Game-enriched categories

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1 Brief introduction to game semantics

2 Ouch, we have $m \times n$ categories of games

3 Hyland’s co-Kleisli category of games

4 Game-enrichment gives $m + n$ jobs
We want to model a higher-order programming language, some version of typed $\lambda$-calculus.
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Play alternates between **Proponent** (the program) and **Opponent** (the environment).

Moves can be **Questions** or **Answers**.
In call-by-value $\lambda$-calculus with recursion and side-effects, here is a term:

\[
f : \text{bool} \to \text{bool} \vdash
\begin{align*}
  & \text{if } f(\text{true}) \\
  & \text{then false} \\
  & \text{else if } f(\text{true}) \text{ then true else diverge} \\
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Example play:

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<tr>
<th>PQ</th>
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<tbody>
<tr>
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\[
\begin{array}{cccccccc}
PQ & OA & PQ & OA & PA \\
\text{f(true)} & \text{false} & \text{f(true)} & \text{true} & \text{true}
\end{array}
\]

The strategy is deterministic but partial, i.e. can diverge.
It is common practice to organise a game model into a category. Here are some examples:

1. The category of arenas and OP-visible strategies. Useful for modelling Idealized Algol.  
   (Abramsky and McCusker)
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Game categories for PL semantics

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2. The category of arenas and strategies.
   Useful for modelling general references.
   (Abramsky, Honda and McCusker)

3. The category of strong nominal arenas and equivariant strategies.
   Useful for modelling good general references.
   (Murawski and Tzevelekos)
Obligations

To set up each category of games, we have to

- define the objects
- define the homsets
- define composition
- prove associativity.

Note Depending on the specific categorical structure we want, there may be more operations on strategies, and more equations to prove. For example, unital laws.
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Beyond determinism

Extend each language with nondeterministic choice.

Or with probabilistic choice.
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Extend each language with nondeterministic choice.

Or with probabilistic choice.

Then the categories need to be adapted.

Same objects, more morphisms.
## Deterministic / nondeterministic / probabilistic strategies

### Deterministic strategy
A set of finite plays that satisfies a determinacy condition.

### Nondeterministic strategy
- To model **may-testing**, drop the determinacy condition.

### Probabilistic strategy
For trace equivalence, a probabilistic strategy associates a real to each finite play. (Danos and Harmer)

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Need to construct \( m \times n \) categories (for \( m \) rows and \( n \) columns). Each requires homsets, composition, associativity etc. Can't we do this just once for each row?
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Each requires homsets, composition, associativity etc.

**Can't we do this just once for each row?**
We define a cartesian closed category.
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The category $G$!

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An object is a forest, representing a game where Opponent starts.

A morphism $A \rightarrow B$ is a strategy for the following game $B^A$.

- Opponent starts playing $B$.
- Proponent can initiate an $A$-thread any number of times.
- Only Proponent can switch between threads.
By varying the notion of strategy, we have nondeterministic variations

\[ \mathcal{G}_1^{\text{may}} \quad \mathcal{G}_1^{\text{must}} \quad \mathcal{G}_1^{\text{inftrace}} \quad \mathcal{G}_1^{\text{bisim}} \]

and probabilistic variations

\[ \mathcal{G}_1^{\text{protrace}} \quad \mathcal{G}_1^{\text{probisim}} \]
By varying the notion of strategy, we have nondeterministic variations

\[ \mathcal{G}^\text{may}, \mathcal{G}^\text{must}, \mathcal{G}^\text{intrace}, \mathcal{G}^\text{bisim} \]

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Each is cartesian closed
and has \( \mathcal{G} \) embedded as a cartesian closed wide subcategory.
Nondeterministic and probabilistic variations

By varying the notion of strategy, we have nondeterministic variations

\[ G^\text{may}, G^\text{must}, G^\text{inftrace}, G^\text{bisim} \]

and probabilistic variations

\[ G^\text{protrace}, G^\text{probbisim} \]

Each is cartesian closed
and has \( G \) embedded as a cartesian closed wide subcategory.

Setting these up is \( n \) jobs.
Recall we have $m = 3$ categories for modelling deterministic languages.

1. arenas and OP-visible strategies
2. arenas and strategies
3. strong nominal arenas and equivariant strategies.
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Calling them “categories” is an understatement.

They are $G_\Gamma$-enriched categories. (I hope.)
Objects are arenas, as before.

For arenas $A$ and $B$, we give a hom-game $C(A, B)$ whose strategies are precisely the morphisms from $A$ to $B$. 
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Example

In the case of OP-visible strategies, $C(A, B)$ is the pointer-game on $B^A$ where both Players must obey the visibility constraint.
For arenas $A, B, C$, composition is no longer an operation on strategies but a single strategy:

a $\mathcal{G}$-morphism $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$.
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For arenas $A, B, C, D$, associativity is no longer a universally quantified equation but a single equation:

left-associated and right-associated composition are the same strategy.
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Traditional composition and associativity follow.
Variation: nondeterministic strategies

For each of our \( m = 3 \) rows, we want

- a \( G_i^{\text{may}} \)-enriched category
- a \( G_i^{\text{must}} \)-enriched category
- ...
Variation: nondeterministic strategies

For each of our $m = 3$ rows, we want

- a $\mathcal{G}_i^{\textit{may}}$-enriched category
- a $\mathcal{G}_i^{\textit{must}}$-enriched category
- $\ldots$

So we still have $m \times n$ jobs, right?
Change of base: the principle

Given a (lax) monoidal functor $\mathcal{V} \rightarrow \mathcal{W}$, any $\mathcal{V}$-enriched category becomes a $\mathcal{W}$-enriched one with the same objects.
Since we have an embedding $G_! \rightarrow G_!^{\text{may}}$

any $G_!$-enriched category gives a $G_!^{\text{may}}$-enriched one with the same objects.
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And so on.
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And so on.

Hom-games, composition, associativity etc. come for free.
For each row, we define a $G$-enriched category.
For each row, we define a $G_!$-enriched category.

For each column, we define a category $\mathcal{H}_!$ (actually a co-Kleisli category) and an identity-on-objects functor $G_! \to \mathcal{H}_!$. 
For each row, we define a $G!$-enriched category.

For each column, we define a category $H!$ (actually a co-Kleisli category) and an identity-on-objects functor $G! \to H!$.

Overall, this amounts to $m + n$ tasks.
Don’t say “category” when you actually have a game-enriched category.
Take-home message

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This will pay off.