The Markov category of a random graph

Sean Moss
SYCO 12, Birmingham, 16 April 2024
University of Birmingham
Probabilistic Programming Interfaces for Random Graphs: Markov Categories, Graphons, and Nominal Sets

NATE ACKERMAN, Harvard University, USA
CAMERON E. FREER, Massachusetts Institute of Technology, USA
YOUNESSE KADDAR, University of Oxford, UK
JACEK KARWOWSKI, University of Oxford, UK
SEAN MOSS, University of Birmingham, UK
DANIEL ROY, University of Toronto, Canada
SAM STATON, University of Oxford, UK
HONGSEOK YANG, KAIST, South Korea

We study semantic models of probabilistic programming languages over graphs, and establish a connection to graphons from graph theory and combinatorics. We show that every well-behaved equational theory for our graph probabilistic programming language corresponds to a graphon, and conversely, every graphon arises in this way.

We provide three constructions for showing that every graphon arises from an equational theory. The first is an abstract construction, using Markov categories and monoidal indeterminates. The second and third are more concrete. The second is in terms of traditional measure theoretic probability, which covers 'black-and-white' graphons. The third is in terms of probability monads on the nominal sets of Gabbay and Pitts. Specifically, we use a variation of nominal sets induced by the theory of graphis, which covers Erdős-Rényi graphons. In this way, we build new models of graph probabilistic programming from graphons.


Additional Key Words and Phrases: probability monads, exchangeable processes, graphons, nominal sets, Markov categories, probabilistic programming

ACM Reference Format:
Outline

1. Large graphs and random graph models
2. Probability theories over finite sets
3. (Distributive) Markov categories and Bernoulli bases
4. A Markov category for random graphs
A simple graph $G = (V, E)$ is

- a set $V = V(G)$ of nodes,
- a symmetric, irreflexive binary relation $E = E(G) \subseteq V \times V$ (edges).

Write $\mathcal{G}_n \in \text{Set}$ for the set of simple graphs $G$ with $V(G) = [n] = \{1, \ldots, n\}$. Thus $|\mathcal{G}_n| = 2^{\binom{n}{2}}$. 
A large graph: proximity on a sphere

For a fixed $\theta$, let $x, y \in S^2$ be connected if the angle between them is $< \theta$.

\[
\theta = \pi/3
\]

\[
\theta = \pi/6
\]
Graphons — a model of large graphs

A graphon* consists of

- a probability space \((\Omega, \mathcal{F}, \mu)\),
- a symmetric measurable function \(W : \Omega^2 \to [0, 1]\).

- E.g. \(\Omega = S^2\), \(\mu =\) uniform distribution, \(W(x, y) = \lceil x \cdot y > \cos \theta \rceil\).
- E.g. Any finite graph \(G\): let \(\Omega = V(G)\), \(\mu =\) uniform distribution, \(W(u, v) = \lceil uv \in E(G) \rceil\).
- Models dense rather than sparse graphs.
- Equivalence of graphons is non-trivial.

*Usually restricted to \((\Omega, \mathcal{F}, \mu) = ([0, 1], B([0, 1]), \lambda)\).
A graphon consists of

- a probability space $(\Omega, \mu)$,
- a symmetric measurable function $W : \Omega^2 \to [0, 1]$.

Induce a distribution on graphs $G_n$ as follows.

- Sample a vector $\vec{x} \in \Omega^n$ using $\mu^\otimes n$.
- For each $1 \leq i < j \leq n$, let there be an edge $ij \in E(G)$ with probability $W(x_i, x_j)$.

I.e.,

$$P_{W,n}(G) = \int_{x_1 \in \Omega} \ldots \int_{x_n \in \Omega} \prod_{ij \in E(G)} W(x_i, x_j) \times \prod_{ij \notin E(G)} (1 - W(x_i, x_j)) \mu(dx_n) \ldots \mu(dx_1).$$
Random graph models

A random graph model is \((p_n \in D(G_n) \mid n \in \mathbb{N})\) which is:

- **exchangeable**: each \(p_n\) is invariant under permutations of \(\{1, \ldots, n\}\),
- **consistent**: \(p_n(G) = \sum p_{n+1}(G')\) where \(G'\) varies over the \(2^n\) one-node extensions of \(G\),
- **local**: \(p_h(G_1)p_k(G_2) = \sum p_{h+k}(G')\) where \(G'\) varies over all \(2^{hk}\) graphs on \(\{1, \ldots, h+k\}\) that restrict to \(G_1\) on \(\{1, \ldots, h\}\) and \(G_2\) on \(\{h+1, \ldots, h+k\}\).

Basic result: a graphon gives rise to a random graph model, and conversely every random graph model arises from a graphon.
Aside: countable random graph models and graph limits

A random graph model is \( (p_n \in D(G_n) \mid n \in \mathbb{N}) \) which is:

▶ exchangeable, ▶ consistent, ▶ local.

▶ We could also use a graphon to sample a graph on vertex set \( \mathbb{N} \).

▶ Can also consider graphons as ‘graph limits’:

\[
\text{FinGraph} \to [0, 1]^\text{FinGraph} \\
G \mapsto \lambda H. \frac{|\text{hom}(H, G)|}{|V(G)||V(H)|}.
\]
The Rado graph

The Rado graph is the unique countable graph with the extension property:

For all finite disjoint $A, B \subseteq V(G)$, $\exists x \in V(G) \setminus (A \cup B)$ with $\forall y \in A. xy \in E(G)$ and $\forall z \in B. xz \notin E(G)$. 

\[
\begin{array}{c}
A \quad x \\
B
\end{array}
\]
The Erdős-Rényi model

For $0 < \alpha < 1$, let $\Omega = \{\ast\}$ and $W_\alpha(\ast, \ast) = \alpha$.

**Theorem (Erdős-Rényi)**
With probability 1, the countable graph sampled from $W_\alpha$ is isomorphic to the Rado graph.

▶ N.B. The $W_\alpha$ are different random graph models:

$$\mathbb{P}_{W_\alpha,n}(G) = \alpha^{|E(G)|} \cdot (1 - \alpha)^{\binom{n}{2} - |E(G)|}.$$

▶ Impossible to present with a black-and-white ($\{0,1\}$-valued) graphon.
A graphon consists of
▶ a probability space \((\Omega, \mathcal{F}, \mu)\),
▶ a symmetric measurable function \(W : \Omega^2 \to [0, 1]\).

A random graph model is \((p_n \in D(\mathcal{G}_n) \mid n \in \mathbb{N})\) which is:
▶ exchangeable, ▶ consistent, ▶ local.

We have a model of ‘large graphs with a measure on the space of vertices’.
Next: we give a more categorical presentation of random graph model.
Then: identify random graph models with black-and-white graphons internal to Markov categories, i.e. with equational semantics for programming language with a ‘graph interface’. 
Outline

1. Large graphs and random graph models
2. Probability theories over finite sets
3. (Distributive) Markov categories and Bernoulli bases
4. A Markov category for random graphs
Recap of finite distributions monad

- Let $\text{Fin} = \text{a skeleton of the category of finite sets and functions.}$
- **Finite distributions** form a functor $D : \text{Fin} \to \text{Set}$ where
  
  $D(n) = \{\alpha \in [0, 1]^n \mid \sum \alpha(i) = 1\}.$

- **(Abstract) clone** with
  
  $\eta_n : n \to D(n)$ \hspace{1cm} $\eta_n(i) = \delta_i = \lambda j. [i = j]$ 

  $(\ggg) : D(m) \times D(n)^m \to D(n)$ \hspace{1cm} $\alpha \ggg (\beta_i)_i = \sum_i \alpha(i) \cdot \beta_i$

- Clones $\simeq$ finitary monads (left Kan extend to an endofunctor on Set).
Commutative clones

A clone $T$ is **commutative** if for all $\alpha \in T(m), \beta \in T(n)$

$$\alpha \gg (\lambda i.\beta \gg (\lambda j.\eta(i, j))) = \beta \gg (\lambda j.\alpha \gg (\lambda i.\eta(i, j)))$$

in $T(m \times n)$.

Define $\alpha \star \beta$ to be the common value of the two sides. Then

$$(\star) : T(m) \times T(n) \to T(m \times n)$$

with $\eta_1 : 1 \to T(1)$ makes $T$ a symmetric monoidal functor.

For finite distributions, actually $1 \cong D(1)$ and

$$\alpha \star \beta = \lambda(i, j).\alpha(i)\beta(j)$$
The category of finite kernels

Let \( \text{FinStoch} = \text{Fin}_D \) be the Kleisli category of \( D \).

This is a symmetric monoidal category with a terminal unit, with an identity-on-objects symmetric monoidal functor \( \text{Fin} \rightarrow \text{FinStoch} \).

- Maps \( m \rightarrow n \) in \( \text{FinStoch} \) are matrices \( A : [m] \times [n] \rightarrow [0, 1] \) with \( \sum_j A_{ij} = 1 \) for all \( i \in [m] \).
- Composition of \( A : l \rightarrow m \) with \( B : m \rightarrow n \) is given by

\[
(B \circ A)_{ik} = \sum_{j \in [m]} A_{ij} B_{jk}.
\]

- The tensor of \( A : m \rightarrow n \) with \( B : m' \rightarrow n \) is given by

\[
(A \otimes B)_{(i,i')(j,j')} = A_{ij} B_{i'j'}.
\]
A random graph model is $(p_n \in D(G_n) \mid n \in \mathbb{N})$ which is:

- exchangeable: [...] 
- consistent: [...] 
- local: [...] 

In particular, it is a sequence $(p_n : 1 \rightarrow G_n)$ of maps in FinStoch.
A random graph model is \((p_n \in D(G_n) \mid n \in \mathbb{N})\) which is:

- **exchangeable**: each \(p_n\) is invariant under permutations of \(\{1, \ldots, n\}\),
- **consistent**: \(p_n(G) = \sum p_{n+1}(G')\) where \(G'\) varies over the \(2^n\) one-node extensions of \(G\),
- **local**: [...]

Let \(\text{Inj} \hookrightarrow \text{Fin}\) be the wide subcategory of injections.

\(\mathcal{G}_{(-)}\) extends to a functor \(\text{Inj}^{\text{op}} \to \text{Fin} \to \text{FinStoch}\).

A cone \(p : 1 \Rightarrow \mathcal{G}_{(-)}\) in \(\text{FinStoch}\) is an exchangeable, consistent sequence.
A random graph model is \((p_n \in D(G_n) \mid n \in \mathbb{N})\) which is:

- **exchangeable:** 
  
- **consistent:** 
  
- **local:** 
  \[ p_h(G_1)p_k(G_2) = \sum p_{h+k}(G') \text{ where } G' \text{ varies over all } 2^{hk} \text{ graphs on } \{1, \ldots, h + k\} \text{ that restrict to } G_1 \text{ on } \{1, \ldots, h\} \text{ and } G_2 \text{ on } \{h + 1, \ldots, h + k\}. \]

- **+** on \(\text{Fin}\) restricts to monoidal structure on \(\text{Inj}\).

- \(\mathcal{G}(-) : \text{Inj}^{\text{op}} \to \text{Fin}\) is oplax monoidal:
  \[
  (G_{t_1}, G_{t_2}) : G_{m+n} \to G_m \times G_n
  \]

- A cone \(p : 1 \Rightarrow \mathcal{G}(-)\) in \(\text{FinStoch}\) is **monoidal** iff \((p_n)\) is **local**.
Cones are equivalently:

\[ \lim_{n \in \text{Inj}^{\text{op}}} D(G_n) \cong \lim_{n \in \text{Inj}^{\text{op}}} [\text{Fin, Set}](\text{Fin}(G_n, -), D) \]

\[ \cong [\text{Fin, Set}](\text{colim}_{n \in \text{Inj}^{\text{op}}} \text{Fin}(G_n, -), D) \]

Define \( \mathcal{G} := \text{colim}_{n \in \text{Inj}^{\text{op}}} \text{Fin}(G_n, -) : \text{Fin} \to \text{Set} \).

\( \mathcal{G} \) is a commutative clone

\[ \mathcal{G}(m) \times \mathcal{G}(n)^m \cong \text{colim}_h \text{Fin}(G_h, m) \times (\text{colim}_k \text{Fin}(G_k, n))^m \]

\[ \to \text{colim}_{h,k} \text{Fin}(G_{h+k}, m \times n^m) \]

\[ \to \mathcal{G}(n). \]

Monoidal cone \( \iff \) monoidal natural transformation \( \mathcal{G} \to D \)
Morphisms of clones

- If $T$ is a commutative clone with $T(1) \cong 1$ ("affine"), then
  \[ \forall \alpha \in T(m), \beta_i \in T(n) \]
  \[ \alpha \Rightarrow (\beta_i)_i = T(\text{ev})(\alpha \star \beta_1 \star \ldots \star \beta_m) \]
  where $\text{ev} : m \times n^m \to n$ is the evaluation map, and $\eta_n : n \to T(n)$ is
  \[ n \cong 1 + \ldots + 1 \cong T(1) + \ldots + T(1) \to T(n). \]

- Therefore, a monoidal natural transformation $G \to D$ is the same as a morphism of clones $G \to D$ (i.e. a morphism of monads).
A ‘theory of probability over finite sets’ is a commutative, affine clone.

The Kleisli category is symmetric monoidal with terminal unit.

Random graph models are equivalent to theory morphisms $\mathcal{G} \to D$.

We would like to characterize random graph models in terms of programs that interface with the sets of vertices and their edge relation.

Next: we will recap Markov categories.
1. Large graphs and random graph models
2. Probability theories over finite sets
3. (Distributive) Markov categories and Bernoulli bases
4. A Markov category for random graphs
A **Markov category** is a symmetric monoidal category \( \mathcal{C} \), where

- every object \( X \) is equipped with the structure of a commutative comonoid
  \[ \text{copy}_X : X \rightarrow X \otimes X, \quad \text{disc}_X : X \rightarrow I, \]
- the \( \text{copy}_X, \text{disc}_X \) maps are compatible with the monoidal structure,
- \( \text{disc}_X \) is natural (which implies \( I \) is a terminal object).

**Basic fact:** There is a subcategory \( \mathcal{C}_{\text{det}} \) of deterministic maps: those \( f : X \rightarrow Y \) such that \( \text{copy}_Y \circ f = (f \otimes f) \circ \text{copy}_X \).

- \( \mathcal{C}_{\text{det}} \) is cartesian (has chosen finite products).
- \( \mathcal{C}_{\text{det}} \hookrightarrow \mathcal{C} \) is symmetric and strict monoidal.
Let \( \mathcal{V} \) be a cartesian category.

A “probability theory over”\* \( \mathcal{V} \) is an identity-on-objects symmetric strict monoidal functor \( i : \mathcal{V} \to \mathcal{C} \) into a symmetric monoidal category whose unit is a terminal object.

- For \( X \in \mathcal{V} \), the images of \( X \to X \times X \) and \( X \to 1 \) under \( i \) make \( X \) a commutative comonoid in \( \mathcal{C} \).
- \( \mathcal{C} \) becomes a Markov category.
- Get a factorization \( \mathcal{V} \to \mathcal{C}_{\text{det}} \hookrightarrow \mathcal{C} \).

\* = commutative Freyd category over \( \mathcal{V} \) with terminal unit.
Markov categories and monads

- Basic example: $V \to V_T$, for $T$ a commutative affine monad on $V$.

Conversely, let $i : V \to C$ be a ‘probability theory’.

The adjunction

\[
\begin{array}{ccc}
[V^{op}, \text{Set}] & \cong & [C^{op}, \text{Set}] \\
\downarrow & & \downarrow \\
i! & & i^* \\
i^* & & i!
\end{array}
\]

induces a commutative affine monad on $T$ on the cartesian category $[V^{op}, \text{Set}]$, such that the bijective-on-objects factorization of

\[
V \to [V^{op}, \text{Set}] \to [V^{op}, \text{Set}]_T
\]

is isomorphic to $i : V \to C$. 

27
**Summary of Markov categories**

*Roughly, a Markov category is*

- a symmetric monoidal category,
- whose unit is terminal,
- which contains a wide cartesian subcategory.

- We ‘forget’ exactly which cartesian subcategory. Maximal possibility is $\mathcal{C}_{\text{det}}$.
- Sufficient to consider commutative affine monads on cartesian categories.
- E.g.
  - Finite distributions $\text{Fin} \to \text{FinStoch}$.
  - $\text{Meas} \to \text{Stoch} = \text{Meas}_P$, the Kleisli category of the Giry monad $P$ on measurable spaces.
Coproducts in a Markov category

Consider a coproduct diagram in a Markov category $\mathcal{C}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & A + B \\
& & \leftarrow \xleftarrow{\iota_B} B
\end{array}
$$

The following are equivalent.

- $\iota_A, \iota_B$ induce $\mathcal{C}_{\text{det}}(A, -) \times \mathcal{C}_{\text{det}}(B, -) \cong \mathcal{C}_{\text{det}}(A + B, -)$,
- $\iota_A, \iota_B$ are deterministic,
- “+ is chosen compatibly with the comonoid structures”, i.e.

$$
\text{copy}_{A+B} = [(\iota_A \otimes \iota_A) \circ \text{copy}_A, (\iota_B \otimes \iota_B) \circ \text{copy}_B].
$$
A **distributive symmetric monoidal category** is a symmetric monoidal category $\mathcal{C}$, where

- $\mathcal{C}$ has chosen finite coproducts,
- The canonical maps

\[ X \otimes Z + Y \otimes Z \to (X + Y) \otimes Z \]

\[ 0 \to 0 \otimes Z \]

are isomorphisms.

A **distributive category** is a *cartesian* distributive symmetric monoidal category.
A **distributive Markov category** is a Markov category $\mathcal{C}$, which is
distributive as a symmetric monoidal category, and moreover
the chosen coproduct inclusions are deterministic.
Let $\mathcal{V}$ be a distributive category.

A "(distributive) probability theory over"* $\mathcal{V}$ is an identity-on-objects symmetric strict monoidal functor $i : \mathcal{V} \to \mathcal{C}$ into a symmetric monoidal category whose unit is a terminal object such that $i$ preserves finite coproducts.

- $\mathcal{C}$ becomes a distributive Markov category.
- Get a factorization $\mathcal{V} \to \mathcal{C}_{\text{det}} \hookrightarrow \mathcal{C}$.
- Basic example: $\mathcal{V} \to \mathcal{V}_T$, for $T$ a commutative affine monad on a distributive $\mathcal{V}$.

* = commutative distributive Freyd category over $\mathcal{V}$ with terminal unit.
Conversely, let \( i : \mathcal{V} \to \mathcal{C} \) be a ‘distributive probability theory’.

There is an adjunction

\[
\begin{array}{ccc}
\mathsf{FP}(\mathcal{V}^{\text{op}}, \text{Set}) & \cong & \mathsf{FP}(\mathcal{C}^{\text{op}}, \text{Set}) \\
\downarrow & & \downarrow \\
\mathsf{FP}(\mathcal{V}^{\text{op}}, \text{Set}) & \cong & \mathsf{FP}(\mathcal{C}^{\text{op}}, \text{Set})
\end{array}
\]

which induces a commutative affine monad on \( T \) on the distributive category \( \mathsf{FP}(\mathcal{V}^{\text{op}}, \text{Set}) \), such that the bijective-on-objects factorization of

\[
\mathcal{V} \to \mathsf{FP}(\mathcal{V}^{\text{op}}, \text{Set}) \to \mathsf{FP}(\mathcal{V}^{\text{op}}, \text{Set})_T
\]

is isomorphic to \( i : \mathcal{V} \to \mathcal{C} \).

\[\Rightarrow\] If \( \mathcal{V} = \text{Fin} \) then precisely a commutative affine clone.
Roughly, a distributive Markov category is
- a distributive symmetric monoidal category,
- whose unit is terminal,
- which contains a wide distributive subcategory.

Sufficient to consider commutative affine monads on distributive categories.

E.g.
- Finite distributions $\text{Fin} \rightarrow \text{FinStoch}$.
- Kleisli of Giry monad $\text{Meas} \rightarrow \text{Stoch}$.
- Over $\text{Fin}$, equivalent to finitary commutative affine clones.
The numerals

\[
0, \quad 1, \quad 2 := 1 + 1, \quad 3 := 2 + 1, \quad \ldots
\]

are the image of the canonical distributive symmetric monoidal functor $\text{Fin} \to \mathbb{C}$.

**Definition**

For a distributive Markov category $\mathbb{C}$, write $\mathbb{C}_N$ for the induced clone.

I.e.

\[
\mathbb{C}_N(m) := \mathbb{C}(1, 1 + \ldots + 1).
\]

E.g. $\text{FinStoch}_N \cong \text{Stoch}_N \cong D$. 
Bernoulli bases

**Definition**

A Bernoulli base for a distributive Markov category $\mathbb{C}$ is an injective monoidal natural transformation (equivalently, clone morphism),

$$\mathbb{C}_N \hookrightarrow D.$$

Concretely, each $n$-measurement of $\mathbb{C}$ can be identified with classical probability distribution:

$$\Phi_n : \mathbb{C}(1, n) \hookrightarrow D(n).$$

Since $\text{FinStoch}_N \cong \text{Stoch}_N \cong D$, $\text{FinStoch}$ and $\text{Stoch}$ each have a unique Bernoulli base.
1 Large graphs and random graph models
2 Probability theories over finite sets
3 (Distributive) Markov categories and Bernoulli bases
4 A Markov category for random graphs
The graph programming interface

Let $\mathcal{C}$ be a distributive Markov category.

**Definition**

A graph interface in $\mathcal{C}$ consists of

$$V \in \mathcal{C} \quad \text{edge} : V \otimes V \to 2 \quad \text{new} : 1 \to V$$

such that

- edge is deterministic: $(\text{edge} \otimes \text{edge}) \circ \text{copy}_V \otimes V = \text{copy}_2 \circ \text{edge}$
- edge is irreflexive and symmetric:

  $$\text{edge} \circ \text{copy}_V = \iota_0 \circ \text{disc}_V : V \to 2 \quad \text{edge} \circ \text{swap} = \text{edge} : V \otimes V \to 2.$$
Let \((V, \text{edge}, \text{new})\) be a graph interface in a distributive Markov category \(\mathcal{C}\). As before, \(\text{Inj}^{\text{op}} \xrightarrow{G(-)} \text{Fin} \rightarrow \mathcal{C}\) is an oplax monoidal functor.

Define a monoidal cone over \(G(-)\) where \(p_n\) is

\[
1 \cong 1 \otimes_n \xrightarrow{\text{new} \otimes_n} V \otimes_n \rightarrow (V \otimes^2)^{\binom{n}{2}} \xrightarrow{\text{edge}(\pi_i, \pi_j) | i < j} 2 \otimes^{\binom{n}{2}} \cong G_n.
\]
Define a monoidal cone over $G(-)$ where $p_n$ is

$$1 \cong 1 \otimes n \xrightarrow{\text{new} \otimes n} V \otimes n \rightarrow (V \otimes 2) \otimes \binom{n}{2} \xrightarrow{(\text{edge}(\pi_i, \pi_j)|i<j)} 2 \otimes \binom{n}{2} \cong G_n.$$ 

As before, a monoidal cone is the same as a monoidal transformation $G \rightarrow \mathbb{C}_N$.

Suppose that $\mathbb{C}$ has a Bernoulli base $\Phi : \mathbb{C}_N \rightarrow D$.

Then we get a random graph model by composition.

$$G \rightarrow \mathbb{C}_N \xrightarrow{\Phi} D$$
A graphon consists of

- a probability space $(\Omega, \mathcal{F}, \mu)$,
- a symmetric measurable function let $W : \Omega^2 \to [0, 1]$.

A graphon is black-and-white (or random-free) if $W : \Omega^2 \to \{0, 1\}$.

Gives a graph interface in Stoch:

$$V = \Omega \quad \text{edge} = W \quad \text{new} = \mu$$

Bernoulli-based: $\text{Stoch}_N \cong \text{FinStoch}$.

We recover the same random graph model.

What about general graphons?
Let $\mathcal{G}$ be the category of finite simple graphs and functions that preserve and reflect the edge relation.

Then $\text{Fam}(\mathcal{G}^{\text{op}})$ has

- as objects, sequence $(G_1, \ldots, G_n)$ of finite simple graphs;
- as morphisms $(G_1, \ldots, G_m) \to (H_1, \ldots, H_n)$, functions $f : m \to n$ with $\mathcal{G}$-morphisms $f_i : H_{f(i)} \to G_i$ for $1 \leq i \leq m$.

**Proposition**

$\text{Fam}(\mathcal{G}^{\text{op}})$ is a distributive category modelling the $(V, \text{edge})$ part of the graph interface.
Programming with an edge relation

**Proposition**

\( \text{Fam}(\mathcal{G}^{\text{op}}) \) is a distributive category modelling the \((V,\text{edge})\) part of the graph interface.

Let \( V = \bullet \) the one-vertex graph. Cartesian products:

\[
\begin{align*}
    (\bullet)^2 &= \bullet + \bullet \\
    (\bullet)^3 &= \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \draw (v1) to (v2);
    \end{tikzpicture} + 3 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1.5,0) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v2) to (v3);
    \end{tikzpicture} + 3 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
    \end{tikzpicture} + \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
    \end{tikzpicture} \\
    (\bullet)^n &= \sum_{G \in \mathcal{G}_n} G \\
    \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
    \end{tikzpicture} + 4 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \node (v5) at (1.5,1) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
        \draw (v3) to (v5);
    \end{tikzpicture} + 2 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \node (v5) at (1.5,1) {$\bullet$};
        \node (v6) at (1.5,2) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
        \draw (v3) to (v5);
        \draw (v5) to (v6);
    \end{tikzpicture} + 2 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \node (v5) at (1.5,1) {$\bullet$};
        \node (v6) at (1.5,2) {$\bullet$};
        \node (v7) at (1.5,2.5) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
        \draw (v3) to (v5);
        \draw (v5) to (v6);
        \draw (v5) to (v7);
    \end{tikzpicture} + 4 \cdot \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \node (v5) at (1.5,1) {$\bullet$};
        \node (v6) at (1.5,2) {$\bullet$};
        \node (v7) at (1.5,2.5) {$\bullet$};
        \node (v8) at (1.5,3) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
        \draw (v3) to (v5);
        \draw (v5) to (v6);
        \draw (v5) to (v7);
        \draw (v5) to (v8);
    \end{tikzpicture} \\ 
    \begin{tikzpicture}[baseline=-.5ex]
        \node (v1) at (0,0) {$\bullet$};
        \node (v2) at (1,0) {$\bullet$};
        \node (v3) at (1,1) {$\bullet$};
        \node (v4) at (1,1.5) {$\bullet$};
        \node (v5) at (1.5,1) {$\bullet$};
        \node (v6) at (1.5,2) {$\bullet$};
        \node (v7) at (1.5,2.5) {$\bullet$};
        \node (v8) at (1.5,3) {$\bullet$};
        \node (v9) at (1.5,3.5) {$\bullet$};
        \draw (v1) to (v2);
        \draw (v1) to (v3);
        \draw (v3) to (v4);
        \draw (v3) to (v5);
        \draw (v5) to (v6);
        \draw (v5) to (v7);
        \draw (v5) to (v8);
        \draw (v5) to (v9);
    \end{tikzpicture} & \cdots
\end{align*}
\]

The edge map is \( \begin{tikzpicture}[baseline=-.5ex]
    \node (v1) at (0,0) {$\bullet$};
    \node (v2) at (1,0) {$\bullet$};
    \draw (v1) to (v2);
\end{tikzpicture} \rightarrow 1 + 1 = 2 \).
Sampling a fresh vertex

We want to add a new global element $1 \to V$.

Define $\text{Fam}(\mathcal{G}^{\text{op}}) \to \text{Fam}(\mathcal{G}^{\text{op}})[\nu]$ by

$$\text{Fam}(\mathcal{G}^{\text{op}})[\nu](G, \vec{H}) := \text{colim}_{k \in \text{Inj}} \text{Fam}(\mathcal{G}^{\text{op}})((\cdot)^k \times G, \vec{H})$$

and extend to $\text{Fam}(\mathcal{G}^{\text{op}})[\nu](\vec{G}, \vec{H})$ by distributivity.

This is indeed a distributive probability theory over $\text{Fam}(\mathcal{G}^{\text{op}})$.

Define new $: 1 \to (\cdot)$ by $[1, \pi_1 : \cdot \times 1 \to \cdot]$.

See [Hermida & Tennent 2012] for the non-distributive case.

Cf. the ‘para construction’.
Define $\text{Fam}(\mathcal{G}^{\text{op}}) \to \text{Fam}(\mathcal{G}^{\text{op}})[\nu]$ by

$$\text{Fam}(\mathcal{G}^{\text{op}})[\nu](G, \vec{H}) := \text{colim}_{k \in \text{Inj} \text{Fam}(\mathcal{G}^{\text{op}})} \text{Fam}(\mathcal{G}^{\text{op}})((\bullet)^k \times G, \vec{H})$$

and extend to $\text{Fam}(\mathcal{G}^{\text{op}})[\nu](\vec{G}, \vec{H})$ by distributivity.

The numerals of this distributive Markov category are familiar:

$$\text{Fam}(\mathcal{G}^{\text{op}})[\nu]_{\mathbb{N}}(n) \cong \text{colim}_{k \in \text{Inj}} \text{Fam}(\mathcal{G}^{\text{op}})((\bullet)^k, n) \cong \text{colim}_{k \in \text{Inj}} \text{Set}(\mathcal{G}_k, n) = \mathcal{G}(n)$$
Theorem

Suppose

- $\mathcal{C}$ is a distributive Markov category,
- $\Psi : \mathcal{C}_N \to D$ is a monoidal natural transformation,
- $\forall X \in \mathcal{C}$, either $X \cong 0$ or $\mathcal{C}(1, X) \neq \emptyset$.

Then there is a distributive Markov category $\mathcal{C}/\Psi$ and an identity-on-objects strict distributive Markov functor $\mathcal{C} \to \mathcal{C}/\Psi$ inducing a factorization

$$\mathcal{C}_N \twoheadrightarrow (\mathcal{C}/\Psi)_N \hookrightarrow D.$$ 

Idea: for $f, g \in \mathcal{C}(X, Y)$, say $f \sim g$ iff

$$\forall Z, n, 1 \xrightarrow{h} X \otimes Z, Y \otimes Z \xrightarrow{k} n.\Psi(k \circ (f \otimes Z) \circ h) = \Psi(k \circ (g \otimes Z) \circ h)$$
in $D(n)$. 

Quotients of distributive Markov categories
A random graph model corresponds to a monoidal transformation
\[ \Psi : (\text{Fam}(G^{\text{op}})[\nu])_N \rightarrow D. \]

We can quotient \( \text{Fam}(G^{\text{op}})[\nu] \) to another distributive Markov category with graph interface so that \( \Psi \) becomes injective.

**Theorem**

Every random graph model arises from a graph interface* in some Bernoulli-based distributive Markov category.

* = internal notion of a black-and-white graphon.
Conclusions

- Categorical structures are a good match for the algebraic constraints of random graph models.
- Distributivity is natural for Markov categories.
- In the paper: another way to obtain the Erdős-Rényi model using the topos of Rado-nominal sets.
Conclusions

- Categorical structures are a good match for the algebraic constraints of random graph models.
- Distributivity is natural for Markov categories.
- In the paper: another way to obtain the Erdős-Rényi model using the topos of Rado-nominal sets.

Thank you!