On the Centre of Strong Graded Monads

Flavien BREUVART
Quan LONG
Vladimir ZAMDZHIEV

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Motivation
A first computation example

Effectful computations

\[ f: X_1 \rightarrow X_2 \quad \& \quad g: Y_1 \rightarrow Y_2 \]

Two sequential compositions

\[ p_1 : \text{do } x_2 \leftarrow f(x_1); \quad y_2 \leftarrow g(y_1); \quad h(x_2, y_2) \]

\[ p_2 : \text{do } y_2 \leftarrow g(y_1); \quad x_2 \leftarrow f(x_1); \quad h(x_2, y_2) \]

- Monads as model of effects.
- **Centre**: the elements which commute with all other elements.

- If \( f \) or \( g \) is central, then \( p_1 \) and \( p_2 \) are equal!
**Background : Centre of Monads**

**Strong Monad and Premonoidal Category**

**Strong monad** : combines Effects (monads) and Pairing (monoidal structure \( \otimes \)) with **strength** \( \tau \).

\[
\tau_{X,Y} : X \otimes TY \rightarrow T(X \otimes Y)
\]

Input : \((x, M)\)

Output : \(\text{do } y \leftarrow M; \text{ return } (x, y)\)

**Remark**

A premonoidal category \( \mathcal{P} \) has a **centre**, a monoidal subcategory \( Z(\mathcal{P}) \). [Power and Robinson, 1997]

Where the tensor \( \otimes \) is not a bi-functor

Where the tensor \( \otimes \) is a bi-functor

- The Kleisli category of a strong monad is a premonoidal category!
Background: Centre of Monads

Central cone

A central cone of $T$ at $X$ is given by a pair $(Z, i : Z \to TX)$ s.t.

$$
Z \otimes \mathcal{T}Y \xrightarrow{i \otimes \mathcal{T}Y} \mathcal{T}X \otimes \mathcal{T}Y \xrightarrow{\tau'_{X,Y}} \mathcal{T}(X \otimes \mathcal{T}Y)
$$

$$
\mathcal{T}(\mathcal{T}X \otimes \mathcal{T}Y) \xrightarrow{\mathcal{T}\tau_{X,Y}} \mathcal{T}^2(X \otimes Y)
$$

Commutative Monad

Morphism between Monads

Commutes for every object $Y$.

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]
Centre of Monads

Centralisability

Theorem

Equivalent conditions for a strong monad to be centralisable:

1. Existence of all universal (terminal) central cones.
2. Existence of a commutative monad $\mathcal{E}$ s.t. $C_\mathcal{E} \cong Z(C_T)$.
3. Left adjoint $C \to C_T$ corestricts to a left adjoint $C \to Z(C_T)$.

Proof strategy

All morphisms are central in premonoidal categories.

(No direct construction on the monadic structure)

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]
Examples on Set
Examples of Centre of Monads

Commutative Monad:
- Centre: itself.

Writer Monad induced by a monoid $M$ with a centre $Z(M)$:
- Monad: $(M \times -): \text{Set} \to \text{Set}$.
- Centre: $(Z(M) \times -): \text{Set} \to \text{Set}$.

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]
Background: Pomonoid Graded Monads

Motivation & A quick reminder

Why Graded Monad? — Generalization of Monad.

Why Pomonoid? — Poset structure is especially interesting in computations.

A pomonoid graded monad on category $C$ is:

- For any $a \in G$ (the pomonoid), an endofunctor $T^a : C \to C$;
- A natural transformation $\eta : id \to T^i$;
- For any $a, b \in G$, a natural transformation $\mu^{a,b} : T^a \cdot T^b \to T^{a \cdot b}$;
- For any $a \leq a' \in G$, a natural transformation $T^a : T^a \to T^{a'}$ with commutative diagrams.

endoFunctor over $C$  \rightarrow  lax monoidal Functor from pomonoid to endoFunctor category of $C$
Construct the Centre of Graded Monads

Morphisms between Pomonoid Graded Strong Monads

The morphism between
\[ \mathcal{G} : ((G, \leq), i, \ast) \quad -\text{graded monad } T \quad \& \quad \mathcal{H} : ((H, \sqsubseteq), e, \otimes) \quad -\text{graded monad } P : \]

- A pomonoid morphism \( \phi : \mathcal{G} \to \mathcal{H} \)
- A family of natural transformations \( \iota : T \Rightarrow P \) indexed by elements of \( G \), s.t. for all \( a, b \in G \) and \( X, Y \in C \), the diagrams commute.
Construct the **Centre of Graded Monads**

**Morphisms between Pomonoid Graded Strong Monads**

The morphism between

\[ \mathcal{G} : ((G, \leq), i, \ast) \text{ -graded monad } T \quad \& \quad \mathcal{H} : ((H, \subseteq), e, \otimes) \text{ -graded monad } P : \]

\[ (G, \ast, i) \rightarrow (T, \mu^T, \nu^T) \]

\[ (\phi, \mu^\phi, \nu^\phi) \quad \downarrow \quad \downarrow \quad l \]

\[ (\mathcal{H}, \otimes, e) \rightarrow ([C, C], \cdot, I) \]

\[ (P, \mu^P, \nu^P) \]
Construct the **Centre of Graded Monads**

**Graded Central Cone**

A graded central cone of $\mathcal{G}$-graded $T$ at $(z \in Z(G), X)$ is given by a pair $(Z, \iota : Z \to TX)$ s.t.

Commutes for any object $Y \in C, b \in G$. 

\[
\begin{align*}
Z \otimes TY & \xrightarrow{\iota \otimes TY} \hat{TX} \otimes TY \xrightarrow{\tau'} \hat{T}(X \otimes TY) \\
\hat{TX} \otimes TY & \xrightarrow{\tau} \hat{T}(X \otimes TY)
\end{align*}
\]
Construct the Centre of Graded Monads

Graded Centralisability

Centralisable — Terminal Central Cone exists.

Theorem (Centre)

If centralisable, then we can find a commutative graded submonad as its centre.

Proof Strategy

Now by direct construction on the monadic structure. Independent of premonoidal and Kleisli properties.

(New proof, of course works on non-graded version too !)

• Link to Graded premonoidal centre, Kleisli Graded Monad still unkown.
Let $T$ be a (non-graded) strong monad, and $Z$ its centre.

Construct a monad graded by pomonoid $((\text{Bool}, \texttt{tt} \leq \texttt{ff}), \texttt{tt}, \land)$: $\hat{T} = Z$ and $\bar{T} = T$.

Morphism $\overline{T} : Z \rightarrow T$ is the subset inclusion.

- It tracks whether an effectful operation $\odot : TA \times TB \rightarrow TC$ can be evaluated in an order-independent manner via the grading $\hat{\odot} : TA \times TB \rightarrow T C$.
Remark

The morphism between gradations is only simply the subset inclusion:

$$\phi : Z(\mathcal{G}) \subseteq \mathcal{G}$$

- Simply indexing and tracking the effects and make them explicit, applications are limited!
Limitations of the Centre of Graded Monads

(Future) Work

• Try to define the centre of monads with gradation being more complicated, e.g. lax monoidal functors $\mathcal{F}$ instead of $Z(\mathcal{G})$, maybe homomorphic to $Z(\mathcal{G})$?

• Actually go graded premonoidal center and find an universal property.
  — construct the centre on graded Kleisli construction

• Relax the commutativity of graded monads to broader the application power. (What we are trying to do in the second half of the paper).
  — Introduce duoid as gradation, half sequential and half parallel.
Limitations of the Centre of Graded Monads

Possible solution - lax commutative and duoid

A monad $T$ on category $C$ (order-enriched) is **Lax commutative** iff there exists a nature transformation

$$m_{X,Y} : TX \otimes TY \rightarrow T(X \otimes Y)$$

s.t.

$$
\begin{array}{c}
TTX \otimes TTY \xrightarrow{m} T(TX \otimes TY) \xrightarrow{Tm} TT(X \otimes Y) \\
\downarrow \mu \otimes \mu \quad \quad \quad \quad \quad \quad \quad \downarrow \mu \\
TX \otimes TY \xrightarrow{m} T(X \otimes Y)
\end{array}
$$

along with other coherence conditions.

A duoid is a pomonoid $(G, \leq, i, *)$ with additional symmetric monoidal structure $(j, ||)$ s.t.:

$$(a || c) * (b || d) \leq (a * b) || (c * d)$$
Limitations of the Centre of Graded Monads

Possible solution - duoidal graded monad

Let $\mathcal{G} : (G, \leq, i, \ast, j, \parallel)$ be a duoid, a $\mathcal{G}$-graded monad $T$ on category $C$ (order-enriched) is

- an ordered $(G, \leq, i, \ast)$-graded monad $T$;

- a natural transformation $m_{a,b,X,Y} : \overset{a}{TX} \otimes \overset{b}{TY} \to \overset{a \parallel b}{T(X \otimes Y)}$, s.t.

\[
\begin{array}{ccc}
\overset{a \ast b}{TX} \otimes \overset{c * d}{TY} & \xrightarrow{m} & \overset{(a \parallel c) \ast (b \parallel d)}{T(X \otimes Y)} \\
\overset{\mu \otimes \mu}{\downarrow} & & \overset{\delta}{\downarrow}
\end{array}
\]

along coherence conditions.
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