Pushing monads forward

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1. Pushforward monads

2. Pushing forward along $\text{FinSet} \leftrightarrow \text{Set}$

3. The codensity monad of $\text{Field} \leftrightarrow \text{Ring}$
Pushforward monads
Let $T$ be a monad on $\mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$. Under what conditions do we get a monad on $\mathcal{D}$?
Let $T$ be a monad on $C$ and $G: C \to D$. Under what conditions do we get a monad on $D$?

**Well-known answer**

If $F \dashv G$, then $GTF$ is a monad on $D$.

If $T$ is the identity monad, then this is the usual monad induced by the adjunction $F \dashv G$. 
Pushing a monad forward along a functor

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If $F \dashv G$, then $GTF$ is a monad on $\mathcal{D}$.

If $T$ is the identity monad, then this is the usual monad induced by the adjunction $F \dashv G$.

**Little-known answer**

If a certain Kan extension exists, then we get a monad on $\mathcal{D}$. 
The pushforward monad

Even when $G : C \to D$ doesn’t have a left adjoint, we can consider the following right Kan extension.
The pushforward monad

Even when $G : \mathcal{C} \to \mathcal{D}$ doesn’t have a left adjoint, we can consider the following right Kan extension.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
\downarrow T & \kappa_{G,T} & \downarrow \text{Ran}_G GT \\
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\end{array}
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\textbf{Definition}

The \textbf{pushforward} of $T$ along $G$ is $G_* T := \text{Ran}_G GT$, when the latter exists.
Even when $G : \mathcal{C} \to \mathcal{D}$ doesn’t have a left adjoint, we can consider the following right Kan extension.

\[ \begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
\downarrow T & \underset{\kappa^G,T}{\swarrow} & \downarrow \text{Ran}_G GT \\
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\end{array} \]

**Definition**

The **pushforward** of $T$ along $G$ is $G_\ast T := \text{Ran}_G GT$, when the latter exists.

This comes with a monad structure, which I will now describe.
We have a strict monoidal category $\mathcal{K}(G, T)$, where objects are pairs $(S, \sigma)$ fitting into a diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
\downarrow T & & \downarrow S \\
\mathcal{C} & \xleftarrow{G} & \mathcal{D}
\end{array}
$$

and a morphism $(S, \sigma) \to (S', \sigma')$ is a natural transformation $\alpha: S \Rightarrow S'$ such that $\sigma = \sigma' \circ \alpha G$. 

The monad structure
The monoid structure

The monoidal product of \((S, \sigma)\) and \((S', \sigma')\) and the monoidal unit are

\[
\begin{array}{ccc}
C & \xrightarrow{G} & D \\
\downarrow T & \sigma' & \downarrow S' \\
C & \xrightarrow{T} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
\mu^T & \leftarrow & C \\
\downarrow T & \sigma & \downarrow S \\
C & \xrightarrow{T} & D \\
\end{array}
\]

and

\[
\begin{array}{ccc}
C & \xrightarrow{G} & D \\
\downarrow \eta^T & 1_C & \downarrow 1_D \\
C & \xrightarrow{G} & D \\
\end{array}
\]
The monoidal product of \((S, \sigma)\) and \((S', \sigma')\) and the monoidal unit are

\[
\begin{align*}
\text{Ran}_G GT & \text{ is, by definition, the terminal object of } \mathcal{K}(G, T), \text{ and hence it has a unique monoid structure. This gives it a canonical monad structure.}
\end{align*}
\]
Proposition

If $G$ has a left adjoint $F$, then $G_\ast T = GTF$. 
Proposition

If $G$ has a left adjoint $F$, then $G^*T = GTF$.

Proof sketch. This follows from the fact that right Kan extending along a right adjoint is the same as precomposing with the left adjoint:

$$G^*T = \text{Ran}_G GT = GTF$$
Recall the limit formula for a right Kan extension:

$$(\text{Ran}_G GT)(d) = \lim_{d \to Gc} GTc,$$

where the limit is indexed by the comma category $(d \downarrow G)$. 

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**Examples**

- Let $G : 0 \to D$ and $D$ have a terminal object $\mathbb{1}$. Then $G \ast 1$ is constant at $\mathbb{1}$ with its unique monad structure.
Some easy examples

Recall the limit formula for a right Kan extension:

$$(\text{Ran}_G GT)(d) = \lim_{d \to Gc} GTc,$$

where the limit is indexed by the comma category $(d \downarrow G)$.

Examples

- Let $G : 0 \to D$ and $D$ have a terminal object $1$. Then $G \cdot 1$ is constant at $1$ with its unique monad structure.
- Let $d : 1 \to D$ and $D$ have powers. Then $A \cdot 1$ is the endomorphism monad of $d$, given by $d' \mapsto [D(d', d), d]$. 
Codensity monads

<table>
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<tr>
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Codensity monads

Definition
For any functor $G: \mathcal{C} \to \mathcal{D}$, if $G_* \mathbf{1}_\mathcal{C}$ exists, it is called the codensity monad of $G$.

Many codensity monads have been studied in the literature.

Examples
- The codensity monad of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ is the ultrafilter monad, whose algebras are compact Hausdorff spaces.
## Codensity monads

### Definition

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### Examples

- The codensity monad of $\text{FinSet} \hookrightarrow \text{Set}$ is the *ultrafilter monad*, whose algebras are compact Hausdorff spaces.

- The codensity monad of $\text{Vect}_{k}^{\text{fd}} \hookrightarrow \text{Vect}_{k}$ is the *double dualisation monad*.
## Codensity monads

### Definition

For any functor $G: \mathcal{C} \to \mathcal{D}$, if $G_*1_\mathcal{C}$ exists, it is called the **codensity monad** of $G$.

Many codensity monads have been studied in the literature.

### Examples

- The codensity monad of $\text{FinSet} \hookrightarrow \text{Set}$ is the *ultrafilter monad*, whose algebras are compact Hausdorff spaces.

- The codensity monad of $\text{Vect}^{\text{fd}}_k \hookrightarrow \text{Vect}_k$ is the *double dualisation monad*.

- The codensity monad of $\text{FinGrp} \hookrightarrow \text{Grp}$ is the *profinite completion monad*, whose algebras are profinite groups.
A universal property of the pushforward

The comparison transformation $\kappa^{G,T} : G_* T \circ G \rightarrow G T$ of the Kan extension gives a functor $K^{G,T}$ making the following square commute

$\begin{array}{ccc}
CT & \xrightarrow{K^{G,T}} & DG_* T \\
\downarrow U^T & & \downarrow U^{G_* T} \\
C & \xrightarrow{G} & D
\end{array}$

We can hence see $K^{G,T}$ as an arrow in $\text{CAT}/D$. 
A universal property of the pushforward

The comparison transformation $\kappa_{G, T}^G : G_* T \circ G \to G T$ of the Kan extension gives a functor $K_{G, T}$ making the following square commute

$$
\begin{array}{ccc}
C^T & \xrightarrow{K_{G, T}^G} & D^{G_* T} \\
\downarrow U^T & & \downarrow U^{G_* T} \\
C & \xrightarrow{G} & D
\end{array}
$$

We can hence see $K_{G, T}^G$ as an arrow in $\text{CAT}/\mathcal{D}$.

Recall that we have a functor $\text{Alg} : \text{Mnd}(\mathcal{D})^\text{op} \to \text{CAT}/\mathcal{D}$, which sends a monad $S$ on $\mathcal{D}$ to its category of algebras, $\mathcal{D}^S$. Then:

**Theorem**

$K_{G, T}^G$ is a universal arrow from $GU^T$ to $\text{Alg}$. 

More explicitly, we have an isomorphism, natural in $S$, 

$$\text{Mnd}(\mathcal{D})(S, G_* T) \cong (\text{CAT} / \mathcal{D}) \begin{pmatrix} \mathcal{C}^T & \mathcal{D}^S \\ \downarrow_{G^T} & \downarrow_{U^S} \\ \mathcal{D} & \mathcal{D} \end{pmatrix}$$

sending $\theta$ to $\text{Alg}(\theta) \circ K^{G, T}$. Hence, $U^{G_* T}$ is the universal monadic replacement of $GU^T$. 
A universal property of the pushforward

Theorem (continued)

More explicitly, we have an isomorphism, natural in $S$,

$$\text{Mnd}(\mathcal{D})(S, G_* T) \cong (\text{CAT}/\mathcal{D}) \begin{pmatrix} \mathcal{C}^T & \mathcal{D}^S \\ \Downarrow_{G^T U} & \Downarrow_{U^S} \\ \mathcal{D} & \mathcal{D} \end{pmatrix}$$

sending $\theta$ to $\text{Alg}(\theta) \circ K^{G, T}$. Hence, $U^{G_* T}$ is the universal monadic replacement of $G^T U$.

Putting $G \mapsto G^T U$ and $T \mapsto 1$ in the last sentence, we get:

Corollary

$G_* T \cong (G^T U)_* 1$, i.e. $G_* T$ is the codensity monad of $UG^T$. 
Some functoriality properties

Proposition

If $G_* T$ exists for all $T \in \text{Mnd}(\mathcal{C})$, then $G_*$ becomes a functor $\text{Mnd}(\mathcal{C}) \to \text{Mnd}(\mathcal{D})$.

This is the case, for example, if $\mathcal{C}$ is small and $\mathcal{D}$ is complete.
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Proposition
If $G_\ast T$ exists for all $T \in \text{Mnd}(\mathcal{C})$, then $G_\ast$ becomes a functor $\text{Mnd}(\mathcal{C}) \to \text{Mnd}(\mathcal{D})$.

This is the case, for example, if $\mathcal{C}$ is small and $\mathcal{D}$ is complete.

If we further have $H : \mathcal{D} \to \mathcal{E}$, then:

Proposition
If $H$ preserves limits, or if $G$ is a right adjoint, then

$$(HG)_\ast T \simeq H_\ast (G_\ast T),$$

and both of these conditions are sharp.
Pushing forward along $\text{FinSet} \rightarrow \text{Set}$
Consider the following endofunctors of \textbf{Set}:

- For a finite set $E$, the functor $P_E := (-) + E$ has a monad structure, whose algebras are $E$-pointed sets.
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Each of these monads preserves finiteness, so they descend to monads on \textbf{FinSet}, which we denote $P^f_E$, $A^f_M$ and $\mathcal{P}^f$, respectively.
Let \( i : \text{FinSet} \hookrightarrow \text{Set} \) denote the obvious inclusion. What is \( i_* \mathcal{T}^f \), for \( \mathcal{T}^f \) each of the monads in the previous slide?
Pushing forward along $\text{FinSet} \hookrightarrow \text{Set}$

Let $i: \text{FinSet} \hookrightarrow \text{Set}$ denote the obvious inclusion. What is $i_* T^f$, for $T^f$ each of the monads in the previous slide?

The unit $\eta^T$ is always a map of monads $1 \rightarrow T$. Using the functoriality of $i_*$, get a map of monads $i_1 \rightarrow i_* T^f$. 

Recall that $\mathcal{U} = i_* 1$ is the ultrafilter monad, whose algebras are compact Hausdorff spaces. Moreover, each $T^f$ is the restriction of a monad $T$ on $\text{Set}$, which gives a map of monads $T \rightarrow i_* T^f$.

Intuition

Thus, $i_* T^f$-algebras have an underlying $T$-algebra structure and compact Hausdorff topology, which are compatible in some way.
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Thus, $i_* T^f$-algebras have an underlying $T$-algebra structure and compact Hausdorff topology, which are compatible in some way.
The case of $P_E^f$ and $A_M^f$

**Proposition**

$U$ preserves finite coproducts. In particular, $U P_E \cong P_E U$ and $U A_M \cong A_M U$.

Moreover, these isomorphisms are distributive laws.
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$U$ preserves finite coproducts. In particular,

$$UP_E \cong P_E U \quad \text{and} \quad UA_M \cong A_M U.$$ 

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This makes $UP_E$ and $UA_M$ monads on $\textbf{Set}$, whose algebras are $E$-pointed compact Hausdorff spaces, and compact Hausdorff spaces with a continuous (left) $M$-action, respectively.
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These seem to fit the bill for $i^{*}P_{E}^{f}$ and $i^{*}A_{M}^{f}$-algebras!
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**Theorem**

We have isomorphisms of monads

$$i_*P^f_E \cong UP_E \quad \text{and} \quad i_*A^f_M \cong UA_M.$$
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**Proof sketch.** A general construction gives a transformation $\alpha : UP_E \to i_* P^f_E$. For $X \in \textbf{Set}$, this is

$$\alpha_X : \lim_{P_E X \to N} N \to \lim_{X \to N} P_E N,$$

where, for $f : X \to N$, we have $\lambda_f \alpha_X = \lambda_{P Ef}$.
The case of $P^f_E$ and $A^f_M$

**Theorem**

We have isomorphisms of monads

$$i_* P^f_E \simeq UP_E \quad \text{and} \quad i_* A^f_M \simeq UA_M.$$  

**Proof sketch.** A general construction gives a transformation $\alpha : UP_E \to i_* P^f_E$. For $X \in \textbf{Set}$, this is

$$\alpha_X : \lim_{P_EX \to N} N \to \lim_{X \to N} P_EN,$$

where, for $f : X \to N$, we have $\lambda_f \alpha_X = \lambda_{P Ef}$. We will construct an inverse for $\alpha_X$.  

Proof sketch.

\[ \alpha_{X} : \lim_{P_{E}X \to N} N \to \lim_{X \to N} P_{E}N. \]

For \( f : X \to N \), we have \( \lambda_{f} \alpha_{X} = \lambda_{P_{E}f} \). We will construct an inverse for \( \alpha_{X} \).
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Proof sketch.

$$\alpha_X : \lim_{P^f_EX \to N} N \to \lim_{X \to N} P^f_EN.$$  

For $f : X \to N$, we have $\lambda_f \alpha_X = \lambda_{P^f_E}$. We will construct an inverse for $\alpha_X$.

Given $x \in i_*P^f_EX$, consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & N \\
\downarrow! & & \downarrow! \\
1 & \xrightarrow{P^f_E!} & P^f_E1 \\
\end{array}
\]

We see that $\lambda_f x \in E$ iff $\lambda_! x \in E$. 


The case of $P_E^f$ and $A_M^f$

Proof sketch.

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For $f : X \to N$, we have $\lambda_f \alpha_X = \lambda_{P_E^f}$. We will construct an inverse for $\alpha_X$.

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```
X \xrightarrow{f} N \\
\downarrow ! \quad \downarrow ! \\
1 \quad \downarrow P_E! \\
\quad P_E 1 \ni \lambda_{!} x
```

We see that $\lambda_f x \in E$ iff $\lambda_{!} x \in E$. Hence, either $x$ is constant at $\lambda_{!} x \in E$, or $x$ can be seen as an element of $UX$. This gives an element of $P_E UX \simeq UP_E X$.
The case of $\mathcal{P}^f$

There is no distributive law between $U$ and $\mathcal{P}$. But there is a well known monad on $\textbf{Set}$ that restricts to $\mathcal{P}^f$ on $\textbf{FinSet}$, the filter monad $F$. 
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There is no distributive law between $U$ and $\mathcal{P}$. But there is a well known monad on $\textbf{Set}$ that restricts to $\mathcal{P}^f$ on $\textbf{FinSet}$, the filter monad $F$. This gives us a map $F \to i_*\mathcal{P}^f$, and:

**Theorem**

This map is an isomorphism of monads $F \simeq i_*\mathcal{P}^f$. 
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**Theorem**

This map is an isomorphism of monads $F \cong i_* \mathcal{P}^f$.

The algebras for $F$ are *continuous lattices*, which are a certain kind of complete lattices with a compatible compact Hausdorff topology.
The codensity monad of Field $\rightarrow$ Ring
For this last section, let $i: \text{Field} \to \text{Ring}$ be the obvious inclusion, and let $K := i_*1$ be its codensity monad.
The monad $K$

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For $R \in \text{Ring}$, we have

$$KR = \lim_{R \to k} k.$$  

Any map from a ring to a field factors through a fraction field $\text{Frac}(R/p)$ for a unique prime ideal $p$. This means that:

$$KR \cong \prod_{p \in \text{Spec } R} \text{Frac}(R/p).$$
The monad $K$

The unit $\eta^K_R$ embodies the philosophy of modern algebraic geometry: it realises an element $r \in R$ as a (dependent) function on $\text{Spec } R$. 

Proposition

The prime ideals of a product of fields are all maximal, and they correspond to ultrafilters on the indexing set.

The multiplication $\mu^K_R$ only depends on those components indexed by $p \in \text{Spec } R$ corresponding to principal ultrafilters.
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The category of $K$-algebras

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**Theorem**

There is an isomorphism of categories over $\text{Ring}$

$$\text{Ring}^K \cong \text{Prod}(\text{Field})$$
Pushing forward to Set

Let $R$ denote the free ring monad on $\textbf{Set}$. What happens if we push $K$ forward along $U^R$?

\[
\begin{array}{ccc}
\text{Prod}(\text{Field}) & \xrightarrow{\top} & \text{Ring} \\
& \xleftarrow{FK} & \xrightarrow{FR} \text{Set} \\
& \xrightarrow{UK} & \xrightarrow{UR} \text{Set}
\end{array}
\]

Since we are pushing forward along a right adjoint, $U^R \ast (i \ast 1) \sim = (U^R i) \ast 1$, so this gives the codensity monad of $U^R i$: $\text{Field} \rightarrow \text{Set}$.

Proposition $\text{Prod}(\text{Field})$ has and $U^K$ preserves reflective coequalisers.
Let $R$ denote the free ring monad on $\textbf{Set}$. What happens if we push $K$ forward along $U^R$?

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Since we are pushing forward along a right adjoint, 

$$U^R(i_*1) \cong (U^R i)_*1,$$

so this gives the codensity monad of $U^R i : \textbf{Field} \to \textbf{Set}$.

**Proposition**

$\text{Prod}(\textbf{Field})$ has and $U^R U^K$ preserves reflective coequalisers.
Pushing forward to Set

Corollary

$$U^R U^K : \text{Prod(Field)} \to \text{Set}$$ is monadic.
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The theory of products of fields is the ‘smallest’ algebraic theory containing the theory of fields.

This is an \textit{infinitary theory} with many interesting operations. For example, there are \(n\)-ary operations that vanish on all fields with fewer than \(n\) algebraically independent elements.
Thank you!


### Definition

A **filter** on a set $X$ is a collection $\mathcal{F} \subseteq \mathcal{P}X$ such that

- $X \in \mathcal{F}$;
- if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An **ultrafilter** on $X$ is a filter $\mathcal{U}$ such that

- for each $A \subseteq X$, exactly one of $A$ and $X \setminus A$ is in $\mathcal{U}$.

For example, for $A \subseteq X$, the collection $\uparrow A := \{B \subseteq X \mid A \subseteq B\}$ is a filter on $X$. For $x \in X$, $\uparrow \{x\}$ is an ultrafilter.
• Constants: $\mathbb{Q} \times F_2 \times F_3 \times F_5 \times F_7 \times \cdots$

Given a field $k$, with char $k = p$. The constant $c$ in $k$ is just $c_p$. 
Operations in Prod(Field)

- $n$-ary operations: $\prod_{p \in \text{Spec} \mathbb{Z}[t_1, \ldots, t_n]} \text{Frac}(\mathbb{Z}[t_1, \ldots, t_n]/p)$

Let $k$ be a field, and $\theta$ an $n$-ary operation $\theta$. A choice of $n$ elements of $k$ is equivalent to a ring homomorphism $h: \mathbb{Z}[t_1, \ldots, t_n] \to k$. Then $p := \ker h$ is a prime ideal of $\mathbb{Z}[t_1, \ldots, t_n]$, and applying $\theta$ to the elements $h(t_1), \ldots, h(t_n)$ gives the image of $\theta_p$ under the rightmost morphism of

$$
\begin{array}{c}
\mathbb{Z}[t_1, \ldots, t_n] \xrightarrow{q} \mathbb{Z}[t_1, \ldots, t_n]/p \xrightarrow{l} \text{Frac}(\mathbb{Z}[t_1, \ldots, t_n]/p) \\
\downarrow h \quad \downarrow \quad \downarrow \\
k \quad k \quad k \quad k
\end{array}
$$
Let $\tau \in \prod_{p \in \text{Spec } \mathbb{Z}[t]} \text{Frac}(\mathbb{Z}[t]/p)$ be the unary operation with

- for each $p = 0$ or prime, set $\tau_{(t,p)} = 1$;
- $\tau_p = 0$ for every other $p \in \text{Spec } \mathbb{Z}[t]$.

For $k$ a field and $x \in k$, $\tau(x) = 1$ iff $x$ is transcendental over the prime subfield of $k$. 