(k \circ (g \otimes Z) \circ h)$$

in $D(n)$.

Summary

- A random graph model corresponds to a monoidal transformation $\Psi : (\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu])_{\mathbb{N}} \to D$.
- We can quotient $\mathsf{Fam}(\mathbb{G}^{\mathrm{op}})[\nu]$ to another distributive Markov category with graph interface so that Ψ becomes injective.

Theorem

Every random graph model arises from a graph interface* in some Bernoulli-based distributive Markov category.

* $=$ internal notion of a black-and-white graphon.

Conclusions

- Categorical structures are a good match for the algebraic constraints of random graph models.
- Distributivity is natural for Markov categories.
- In the paper: another way to obtain the Erdős-Rényi model using the topos of Rado-nominal sets.

Conclusions

- Categorical structures are a good match for the algebraic constraints of random graph models.
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Thank you!