On the Centre of Strong Graded Monads

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Motivation

A first computation example

Effectful computations $f\colon X_1 \to X_2$ & $g\colon Y_1 \to Y_2$

Two sequential compositions

$$p_1 : \text{do } x_2 \leftarrow f(x_1);\; y_2 \leftarrow g(y_1);\; h(x_2, y_2)$$

$$p_2 : \text{do } y_2 \leftarrow g(y_1);\; x_2 \leftarrow f(x_1);\; h(x_2, y_2)$$

- Monads as model of effects.
- **Centre** : the elements which commute with all other elements.

—If f **or** g is **central**, then p_1 and p_2 are equal !

Background : Centre of Monads

Strong Monad and Premonoidal Category

Strong monad : combines Effects (monads) and Pairing (monoidal structure $\otimes$) with **strength** τ.

$$\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$$

Input : (x, M)

Output : do $y \leftarrow M;$ return (x, y)

Remark

A premonoidal category $\mathscr{P}$ has a centre , a monoidal subcategory $Z(\mathscr{P})$. [Power and Robinson, 1997]

Where the tensor $\otimes$ is not a bi-functor

Where the tensor $\otimes$ is a bi-functor

- The Kleisli category of a strong monad is a premonoidal category !

Background : Centre of Monads

Central cone

Morphism between Monads

A central cone of T at X is given by a pair $(Z, \iota : Z \to TX)$ s.t.

$$
\begin{array}{ccccc}
Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X,\mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\
\downarrow {\scriptstyle \iota \otimes \mathcal{T}Y} & & & & \downarrow {\scriptstyle \mathcal{T}\tau_{X,Y}} \\
\mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\
\downarrow {\scriptstyle \tau_{\mathcal{T}X,Y}} & & & & \downarrow {\scriptstyle \mu_{X \otimes Y}} \\
\mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow[\mathcal{T}\tau'_{X,Y}]{} & \mathcal{T}^2(X \otimes Y) & \xrightarrow[\mu_{X \otimes Y}]{} & \mathcal{T}(X \otimes Y)
\end{array}
$$

Commutative Monad

Commutes for every object Y.

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]

Background : Centre of Monads

Centralisability

Theorem

Equivalent conditions for a strong monad to be centralisable:

1. Existence of all universal (terminal) central cones.
2. Existence of a commutative monad $\mathscr{Z}$ s.t. $C_{\mathscr{Z}} \cong Z(C_T)$.
3. Left adjoint $C \to C_T$ corestricts to a left adjoint $C \to Z(C_T)$.

Proof strategy

All morphisms are central in premonoidal categories.

(No direct construction on the monadic structure)

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]

Examples on Set

Examples of Centre of Monads

Commutative Monad :

— Centre : itself.

Writer Monad induced by a monoid M with a centre $Z(M)$:

— Monad : $(M \times -)$: Set $\to$ Set.

— Centre : $(Z(M) \times -)$: Set $\to$ Set.

[Carette, T et al.: Central submonads and notions of computation: Soundness, completeness and internal languages, LICS, 2023]

Background : Pomonoid Graded Monads

Motivation & A quick reminder

Why Graded Monad ? — Generalization of Monad.

Why Pomonoid ? — Poset structure is especially interesting in computations.

A **pomonoid graded monad** on category C is :

— For any $a \in G$ (the pomonoid), an endofunctor $\overset{a}{T} : C \to C$;

— A natural transformation $\eta : id \to \overset{i}{T}$;

— For any $a, b \in G$, a natural transformation $\mu^{a,b} : \overset{a}{T} \cdot \overset{b}{T} \to \overset{a*b}{T}$;

— For any $a \leq a' \in G$, a natural transformation $\overset{a \leq a'}{T} : \overset{a}{T} \to \overset{a'}{T}$. with commutative diagrams.

endoFunctor over C ➡ **lax monoidal Functor from pomonoid to endoFunctor category of C**

Construct the Centre of Graded Monads

Morphisms between Pomonoid Graded Strong Monads

The morphism between

$\mathscr{G} : ((G, \leq), i, *)$ -graded monad T &

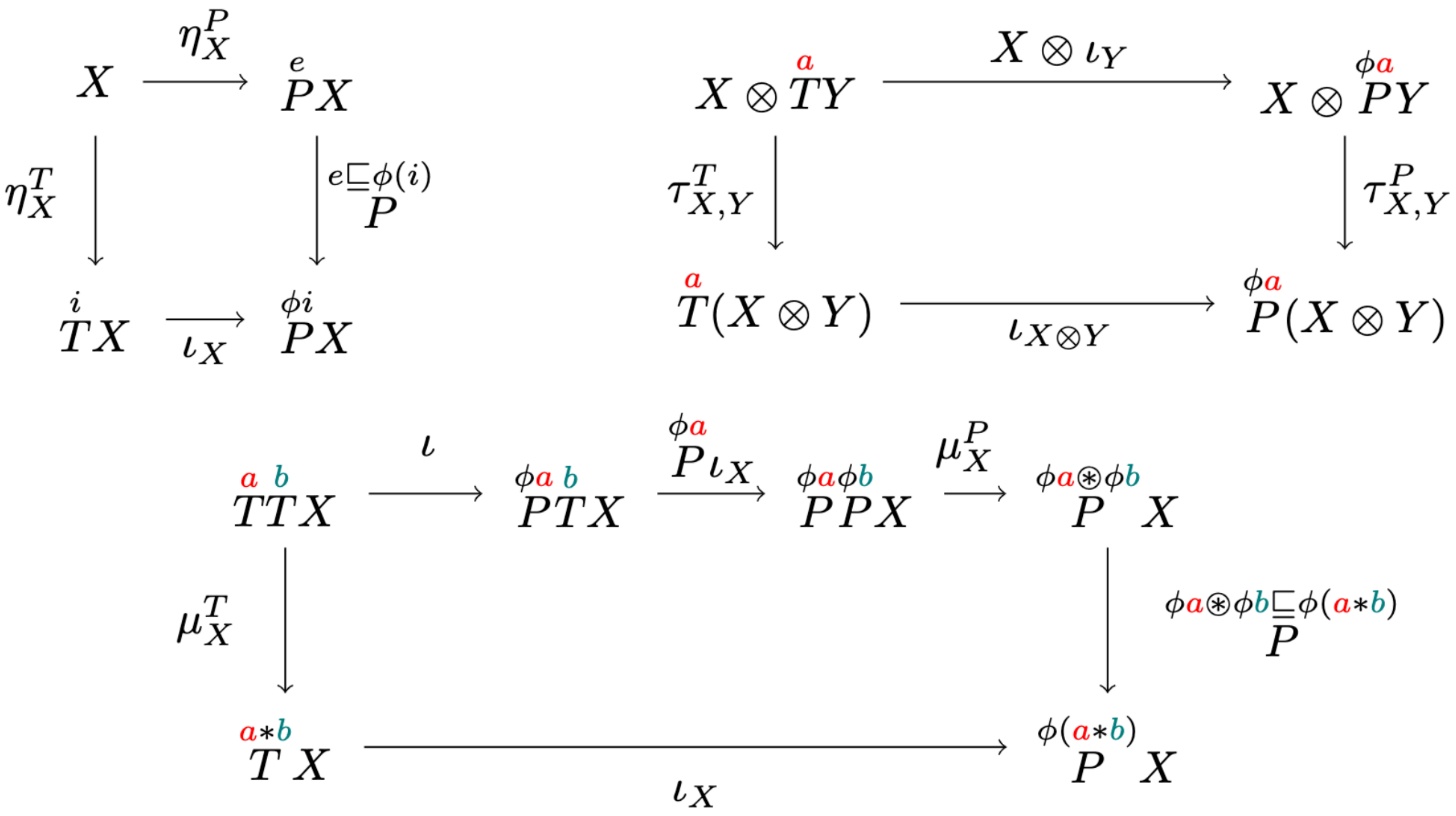
$\mathscr{H} : ((H, \sqsubseteq), e, \circledast)$ -graded monad P :

— A pomonoid morphism $\phi : \mathscr{G} \to \mathscr{H}$

— A family of natural transformations $\overset{a}{\iota} : \overset{a}{T} \Rightarrow \overset{\phi a}{P}$ indexed by elements of $\mathscr{G}$,

s.t. for all $a, b \in G$ and $X, Y \in C$,

the diagrams commute.



Construct the Centre of Graded Monads

Morphisms between Pomonoid Graded Strong Monads

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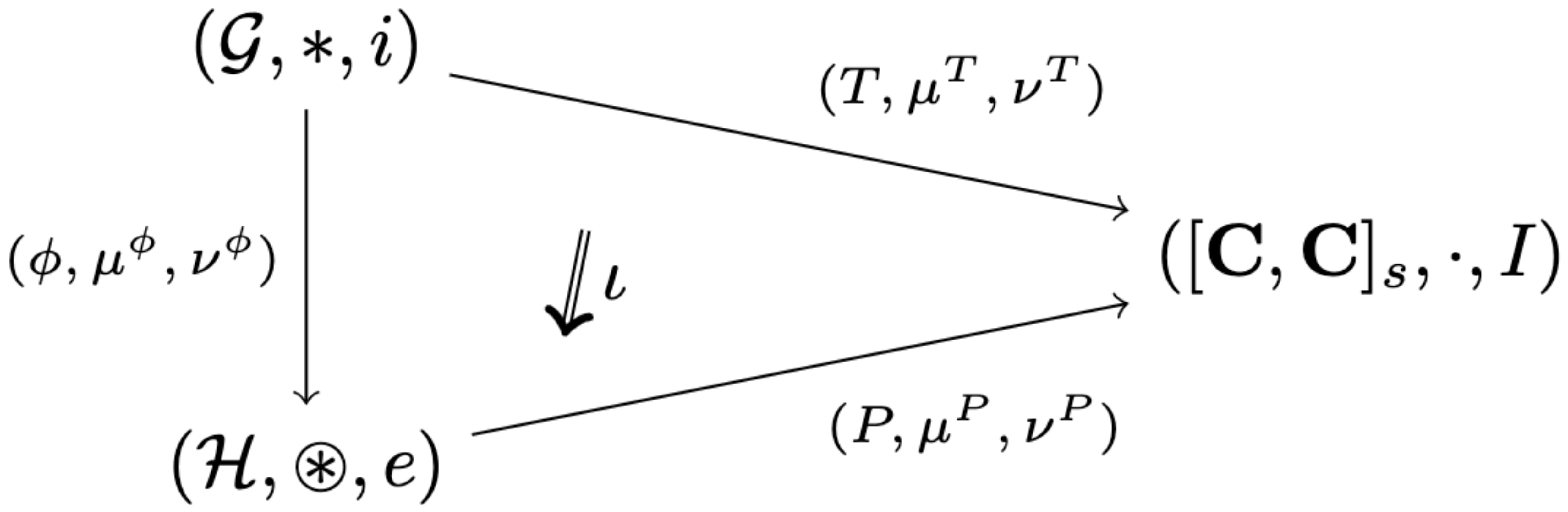


Construct the Centre of Graded Monads

Graded Central Cone

A graded central cone of $\mathscr{G}$-graded T at

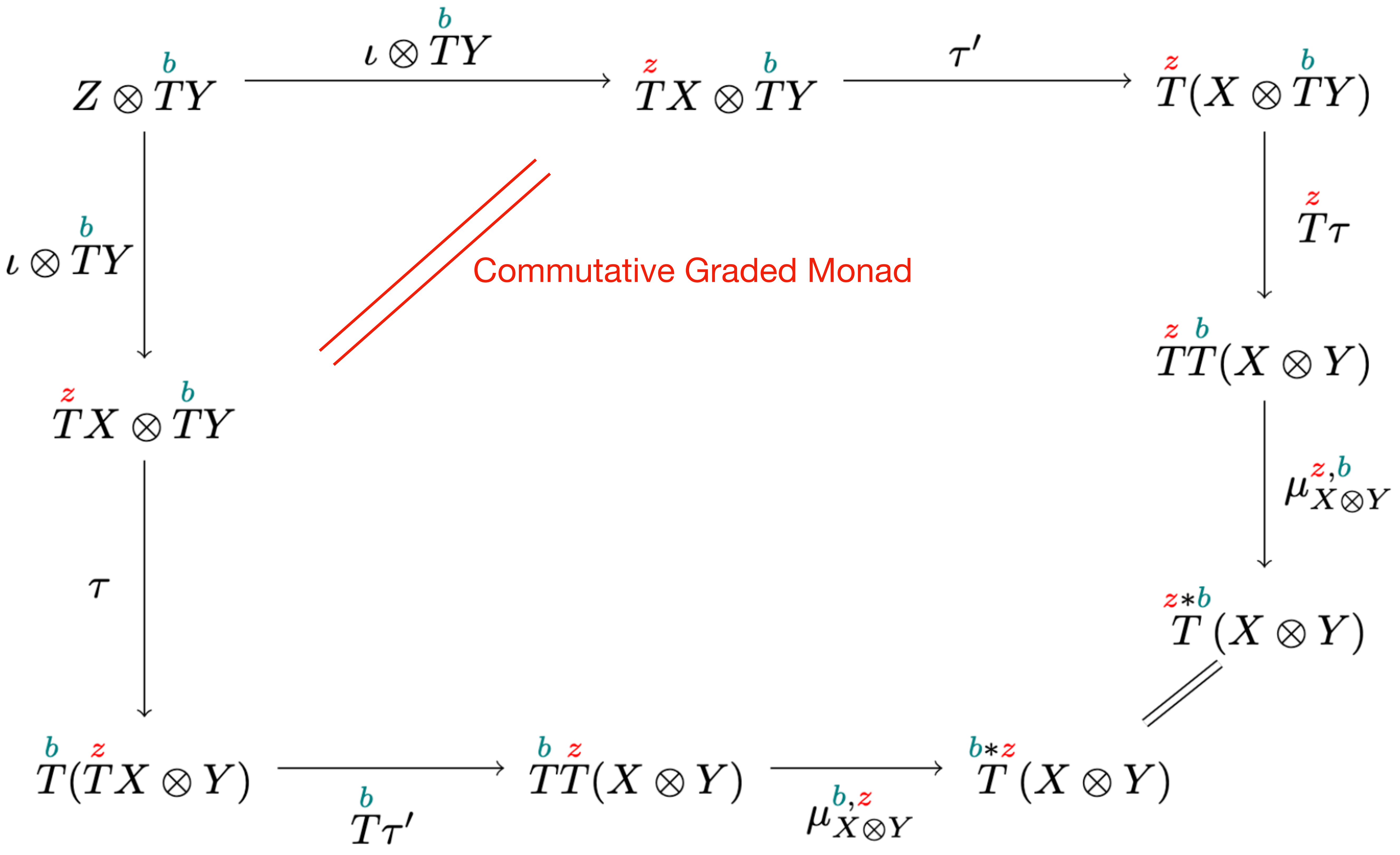
$(z \in Z(G), X)$

is given by a pair

$(Z, \iota : Z \to \overset{z}{T}X)$

s.t.

(Now only subset inclusion)



Commutes for any object $Y \in C, b \in G$.

Construct the Centre of Graded Monads

Graded Centralisability

Centralisable — **Terminal** Central Cone exists.

Theorem (Centre)

If **centralisable**, then we can find a commutative graded submonad as its **centre**.

Proof Strategy

Now by direct construction on the monadic structure. Independent of premonoidal and Kleisli properties.

(New proof, of course works on non-graded version too !)

- Link to Graded premonoidal centre, Kleisli Graded Monad still unkown.

Centre of Graded Monads

Example in computation

Let T be a (non-graded) strong monad, and Z its centre.

Construct a monad graded by pomonoid $((\mathtt{Bool}, \mathtt{tt} \leq \mathtt{ff}), \mathtt{tt}, \wedge)$: $\overset{\mathtt{tt}}{T} = Z$ and $\overset{\mathtt{ff}}{T} = T$.

Morphism $\overset{\mathtt{tt} \leq \mathtt{ff}}{T} : Z \to T$ is the subset inclusion.

- It tracks whether an effectful operation $\odot : TA \times TB \to TC$ can be evaluated in an order-independent manner via the grading : $\hat{\odot} : \overset{\mathtt{a}}{T}A \times \overset{\mathtt{b}}{T}B \to \overset{\mathtt{a} \wedge \mathtt{b}}{T} C$.

Limitations of the Centre of Graded Monads

Limitation in computations

Remark

The morphism between gradations is only simply the **subset inclusion**:

$$\phi : Z(\mathcal{G}) \subseteq \mathcal{G}$$

- Simply indexing and tracking the effects and make them explicit, applications are limited !

Limitations of the Centre of Graded Monads

(Future) Work

- Try to define the centre of monads with gradation being more complicated, e.g. lax monoidal functors ? instead of $Z(\mathscr{G})$, maybe homomorphic to $Z(\mathscr{G})$?

- Actually go graded premonoidal center and find an universal property.
  — construct the centre on graded Kleisli construction

- Relax the commutativity of graded monads to broader the application power. (What we are trying to do in the second half of the paper).

  — Introduce **duoid** as gradation, half sequential and half parallel.

Limitations of the Centre of Graded Monads

Possible solution - lax commutative and duoid

A **monad** T on category C (order-enriched) is **Lax commutative** iff there exists a nature transformation

$$m_{X,Y} : TX \otimes TY \rightarrow T(X \otimes Y)$$

s.t.

$$\begin{array}{ccccc} TTX \otimes TTY & \xrightarrow{m} & T(TX \otimes TY) & \xrightarrow{Tm} & TT(X \otimes Y) \\ \downarrow{\scriptstyle \mu \otimes \mu} & & \Downarrow & & \downarrow{\scriptstyle \mu} \\ TX \otimes TY & & \xrightarrow{\quad m \quad} & & T(X \otimes Y) \end{array}$$

along with other coherence conditions.

A **duoid** is a pomonoid $(G, \leq, i, *)$ with additional symmetric monoidal structure $(j, \|)$ s.t.:

$$(a \,\|\, c) * (b \,\|\, d) \overset{\delta}{\leq} (a * b) \,\|\, (c * d)$$

Limitations of the Centre of Graded Monads

Possible solution - duoidal graded monad

Let $\mathscr{G} : (G, \leq, i, *, j, \|)$ be a duoid, a $\mathscr{G}$- **graded monad** T on category C (order-enriched) is

— an ordered $(G, \leq, i, *)$- graded monad T;

— a natural transformation $m_{a,b,X,Y} : \overset{a}{T}X \otimes \overset{b}{T}Y \to \overset{a\|b}{T}(X \otimes Y)$, s.t.

$$\begin{array}{ccccc}
\overset{a}{T}\overset{b}{T}X \otimes \overset{c}{T}\overset{d}{T}Y & \xrightarrow{m} & \overset{a\|c}{T}(\overset{b}{T}X \otimes \overset{d}{T}Y) & \xrightarrow{Tm} & \overset{a\|c}{T}\,\overset{b\|d}{T}(X \otimes Y) \\
\Big\downarrow{\scriptstyle \mu \otimes \mu} & & & & \Big\downarrow{\scriptstyle \mu} \\
 & & & & \overset{(a*b)\|(c*d)}{T}(X \otimes Y) \\
\overset{a*b}{T}X \otimes \overset{c*d}{T}Y & \xrightarrow{m} & \overset{(a\|c)*(b\|d)}{T}(X \otimes Y) & \xleftarrow{\ \overset{\delta}{T}\ } &
\end{array}$$

along coherence conditions.

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