Pushing monads forward

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1. Pushforward monads

2. Pushing forward along $\textbf{FinSet} \hookrightarrow \textbf{Set}$

3. The codensity monad of $\textbf{Field} \hookrightarrow \textbf{Ring}$

Pushforward monads

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Well-known answer

If $F \dashv G$, then *GTF* is a monad on \mathcal{D} .

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Little-known answer

If a certain Kan extension exists, then we get a monad on \mathcal{D} .

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ & & & \\ T & \swarrow & & \\ \mathcal{C} & \xrightarrow{\kappa^{G,T}} & \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

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Definition

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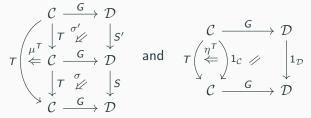
This comes with a monad structure, which I will now describe.

We have a strict monoidal category $\mathcal{K}(G, T)$, where objects are pairs (S, σ) fitting into a diagram

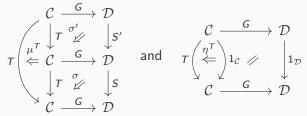
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathsf{G}} & \mathcal{D} \\ \tau & \xrightarrow{\sigma} & \downarrow s \\ \mathcal{C} & \xrightarrow{\sigma} & \mathcal{D} \end{array}$$

and a morphism $(S, \sigma) \rightarrow (S', \sigma')$ is a natural transformation $\alpha \colon S \Rightarrow S'$ such that $\sigma = \sigma' \circ \alpha G$.

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 $\operatorname{Ran}_G GT$ is, by definition, the terminal object of $\mathcal{K}(G, T)$, and hence it has a unique monoid structure. This gives it a canonical monad structure.

Proposition

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Proof sketch. This follows from the fact that right Kan extending along a right adjoint is the same as precomposing with the left adjoint:

$$G_*T = \operatorname{Ran}_G GT = GTF$$

Recall the limit formula for a right Kan extension:

$$(\operatorname{Ran}_G GT)(d) = \lim_{d \to Gc} GTc,$$

where the limit is indexed by the comma category $(d \downarrow G)$.

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Examples

• Let $G: \mathbf{0} \to \mathcal{D}$ and \mathcal{D} have a terminal object $\mathbb{1}$. Then G_*1 is constant at $\mathbb{1}$ with its unique monad structure.

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Examples

- Let G: 0 → D and D have a terminal object 1. Then G_{*}1 is constant at 1 with its unique monad structure.
- Let d: 1 → D and D have powers. Then A_{*}1 is the endomorphism monad of d, given by d' → [D(d', d), d].

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For any functor $G: \mathcal{C} \to \mathcal{D}$, if $G_* 1_{\mathcal{C}}$ exists, it is called the **codensity monad** of G.

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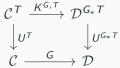
- The codensity monad of FinSet → Set is the *ultrafilter* monad, whose algebras are compact Hausdorff spaces.
- The codensity monad of Vect^{fd}_k → Vect_k is the double dualisation monad.
- The codensity monad of FinGrp → Grp is the profinite completion monad, whose algebras are profinite groups.

The comparison transformation $\kappa^{G,T}$: $G_*T \circ G \to GT$ of the Kan extension gives a functor $K^{G,T}$ making the following square commute

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{\mathcal{K}^{G,T}} & \mathcal{D}^{G_*T} \\ & \downarrow U^T & \qquad \downarrow U^{G_*T} \\ \mathcal{C} & \xrightarrow{\mathcal{G}} & \mathcal{D} \end{array}$$

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Recall that we have a functor $\operatorname{Alg}: \operatorname{Mnd}(\mathcal{D})^{\operatorname{op}} \to \operatorname{CAT}/\mathcal{D}$, which sends a monad S on \mathcal{D} to its category of algebras, \mathcal{D}^S . Then:

Theorem

 $K^{G,T}$ is a universal arrow from GU^T to **Alg**.

A universal property of the pushforward

Theorem (continued)

More explicitly, we have an isomorphism, natural in S,

$$\mathsf{Mnd}(\mathcal{D})(S, G_*T) \cong (\mathsf{CAT}/\mathcal{D}) \left(\begin{array}{cc} \mathcal{C}^T & \mathcal{D}^S \\ \downarrow_{GU^T} & \downarrow_{U^S} \\ \mathcal{D} & \mathcal{D} \end{array} \right)$$

sending θ to $\operatorname{Alg}(\theta) \circ K^{G,T}$. Hence, U^{G_*T} is the *universal* monadic replacement of GU^T .

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Putting $G \mapsto GU^T$ and $T \mapsto 1$ in the last sentence, we get:

Corollary

 $G_*T \cong (GU^T)_*1$, i.e. G_*T is the codensity monad of UG^T .

Some functoriality properties

Proposition

If G_*T exists for all $T \in Mnd(\mathcal{C})$, then G_* becomes a functor $Mnd(\mathcal{C}) \rightarrow Mnd(\mathcal{D})$.

This is the case, for example, if $\mathcal C$ is small and $\mathcal D$ is complete.

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If we further have $H : \mathcal{D} \to \mathcal{E}$, then:

Proposition

If H preserves limits, or if G is a right adjoint, then

 $(HG)_*T\cong H_*(G_*T),$

and both of these conditions are sharp.

Pushing forward along FinSet \hookrightarrow Set

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Each of these monads preserves finiteness, so they descend to monads on **FinSet**, which we denote P_F^f , A_M^f and \mathcal{P}^f , respectively.

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The unit $\eta^{T^{f}}$ is always a map of monads $1 \to T^{f}$. Using the functoriality of i_{*} , get a map of monads $i_{*}1 \to i_{*}T^{f}$.

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Moreover, each T^{f} is the restriction of a monad T on **Set**, which gives a map of monads $T \rightarrow i_{*}T^{f}$.

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Intuition

Thus, i_*T^{f} -algebras have an underlying *T*-algebra structure and compact Hausdorff topology, which are compatible in some way.

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$$UP_E \cong P_E U$$
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These seem to fit the bill for $i_*P_E^f$ and $i_*A_M^f$ -algebras!

Theorem

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Proof sketch. A general construction gives a transformation $\alpha: UP_E \rightarrow i_*P_F^f$. For $X \in \mathbf{Set}$, this is

$$\alpha_X \colon \lim_{P_E X \to N} N \to \lim_{X \to N} P_E N,$$

where, for $f: X \to N$, we have $\lambda_f \alpha_X = \lambda_{P_E f}$.

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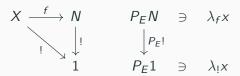
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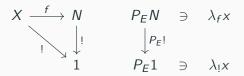
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We see that $\lambda_f x \in E$ iff $\lambda_1 x \in E$. Hence, either x is constant at $\lambda_1 x \in E$, or x can be seen as an element of UX. This gives an element of $P_E UX \cong UP_E X$.

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The algebras for F are *continuous lattices*, which are a certain kind of complete lattices with a compatible compact Hausdorff topology.

The codensity monad of Field \hookrightarrow Ring

For this last section, let $i: \text{Field} \to \text{Ring}$ be the obvious inclusion, and let $K := i_*1$ be its codensity monad.

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For $R \in \mathbf{Ring}$, we have

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Any map from a ring to a field factors through a fraction field Frac(R/p) for a unique prime ideal p. This means that:

$$KR \cong \prod_{\mathfrak{p}\in \operatorname{Spec} R} \operatorname{Frac}(R/\mathfrak{p}).$$

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Proposition

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The multiplication μ_R^K only depends on those components indexed by $\mathfrak{p} \in \operatorname{Spec} KR$ corresponding to *principal ultrafilters*.

The category of *K*-algebras

What might the K-algebras be?

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Theorem

There is an isomorphism of categories over Ring

 $\mathbf{Ring}^{\mathcal{K}} \cong \mathbf{Prod}(\mathbf{Field})$

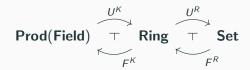
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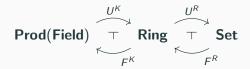
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Proposition Prod(**Field**) has and $U^R U^K$ preserves reflective coequalisers.

Corollary

 $U^{R}U^{K}$: **Prod**(**Field**) \rightarrow **Set** is monadic.

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This is an *infinitary theory* with many interesting operations. For example, there are n-ary operations that vanish on all fields with fewer than n algebraically independent elements.

Thank you!

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Filters and ultrafilters

Definition

A **filter** on a set X is a collection $\mathcal{F} \subseteq \mathcal{P}X$ such that

- $X \in \mathcal{F}$;
- if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An **ultrafilter** on X is a filter \mathcal{U} such that

• for each $A \subseteq X$, exactly one of A and $X \setminus A$ is in \mathcal{U} .

For example, for $A \subseteq X$, the collection $\uparrow A := \{B \subseteq X \mid A \subseteq B\}$ is a filter on X. For $x \in X$, $\uparrow \{x\}$ is an ultrafilter.

• Constants: $\mathbb{Q} \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \cdots$

Given a field k, with char k = p. The constant c in k is just c_p .

• *n*-ary operations: $\prod_{\mathfrak{p}\in \text{Spec }\mathbb{Z}[t_1,...,t_n]} \text{Frac}(\mathbb{Z}[t_1,\ldots,t_n]/\mathfrak{p})$

Let k be a field, and θ an n-ary operation θ . A choice of n elements of k is equivalent to a ring homomorphism $h: \mathbb{Z}[t_1, \ldots, t_n] \to k$. Then $\mathfrak{p} := \ker h$ is a prime ideal of $\mathbb{Z}[t_1, \ldots, t_n]$, and applying θ to the elements $h(t_1), \ldots, h(t_n)$ gives the image of $\theta_{\mathfrak{p}}$ under the rightmost morphism of

Let $au\in\prod_{\mathfrak{p}\in\mathsf{Spec}\,\mathbb{Z}[t]}\mathsf{Frac}(\mathbb{Z}[t]/\mathfrak{p})$ be the unary operation with

• for each p = 0 or prime, set $\tau_{(t,p)} = 1$;

•
$$\tau_{\mathfrak{p}} = 0$$
 for every other $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t]$.

For k a field and $x \in k$, $\tau(x) = 1$ iff x is transcendental over the prime subfield of k.