Monoidal Meta-Theorem

David Forsman
david.forsman@uclouvain.be

Université catholique de Louvain

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Outline

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Theorem (Monoidal Meta-Theorem)

Let $E \cup \{\phi\}$ (cartesian/symmetric) monoidal σ-theory and let C be a (cartesian/symmetric) monoidal category. Then

$$E \vDash_{\mathbf{Set}} \phi \text{ implies } E \vDash_C \phi.$$

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- $E \vDash_{\mathbf{Set}} T$, for $T = \{x +' y \approx x + y, e \approx e', x + y \approx y + x, (x + y) + z \approx x + (y + z)\}$.

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- $E \vDash_{\mathbf{Set}} T$, for $T = \{x +' y \approx x + y, e \approx e', x + y \approx y + x, (x+y)+z \approx x+(y+z)\}$.
- $E \vDash_C T$ for all symmetric monoidal categories C.

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Choose $\sigma = (S = \{a\}, M = \{f, g \colon a \to ()\})$. Now $\emptyset \models_{\mathbf{Set}} f \approx g$ and $\emptyset \not\models_{\mathbf{Ab}} f \approx g$, where **Ab** is the monoidal category of abelian groups.

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Choose $\sigma = (S = \{a\}, M = \{f, g \colon a \to ()\})$. Now $\emptyset \models_{\mathbf{Set}} f \approx g$ and $\emptyset \not\models_{\mathbf{Ab}} f \approx g$, where **Ab** is the monoidal category of abelian groups.

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- $m(f)(x, y) = \min(3, x + y)$, for $x, y \in m(a)$, and $m(g) \equiv 3$.
- $m \models E$ but $m \not\models f \approx g$.

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$$x \xrightarrow{\delta_x} x \otimes x \quad \text{(Diagonal)}$$

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- Let $X = \{x, y\}$. The set of essential indices of $v = ((xe)(y(ex)) \in X^*$ is $I_v = \{0, 2, 4\}$ and $\tau(v) = xeyex$.

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- A morphism $v \to w$ consists of a pair (θ, f), where $\theta\colon I_w \to I_v$ is a function and f is a family of morphisms $f_i\colon v_{\theta(i)} \to w_i$ for $i \in I_w$.

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The category $\mathrm{CM}(I)$ has a cartesian monoidal structure. Furthermore, $\mathrm{CM}(I)$ has two wide subcategories the symmetric monoidal completion $\mathrm{SM}(I)$ and the monoidal completion $\mathrm{M}(I)$ of I defined by morhpisms $(\theta, f), (\phi, g)$, respectively, where θ is a bijection and ϕ is an increasing bijection.

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In addition, if F is constant on all hom-sets, then $\overline{F}$ is constant on hom-sets $Hom(v, w)$, where the directed path components $[v_i], i \in I_v$, are pairwise disjoint in I.

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As an example, we have a unique morphism $\alpha\colon ((xy)z) \to (x(yz))$ in $T(\{x, y, z\})$. Thus we attain a natural transformation between two functors $C^3 \cong C^I \rightrightarrows C$, which is the whole associator itself.

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- $f \colon a \to b \in M$.
- If $a = ()$, then $f \colon b \in M$.

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- $m_2 : M \to \mathrm{Mor}(C)$, where $m_2(f) : \overline{m_1}(\mathrm{rb}(a)) \to m_1(b)$ for all $f : a \to b \in M$.

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A σ-model morphism $m \to n$ in C consists of a family f of morphisms $f_s \colon m(s) \to n(s), s \in S$, where for all morphism symbols $\alpha \colon a \to b$ we have a commuting diagram

$$\begin{array}{ccc} m(rb(a)) & \xrightarrow{m(\alpha)} & m(b) \\ {\scriptstyle f_{rb(a)}}\downarrow & & \downarrow{\scriptstyle f_b} \\ n(rb(a)) & \xrightarrow[n(\alpha)]{} & n(b) \end{array}$$

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- $f(t_0, \ldots, t_n) \colon b \in \mathit{Term}$ for $f \colon a_0 \ldots a_n \to b \in M$ and $t_0 \colon a_0, \ldots, t_n \colon a_n \in \mathit{Term}$.

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We define $\tau\colon Term \to V^+$. For a term $t \in Term$ we form the list of variables

$$\tau(t) = \begin{cases} (), & \text{if } t = c \\ v, & \text{if } t = v \\ \tau(t_1)\cdots\tau(t_n), & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

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- v is a monoidal context for t, if $\tau(t) = v$.
- The term t is called a monoidal term, if $\tau(t)$ is a context.

Monoidal terms can be constructed recursively.

Theories of Universal Algebra

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- A set of (symmetric) monoidal σ-equation is called (symmetric) monoidal σ-theory.

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- We call $m_{v,w}\colon m_v \to m_w$ a canonical morphism.

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- $$\begin{array}{ccc} m_v & \xrightarrow{m_{v,v_1\cdots v_n}} & m_{v_1} \otimes \cdots \otimes m_{v_n} \\ {\scriptstyle m_v(t)}\downarrow & & \downarrow{\scriptstyle m_{v_1}(t_1)\otimes\ldots\otimes m_{v_n}(t_n)} \\ m(b) & \xleftarrow[m(f)]{} & m_a \end{array}$$ for $t = f(t_1, \ldots, t_n)$ and $v_i = \mathrm{rb}(\tau(t_i)), i \leq n$.

Logical Entailment

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Let m be a σ-model in a (cartesian/symmetric) monoidal category C. Let $v, w \in V^*$ be (cartesian/symmetric) monoidal contexts for terms $t, t_1, t_2 : b$ and v is a context for w. Then the following assertions hold:

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- *The diagram*

$$\begin{array}{ccc} m_v & & \\ {\scriptstyle m_{v,w}}\downarrow & \searrow^{m_v(t)} & \\ m_w & \xrightarrow[m_w(t)]{} & m(b) \end{array}$$

  commutes.

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  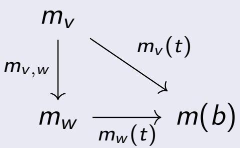

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- *If the equation $m_w(t_1) = m_w(t_2)$ holds, so does $m_v(t_1) = m_v(t_2)$.*

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Definition (Satisfiability and Entailment)

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Let m be a σ-model in a (symmetric) monoidal category C, we define:

- Model m satisfies (symmetric) monoidal equation $t_1 \approx t_2$, iff $m_v(t_1) = m_v(t_2)$ for some (symmetric) monoidal context v. We then denote $m \vDash t_1 \approx t_2$.

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- (Symmetric) monoidal theory E entails ϕ in C, if $m \vDash E$ implies $m \vDash \phi$ for all σ-models m in C. This is denoted $E \vDash_C \phi$.

Examples of Theories

Example (Enriched Category)

Let S be a set. Consider a signature $\sigma = (S \times S, M = \{\circ_{a,b,c} \colon (b,c)(a,b) \to (a,c), id_a \colon () \to (a,a) | a, b, c \in S\})$.

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$$(h \circ_{b,c,d} g) \circ_{a,b,d} f \approx h \circ_{a,c,d} (g \circ_{a,b,c} f)$$
$$f \circ_{a,a,b} id_a \approx f$$
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for all $a, b, c, d \in S$ and distinct variable symbols $f : (a,b), g : (b,c)$ and $h : (c,d)$.

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for all $a,b,c,d \in S$ and distinct variable symbols $f \colon (a,b), g \colon (b,c)$ and $h \colon (c,d)$.
The σ-models satisfying E in a monoidal category C are exactly C-enriched categories with S being the set of objects.

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for $a^1,\ldots,a^n, b = b_1\cdots b_n \in S^+$ and $c,d \in S$.

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for $a^1,\dots,a^n, b = b_1\cdots b_n \in S^+$ and $c,d \in S$. Let E consist of the following equations

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for a suitable choice of sorts and distinct variable symbols h, g_i, f_j^i for $i \leq n$ and $j \leq m_i$.

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for a suitable choice of sorts and distinct variable symbols h, g_i, f_j^i for $i \leq n$ and $j \leq m_i$. Models for E in a symmetric monoidal category V are exactly the V-enriched multi-categories with S as the set of objects. A single object V-enriched multi-category is then an V-enriched operad.

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Let $\sigma = (S, M, V)$ be a signature and let $v = v_1 \cdots v_n$ be a context, where $v_1, \ldots, v_n \in V$. A function $s\colon Var(v) = \{v_1, \ldots, v_n\} \to Term$ that preserves the typing is called a renaming of variables in v.

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Any renaming $s\colon Var(v) \to Term$ extends uniquely to a typing preserving function $\bar{s}\colon Term_v \to Term$, where

$$\bar{s}(t) = \begin{cases} c, \text{ if } t = c \\ s(x), \text{ if } t = x \\ f(\bar{s}(t_1), \ldots, \bar{s}(t_n)), \text{ if } t = f(t_1, \ldots, t_n), \end{cases} \quad \text{for } t \in Term_v.$$

Furthermore, if s is a monoidal renaming, then $\bar{s}$ maps monoidal terms to monoidal terms.

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5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2 \colon Var(t_1) \to Term$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in Var(t_1)$. Then $s(t_1) \approx s(t_2) \in D_E$.

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5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2 : Var(t_1) \to Term$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in Var(t_1)$. Then $s(t_1) \approx s(t_2) \in D_E$.

If $\phi \in D_E$, we write $E \vdash \phi$ and say that ϕ is syntactically deduced from E.

Soundness

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Lemma (Substitution Lemma)

Let m be a σ-model in (cartesian/symmetric) monoidal category C. Assume that $v = v_1 \dots v_n$ is a (cartesian/symmetric) monoidal context for a term $t \colon b$. Assume that $s \colon Var(v) \to Term$ is a (cartesian) monoidal renaming.

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Then for any (cartesian/symmetric) monoidal contexts $w, w_1, \dots w_n$ for $s(t), s(v_1), \dots, s(v_n)$, respectively, where w_i has its variables expressed in w for $i \leq n$, the equation holds

$$m_w(s(t)) = m_v(t) \circ m_{w_1}(s(v_1)) \otimes \dots \otimes m_{w_n}(s(v_n)) \circ m_{w,w_1 \cdots w_n}.$$

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Theorem (Soundness)

Let $E \cup \{\phi\}$ be a (symmetric) monoidal σ-theory. Let C be a (symmetric) monoidal category. Then $E \vdash \phi$ implies $E \vDash_C \phi$.

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Proof.

Let $m \models E$ in C and denote by $T = \{\phi | m \models \phi\}$. For $D_E \subset T$, it suffices to show the substitution condition for T: Assume that v is a (symmetric) monoidal context for terms $t_1, t_2 \colon b$ and $t_1 \approx t_2 \in T$ and w is a (symmetric) monoidal context for $s_1(t), s_2(t)$ for monoidal renamings $s_1, s_2 \colon Var(v) \to Term$ where $s_1(x) \approx s_2(x) \in T$ for all $x \in Var(v)$.
By the Substitution Lemma, we have

$$\begin{aligned} m_w(s_1(t_1)) &= m_v(t_1) \circ m_{w_1}(s_1(v_1)) \otimes \ldots \otimes m_{w_n}(s_1(v_n)) \circ m_{w,w_1 \cdots w_n} \\ &= m_v(t_2) \circ m_{w_1}(s_2(v_1)) \otimes \ldots \otimes m_{w_n}(s_2(v_n)) \circ m_{w,w_1 \cdots w_n} \\ &= m_w(s_2(t_2)) \end{aligned}$$

for $w_i = \tau(s_1(v_i)), i \leq n$. Hence $s_1(t_1) \approx s_2(t_2) \in T$. □

Modified Lindebaum-Tarski-algebras

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- $n(s) = \{*, t | t\colon s \text{ is a monoidal term}\}$ and $m(s) = n(s)/\sim_s$, where $\sim_s = \{(t_1, t_2), (*, *) | E \vdash t_1 \approx t_2, t_1 : s\}$ for sorts s. We denote the quotient map by $q_s\colon n(s) \to m(s)$ for sorts s.
- $n(\alpha)\colon n(a) \to n(b)$,
  $$(u_1, \ldots, u_n) \mapsto \begin{cases} f(u_1, \ldots, u_n), & \text{if } f(u_1, \ldots, u_n) \in n(b) \\ *, & \text{else} \end{cases}$$

Modified Lindebaum-Tarski-algebras

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  $m(\alpha)\colon m(a) \to m(b)$ is the unique map making the commutative diagram
  $$\begin{array}{ccc} n(a) & \xrightarrow{n(\alpha)} & n(b) \\ {\scriptstyle q_a}\downarrow & & \downarrow{\scriptstyle q_b} \\ m(a) & \xrightarrow[m(\alpha)]{} & m(b) \end{array}$$

Completeness

Completeness

Lemma (Term-Naturality of Model Morphisms)

Let $f : m \to n$ be a morphism of σ-models in a (cartesian/symmetric) monoidal category C. Let v be a (cartesian/symmetric) monoidal context for a term $t : b$. Then the diagram

$$\begin{array}{ccc} m_v & \xrightarrow{m_v(t)} & m(b) \\ \downarrow{\scriptstyle f_v} & & \downarrow{\scriptstyle f_b} \\ n_v & \xrightarrow[n_v(t)]{} & n(b) \end{array}$$

commutes.

Completeness

Lemma (Evaluation Lemma)

Completeness

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Let E be a (symmetric) monoidal σ-theory. Let m be the monoidal E-model. Let $v = v_1 \cdots v_n$ be a context for a term $t \colon b$. Denote the variables expressed in t by $v_{i_1}, \ldots, v_{i_k}$. Then

$$n_v(t)(u_1, \ldots, u_n) = \begin{cases} *, & \text{if } * \text{ or a variable twice in } (u_{i_1}, \ldots, u_{i_k}), \\ s(t), & \text{else where } s(v_{i_j}) = u_{i_j}, j \leq k. \end{cases}$$

for $(u_1, \ldots, u_n) \in n_v$.

Completeness

Theorem (Completeness)

Let $E \cup \{\phi\}$ be a (symmetric) monoidal theory and let m be the monoidal E-model. Then $m \vDash \phi$ if and only if $E \vdash \phi$. Especially, $E \vdash \phi$ if and only if $E \vDash_{\mathbf{Set}} \phi$.

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Proof.

$\Rightarrow$: Let $v = v_1 \ldots v_n$ be a (symmetric) monoidal context for terms $t_1, t_2 \colon b$. Assume that $m_v(t_1) = m_v(t_2)$. By the Evaluation Lemma and the term naturality of the quotient $q \colon n \to m$, it follows that

$$\begin{aligned}[t_1] &= m_v(t_1)([v_1], \ldots, [v_n]) \\ &= m_v(t_2)([v_1], \ldots, [v_n]) \\ &= [t_2]\end{aligned}$$

and hence $E \vdash t_1 \approx t_2$.

Completeness

E-Model Completeness (continued).

$\Leftarrow$: Assume then that $E \vdash t_1 \approx t_2$. We show that $m_v(t_1) = m_v(t_2)$. Let $([u_1], \ldots, [u_n]) \in m_v$. Now again by the previous lemmas again

$$
\begin{aligned}
m_v(t_1)([u_1], \ldots, [u_n]) &= [n_v(t_1)(u_1, \ldots, u_n)] \\
&= \begin{cases} [*], & \text{if } * \text{ or a variable twice in } (u_1, \ldots, u_n) \\ [s(t_1)], & \text{else where } s(v_i) = u_i, i \leq n \end{cases} \\
&= \begin{cases} [*], & \text{if } * \text{ or a variable twice in } (u_1, \ldots, u_n) \\ [s(t_2)], & \text{else where } s(v_i) = u_i, i \leq n \end{cases} \\
&= [n_v(t_2)(u_1, \ldots, u_n)] \\
&= m_v(t_2)([u_1], \ldots, [u_n])
\end{aligned}
$$

Thus $m_v(t_1) = m_v(t_2)$. □

Monoidal Meta-Theorem

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Let $E \cup \{\phi\}$ (cartesian/symmetric) monoidal theory. Then $E \vDash_{\mathbf{Set}} \phi$ implies $E \vDash_C \phi$ for all (cartesian/symmetric) monoidal categories C.

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Proof.

If $E \vDash_{\mathbf{Set}} \phi$, then by completeness $E \vdash \phi$ and hence by soundness $E \vDash_C \phi$ for all (cartesian/symmetric) monoidal categories C. □

Thank You

Thank you for your attention!