Pushing monads forward

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Well-known answer

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If T is the identity monad, then this is the usual monad induced by the adjunction $F \dashv G$.

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Little-known answer

If a certain Kan extension exists, then we get a monad on $\mathcal{D}$.

The pushforward monad

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\kappa^{G,T}}{\swarrow}"] & \mathcal{D} \arrow[d, dashed, "\mathrm{Ran}_G\, GT"] \\
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Definition

The **pushforward** of T along G is $G_* T \coloneqq \mathrm{Ran}_G\, GT$, when the latter exists.

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Definition

The **pushforward** of T along G is $G_* T \coloneqq \mathrm{Ran}_G\, GT$, when the latter exists.

This comes with a monad structure, which I will now describe.

The monad structure

We have a strict monoidal category $\mathcal{K}(G, T)$, where objects are pairs (S, σ) fitting into a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\sigma}{\swarrow}"] & \mathcal{D} \arrow[d, "S"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

and a morphism $(S, \sigma) \to (S', \sigma')$ is a natural transformation $\alpha \colon S \Rightarrow S'$ such that $\sigma = \sigma' \circ \alpha G$.

The monad structure

The monoidal product of (S, σ) and (S', σ') and the monoidal unit are

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"] \arrow[dd, bend right=60, "T"'] & \mathcal{D} \arrow[d, "S'"] \arrow[dl, phantom, "\overset{\sigma'}{\swarrow}"] \\
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"] \arrow[l, phantom, "\overset{\mu^T}{\Leftarrow}"] & \mathcal{D} \arrow[d, "S"] \arrow[dl, phantom, "\overset{\sigma}{\swarrow}"] \\
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\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "G"] \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "1_{\mathcal{C}}"] \arrow[d, phantom, "\overset{\eta^T}{\Leftarrow}"] & \mathcal{D} \arrow[d, phantom, "/\!/"] & \mathcal{D} \arrow[d, "1_{\mathcal{D}}"] \\
\mathcal{C} \arrow[rr, "G"] & & \mathcal{D}
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\end{tikzcd}
$$

$\mathrm{Ran}_G\, GT$ is, by definition, the terminal object of $\mathcal{K}(G, T)$, and hence it has a unique monoid structure. This gives it a canonical monad structure.

Reconciling with the adjunction situation

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If G has a left adjoint F, then $G_* T = GTF$.

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Proof sketch. This follows from the fact that right Kan extending along a right adjoint is the same as precomposing with the left adjoint:

$$G_* T = \mathrm{Ran}_G \, GT = GTF$$

Some easy examples

Recall the limit formula for a right Kan extension:

$$(\mathrm{Ran}_G\, GT)(d) = \lim_{d \to Gc} GTc,$$

where the limit is indexed by the comma category $(d \downarrow G)$.

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Examples

- Let $G\colon \mathbf{0} \to \mathcal{D}$ and $\mathcal{D}$ have a terminal object $\mathbb{1}$. Then $G_* 1$ is constant at $\mathbb{1}$ with its unique monad structure.
- Let $d\colon \mathbf{1} \to \mathcal{D}$ and $\mathcal{D}$ have powers. Then $A_* 1$ is the *endomorphism monad* of d, given by $d' \mapsto [\mathcal{D}(d', d), d]$.

Codensity monads

Definition

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Many codensity monads have been studied in the literature.

Examples

- The codensity monad of **FinSet** $\hookrightarrow$ **Set** is the *ultrafilter monad*, whose algebras are compact Hausdorff spaces.

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- The codensity monad of $\mathbf{Vect}_k^{\mathrm{fd}} \hookrightarrow \mathbf{Vect}_k$ is the *double dualisation monad*.
- The codensity monad of $\mathbf{FinGrp} \hookrightarrow \mathbf{Grp}$ is the *profinite completion monad*, whose algebras are profinite groups.

A universal property of the pushforward

The comparison transformation $\kappa^{G,T} \colon G_* T \circ G \to GT$ of the Kan extension gives a functor $K^{G,T}$ making the following square commute

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "K^{G,T}"] \arrow[d, "U^T"] & \mathcal{D}^{G_* T} \arrow[d, "U^{G_* T}"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd}$$

We can hence see $K^{G,T}$ as an arrow in $\mathbf{CAT}/\mathcal{D}$.

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We can hence see $K^{G,T}$ as an arrow in $\mathbf{CAT}/\mathcal{D}$.

Recall that we have a functor $\mathbf{Alg} \colon \mathbf{Mnd}(\mathcal{D})^{\mathrm{op}} \to \mathbf{CAT}/\mathcal{D}$, which sends a monad S on $\mathcal{D}$ to its category of algebras, $\mathcal{D}^S$. Then:

Theorem

$K^{G,T}$ is a universal arrow from GU^T to $\mathbf{Alg}$.

A universal property of the pushforward

Theorem (continued)

More explicitly, we have an isomorphism, natural in S,

$$\mathbf{Mnd}(\mathcal{D})(S, G_* T) \cong (\mathbf{CAT}/\mathcal{D}) \left(\begin{array}{cc} \mathcal{C}^T & \mathcal{D}^S \\ \downarrow_{GU^T} & \downarrow_{U^S} \\ \mathcal{D} & \mathcal{D} \end{array} \right)$$

sending θ to $\mathbf{Alg}(\theta) \circ K^{G,T}$. Hence, $U^{G_* T}$ is the *universal monadic replacement* of GU^T.

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Putting $G \mapsto GU^T$ and $T \mapsto 1$ in the last sentence, we get:

Corollary

$G_*T \cong (GU^T)_*1$, i.e. G_*T is the codensity monad of UG^T.

Some functoriality properties

Proposition

If $G_* T$ exists for all $T \in \mathbf{Mnd}(\mathcal{C})$, then G_* becomes a functor $\mathbf{Mnd}(\mathcal{C}) \to \mathbf{Mnd}(\mathcal{D})$.

This is the case, for example, if $\mathcal{C}$ is small and $\mathcal{D}$ is complete.

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If we further have $H : \mathcal{D} \to \mathcal{E}$, then:

Proposition

If H preserves limits, or if G is a right adjoint, then

$$(HG)_* T \cong H_*(G_* T),$$

and both of these conditions are sharp.

Pushing forward along FinSet $\hookrightarrow$ Set

Some monads on **Set** and **FinSet**

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Each of these monads preserves finiteness, so they descend to monads on **FinSet**, which we denote P_E^{f}, A_M^{f} and $\mathcal{P}^{\mathrm{f}}$, respectively.

Pushing forward along **FinSet** $\hookrightarrow$ **Set**

Let $i\colon \mathbf{FinSet} \hookrightarrow \mathbf{Set}$ denote the obvious inclusion. What is $i_* T^{\mathrm{f}}$, for T^{f} each of the monads in the previous slide?

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Moreover, each T^{f} is the restriction of a monad T on **Set**, which gives a map of monads $T \to i_* T^{\mathrm{f}}$.

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Intuition

Thus, $i_* T^{\mathrm{f}}$-algebras have an underlying T-algebra structure and compact Hausdorff topology, which are compatible in some way.

The case of P_E^{f} and A_M^{f}

Proposition

U preserves finite coproducts. In particular,

$$UP_E \cong P_E U \quad \text{and} \quad UA_M \cong A_M U.$$

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These seem to fit the bill for $i_* P^{\mathrm{f}}_E$ and $i_* A^{\mathrm{f}}_M$-algebras!

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Theorem

We have isomorphisms of monads

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Proof sketch. A general construction gives a transformation $\alpha\colon UP_E \to i_* P^{\mathrm{f}}_E$. For $X \in \mathbf{Set}$, this is

$$\alpha_X \colon \lim_{P_E X \to N} N \to \lim_{X \to N} P_E N,$$

where, for $f : X \to N$, we have $\lambda_f \alpha_X = \lambda_{P_E f}$.

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Given $x \in i_* P_E^{\mathrm{f}} X$, consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "!"'] & N \arrow[d, "!"] & & P_E N \arrow[d, "P_E !"] & \ni & \lambda_f x \\ & 1 & & P_E 1 & \ni & \lambda_! x \end{tikzcd}$$

We see that $\lambda_f x \in E$ iff $\lambda_! x \in E$.

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\end{tikzcd}$$

We see that $\lambda_f x \in E$ iff $\lambda_! x \in E$. Hence, either x is constant at $\lambda_! x \in E$, or x can be seen as an element of UX. This gives an element of $P_E UX \cong UP_E X$.

The case of $\mathcal{P}^{\mathrm{f}}$

There is no distributive law between U and $\mathcal{P}$. But there is a well known monad on **Set** that restricts to $\mathcal{P}^{\mathrm{f}}$ on **FinSet**, the filter monad F.

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Theorem

This map is an isomorphism of monads $F \cong i_*\mathcal{P}^{\mathrm{f}}$.

The algebras for F are *continuous lattices*, which are a certain kind of complete lattices with a compatible compact Hausdorff topology.

The codensity monad of Field $\hookrightarrow$ Ring

The monad K

For this last section, let $i\colon \mathbf{Field} \to \mathbf{Ring}$ be the obvious inclusion, and let $K := i_*1$ be its codensity monad.

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For $R \in \mathbf{Ring}$, we have

$$KR = \lim_{R \to k} k.$$

Any map from a ring to a field factors through a fraction field $\mathrm{Frac}(R/\mathfrak{p})$ for a unique prime ideal $\mathfrak{p}$. This means that:

$$KR \cong \prod_{\mathfrak{p} \in \mathrm{Spec}\, R} \mathrm{Frac}(R/\mathfrak{p}).$$

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To understand μ_R^K, we need to understand $\operatorname{Spec} KR$.

Proposition

The prime ideals of a product of fields are all maximal, and they correspond to ultrafilters on the indexing set.

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The multiplication μ_R^K only depends on those components indexed by $\mathfrak{p} \in \operatorname{Spec} KR$ corresponding to *principal ultrafilters*.

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Field has equalisers, but not products. It turns out that this is all it's missing!

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Theorem

There is an isomorphism of categories over **Ring**

$$\textbf{Ring}^K \cong \textbf{Prod}(\textbf{Field})$$

Pushing forward to Set

Let R denote the free ring monad on **Set**. What happens if we push K forward along U^R?

$$\begin{tikzcd} \mathbf{Prod(Field)} \arrow[r, bend left, "U^K"] \arrow[r, phantom, "\top"] & \mathbf{Ring} \arrow[l, bend left, "F^K"] \arrow[r, bend left, "U^R"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "F^R"] \end{tikzcd}$$

Pushing forward to Set

Let R denote the free ring monad on **Set**. What happens if we push K forward along U^R?

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Since we are pushing forward along a right adjoint,

$$U^R_*(i_*1) \cong (U^R i)_*1,$$

so this gives the codensity monad of $U^R i \colon \mathbf{Field} \to \mathbf{Set}$.

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Proposition

Prod(Field) has and $U^R U^K$ preserves reflective coequalisers.

Pushing forward to Set

Corollary

$U^R U^K : \mathbf{Prod}(\mathbf{Field}) \to \mathbf{Set}$ is monadic.

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This is an *infinitary theory* with many interesting operations. For example, there are n-ary operations that vanish on all fields with fewer than n algebraically independent elements.

Thank you!

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Filters and ultrafilters

Definition

A **filter** on a set X is a collection $\mathcal{F} \subseteq \mathcal{P}X$ such that

- $X \in \mathcal{F}$;
- if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An **ultrafilter** on X is a filter $\mathcal{U}$ such that

- for each $A \subseteq X$, exactly one of A and $X \setminus A$ is in $\mathcal{U}$.

For example, for $A \subseteq X$, the collection $\uparrow A := \{B \subseteq X \mid A \subseteq B\}$ is a filter on X. For $x \in X$, $\uparrow\{x\}$ is an ultrafilter.

Constants in Prod(Field)

- Constants: $\mathbb{Q} \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \cdots$

Given a field k, with $\operatorname{char} k = p$. The constant c in k is just c_p.

Operations in Prod(Field)

- n-ary operations: $\prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t_1,\dots,t_n]} \operatorname{Frac}(\mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p})$

Let k be a field, and θ an n-ary operation θ. A choice of n elements of k is equivalent to a ring homomorphism $h\colon \mathbb{Z}[t_1,\dots,t_n] \to k$. Then $\mathfrak{p} := \ker h$ is a prime ideal of $\mathbb{Z}[t_1,\dots,t_n]$, and applying θ to the elements $h(t_1),\dots,h(t_n)$ gives the image of $\theta_{\mathfrak{p}}$ under the rightmost morphism of

$$\begin{tikzcd}
\mathbb{Z}[t_1,\dots,t_n] \arrow[r, "q"] \arrow[d, "h"] & \mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p} \arrow[r, "l"] \arrow[d] & \operatorname{Frac}(\mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p}) \arrow[d] \\
k \arrow[r, equal] & k \arrow[r, equal] & k
\end{tikzcd}$$

Operations in Prod(Field)

Let $\tau \in \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t]} \operatorname{Frac}(\mathbb{Z}[t]/\mathfrak{p})$ be the unary operation with

- for each $p = 0$ or prime, set $\tau_{(t,p)} = 1$;
- $\tau_{\mathfrak{p}} = 0$ for every other $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t]$.

For k a field and $x \in k$, $\tau(x) = 1$ iff x is transcendental over the prime subfield of k.