Monoidal Meta-Theorem

David Forsman
david.forsman@uclouvain.be

Université catholique de Louvain

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Theorem (Monoidal Meta-Theorem)

Let $E \cup \{\phi\}$ (cartesian/symmetric) monoidal $\sigma$-theory and let $C$ be a (cartesian/symmetric) monoidal category. Then

$$E \models_{\text{Set}} \phi \text{ implies } E \models_{C} \phi.$$
Example (Eckmann-Hilton Argument)

Let $\sigma = (S = \{a\}, M = \{+, +': aa \to a; 0', 0: () \to a\})$
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Let $\sigma = (S = \{a\}, M = \{+,: aa \to a; 0', 0: () \to a\})$ and $E$ consists of

$$
\begin{align*}
    x + 0 &\approx x, & 0 + x &\approx x, \\
    x + 0' &\approx x, & 0' +' x &\approx x, \\
    (x + y) +' (z + w) &\approx (x +' z) + (y +' w)
\end{align*}
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\end{align*}
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- \( E \models_{\text{Set}} T \), for \( T = 
\{ x +' y \approx x + y, e \approx e', x + y \approx y + x, (x + y) + z \approx x + (y + z) \}. \)
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Let $\sigma = (S = \{a\}, M = \{+, +': aa \to a; 0', 0: () \to a\})$ and $E$ consists of

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x + 0' & \approx x, & 0' + x & \approx x, \\
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\end{align*}
\]

- $E \models_{\text{Set}} T$, for $T = \{x+' y \approx x+y, e \approx e', x+y \approx y+x, (x+y)+z \approx x+(y+z)\}$.
- $E \models_{C} T$ for all symmetric monoidal categories $C$. 

David Forsman  david.forsman@uclouvain.be
Non-Examples

Choose $\sigma = (S = \{a\}, M = \{f, g: a \to ()\})$. Now $\emptyset \models \mathsf{Set} f \approx g$ and $\emptyset \not\models \mathsf{Ab} f \approx g$, where $\mathsf{Ab}$ is the monoidal category of abelian groups.

Example $\mathsf{Set} \sigma = (S = \{a\}, M = \{f, g: a \to aa\})$ and $E = \{\begin{array}{l} a \quad a \quad a \\ a \\ f \\ f \\ \Box \end{array} \}$. Now $E \models \mathsf{Set} f \approx g$.

Let $C = \mathsf{Set}^{\text{op}}$ be equipped with its cocartesian structure, $E \not\models C f \approx g$: $m(a) = \{1, 2, 3\}$, $m(f)(x, y) = \min(3, x + y)$, for $x, y \in m(a)$, and $m(g) \equiv 3$. $m \models E$ but $m \not\models f \approx g$.

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Example

Set \( \sigma = (S = \{a\}, M = \{f, g : a \to aa\}) \) and

\[
E = \begin{cases}
  a \xrightarrow{f} aa \\
  g \\
  aa \xrightarrow{g \square 1} aaa
\end{cases} \quad \begin{cases}
  a \xrightarrow{f} aa \\
  g \\
  a \xrightarrow{1 \square g} aaa
\end{cases}.
\]
Non-Examples

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$$E = \begin{cases}
  a \xrightarrow{f} aa & a \xrightarrow{f} aa \\
  g \downarrow & g \downarrow & 1 \downarrow f \\
  aa \xrightarrow{g \Box 1} aaa & a \xrightarrow{1 \Box g} aaa
\end{cases}$$

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\end{cases}
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Now \( E \models_{\text{Set}} f \approx g \). Let \( C = \text{Set}^{\text{op}} \) be equipped with its cocartesian structure, \( E \not\models_C f \approx g \).
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Now $E \models_{\text{Set}} f \approx g$. Let $C = \text{Set}^{op}$ be equipped with its cocartesian structure, $E \not\models_{C} f \approx g$:

- $m(a) = \{1, 2, 3\}$
Non-Examples

Example

Choose $\sigma = (S = \{a\}, M = \{f, g : a \to ()\})$. Now $\emptyset \models_{\text{Set}} f \simeq g$ and $\emptyset \not\models_{\text{Ab}} f \simeq g$, where $\text{Ab}$ is the monoidal category of abelian groups.

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Set $\sigma = (S = \{a\}, M = \{f, g : a \to aa\})$ and

$$E = \left\{ \begin{array}{c}
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    g \downarrow \quad f \Box 1 \\
    aa \xrightarrow{g \Box 1} aaa
\end{array} \right\} \quad \begin{array}{c}
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    g \downarrow \quad 1 \Box f \\
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\end{array} \right\}.$$

Now $E \models_{\text{Set}} f \simeq g$. Let $C = \text{Set}^{\text{op}}$ be equipped with its cocartesian structure, $E \not\models_{C} f \simeq g$:

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- $m(f)(x, y) = \min(3, x + y)$, for $x, y \in m(a)$, and $m(g) \equiv 3$. 
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Choose $\sigma = (S = \{a\}, M = \{f, g: a \to ()\})$. Now $\emptyset \models_{\text{Set}} f \approx g$ and $\emptyset \not\models_{\text{Ab}} f \approx g$, where $\text{Ab}$ is the monoidal category of abelian groups.

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Magmoidal structures

Definition (Structures on Magmoid)
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Let \((C, \otimes: C \times C \to C, I)\) be a pointed magma in the meta-category of categories.
Magmoidal structures

Definition (Structures on Magmoid)

Let \((C, \otimes: C \times C \to C, I)\) be a pointed magma in the meta-category of categories. Consider the natural transformations of the following form:

\[
(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)
\]

(Associator)

\[
I \otimes x \xrightarrow{\lambda_x} x
\]

(Left unitor)

\[
x \otimes I \xrightarrow{\rho_x} x
\]

(Right unitor)

\[
x \otimes y \xrightarrow{\gamma_{x,y}} y \otimes x
\]

(Braiding/Symmetry)

\[
x \xrightarrow{!} I
\]

(Deletor)

\[
x \xrightarrow{\delta_x} x \otimes x
\]

(Diagonal)
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x \otimes I & \overset{\rho_x}{\longrightarrow} x \quad \text{(Right unitor)} \\
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x & \overset{1_x}{\longrightarrow} I \quad \text{(Deletor)}
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\[
x \xrightarrow{1_x} I \quad \text{(Deletor)}
\]

\[
x \xrightarrow{\delta_x} x \otimes x \quad \text{(Diagonal)}
\]
Free Monoid and Pointed Magma

Let $X$ be a set. The free monoid $X^+$ is the set $F_{\in N}X^n$ of finite lists over $X$, where $X^n = \{ f : [n] \to X \}$ and $[n] = \{ i \in N | i < n \}$ for $n \in N$. The length is defined as the canonical map $l : X^+ \to N$.

The free pointed magma $X^*$ over $X$ as a set is defined recursively:

$x, e \in X^*$, for $x \in X | (xy) \in X^*$ for $x, y \in X^*$.

The function $\tau : X^* \to (X \sqcup \{ e \})^+$ is defined by the removal of parenthesis. Right bracketing of a word defines a section $rb : (X \sqcup \{ e \})^+ \to X^*$ to $\tau$.

We denote by $I_v = \{ i < l(v) | v_i \neq e \}$ the set of essential indices of $v \in X^*$. Let $X = \{ x, y \}$. The set of essential indices of $v = ((xe)(y(ex))) \in X^*$ is $I_v = \{ 0, 2, 4 \}$ and $\tau(v) = xeyex$. 

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David Forsman  david.forsman@uclouvain.be
Free Completions

Definition (Cartesian Monoidal Completion)

Let $I$ be a small category. We define the cartesian monoidal completion $\text{CM}(I)$ of $I$ as follows:

The set of objects is $\text{Obj}(I)^*$. A morphism $v \to w$ consists of a pair $(\theta, f)$, where $\theta : I_w \to I_v$ is a function and $f$ is a family of morphisms $f_i : v \theta(i) \to w_i$ for $i \in I_w$. The composition is the natural one.

The category $\text{CM}(I)$ has a cartesian monoidal structure.

Furthermore, $\text{CM}(I)$ has two wide subcategories the symmetric monoidal completion $\text{SM}(I)$ and the monoidal completion $\text{M}(I)$ of $I$ defined by morphisms $(\theta, f)$, $(\phi, g)$, respectively, where $\theta$ is a bijection and $\phi$ is an increasing bijection.
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The category $CM(I)$ has a cartesian monoidal structure. Furthermore, $CM(I)$ has two wide subcategories the symmetric monoidal completion $SM(I)$ and the monoidal completion $M(I)$ of $I$ defined by morphisms $(\theta, f), (\phi, g)$, respectively, where $\theta$ is a bijection and $\phi$ is an increasing bijection.
The categories CM(\text{I}), SM(\text{I}) and M(\text{I}) are cartesian monoidal, symmetric monoidal and monoidal categories via the restriction of the structure of CM(\text{I}).

Theorem (Coherence Theorem)

Let \( F: \text{I} \to \text{UC} \) be a functor, where \( \text{UC} \) is the underlying category of a (cartesian/symmetric) monoidal category \( \text{C} \).

Then there exists a unique strict functor \( F: T(\text{I}) \to \text{C} \) extending \( F \), where \( T(\text{I}) = (\text{C}/\text{S})\text{M}(\text{I}) \).

In addition, if \( F \) is constant on all hom-sets, then \( F \) is constant on hom-sets \( \text{Hom}(v, w) \), where the directed path components \( [v_i], i \in \text{I} \) are pairwise disjoint in \( \text{I} \).

David Forsman  david.forsman@uclouvain.be
Relevant Coherence Theorems

The categories CM(I), SM(I) and M(I) are cartesian monoidal, symmetric monoidal and monoidal categories via the restriction of the structure of CM(I).
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Let $F : I \rightarrow UC$ be a functor, where $UC$ is the underlying category of a (cartesian/symmetric) monoidal category $C$. 

David Forsman  david.forsman@uclouvain.be
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Let $F : I \to UC$ be a functor, where $UC$ is the underlying category of a (cartesian/symmetric) monoidal category $C$. Then there exists a unique strict functor $\overline{F} : T(I) \to C$ extending $F$, where $T(I) = (C/S)M(I)$. 
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In addition, if $F$ is constant on all hom-sets, then $\bar{F}$ is constant on hom-sets $\text{Hom}(v, w)$, where the directed path components $[v_i], i \in I_v$, are pairwise disjoint in $I$. 
Let $I$ be a category and $C$ a (cartesian/symmetric) monoidal category. Let $T(I)$ be the (cartesian/symmetric) monoidal completion of $I$. 

Consider exponential the transposition $I \to [C, C I]$ of the evaluation functor $C I \times I \to C$. We attain a strict functor $T(I) \to [C I, C]$. Thus each arrow in $T(I)$ can be considered a natural transformation. 

As an example, we have a unique morphism $\alpha : ((xy)z) \to (x(yz))$ in $T\{x, y, z\}$. Thus we attain a natural transformation between two functors $C I \cong C \Rightarrow C$, which is the whole associator itself.
Let $I$ be a category and $C$ a (cartesian/symmetric) monoidal category. Let $T(I)$ be the (cartesian/symmetric) monoidal completion of $I$.
Consider exponential the transposition $I \to [C^I, C]$ of the evaluation functor $C^I \times I \to C$. We attain a strict functor $T(I) \to [C^I, C]$. Thus each arrow in $T(I)$ can be considered a natural transformation.

As an example, we have a unique morphism $\alpha: ((xy)z) \to (x(yz))$ in $T(\{x, y, z\})$. Thus we attain a natural transformation between two functors $C^3 \sim = C^I \Rightarrow C$, which is the whole associator itself.
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Definition (Signature)

A multi-sorted signature $\sigma$ of universal algebra consists of the following data:

- A set $S$ of sorts.
- A graph of morphism symbols $M \rightarrow S^+ \times S$.
- A typed set of variable symbols $V \rightarrow S$, where each fiber is countably infinite.

We will often just denote $\sigma = (S, M)$ or $\sigma = (S, M, V)$.

Notation:

- $x : s \in V$.
- $f : a \rightarrow b \in M$.
- If $a = ()$, then $f : b \in M$.
Signature of Universal Algebra

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Models of Universal Algebra

Definition (σ–Model and Morphism)

Let \( \sigma = (S, M) \) be a signature. Let \( C \) be a monoidal category. A \( \sigma \)-model \( m \) in \( C \) consists of associations 

\[ m_1: S \rightarrow \text{Obj}(C) \]

and

\[ m_2: M \rightarrow \text{Mor}(C), \]

where

\[ m_2(f): m_1(r_b(a)) \rightarrow m_1(b) \]

for all \( f: a \rightarrow b \in M \).

A \( \sigma \)-model morphism \( m \rightarrow n \) in \( C \) consists of a family \( f_s: m(s) \rightarrow n(s) \), \( s \in S \), where for all morphism symbols \( \alpha: a \rightarrow b \) we have a commuting diagram

\[ m(r_b(a)) \rightarrow m(b) \]

\[ n(r_b(a)) \rightarrow n(b) \]

\[ f(r_b(a)) \rightarrow f(b) \]

\[ m(\alpha) \rightarrow n(\alpha) \]
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David Forsman  david.forsman@uclouvain.be
Models of Universal Algebra

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\[
\begin{array}{ccc}
  m(rb(a)) & \xrightarrow{m(\alpha)} & m(b) \\
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  n(rb(a)) & \xrightarrow{n(\alpha)} & n(b)
\end{array}
\]
Terms of Universal Algebra

Definition (Terms)

Let \( \sigma = (S, M, V) \) be a signature. The typed set of \( \sigma \)-terms \( \text{Term} \to S \) is defined recursively as follows:

- \( x, c \in \text{Term} \) for constant symbols \( c \in M \) and \( x \in V \) (the type is preserved).
- \( f(t_0, \ldots, t_n) : b \in \text{Term} \) for \( f : a_0, \ldots, a_n \to b \in M \) and \( t_0 : a_0, \ldots, t_n : a_n \in \text{Term} \).

We define \( \tau : \text{Term} \to V^+ \). For a term \( t \in \text{Term} \) we form the list \( \tau(t) = \begin{cases} () & \text{if } t = c \\ v & \text{if } t = v \\ \tau(t_1) \cdots \tau(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases} \).
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We define $\tau : \text{Term} \rightarrow V^+$. For a term $t \in \text{Term}$ we form the list of variables $\tau(t) = \begin{cases} () & \text{if } t = c \in V^+, \\ v & \text{if } t = v \in V, \\ \tau(t_1) \cdots \tau(t_n) & \text{if } t = f(t_1, \ldots, t_n) \in M. \end{cases}$
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David Forsman  david.forsman@uclouvain.be

Monoidal Meta-Theorem
Definition (Terms)

Let $\sigma = (S, M, V)$ be a signature. The typed set of $\sigma$-terms $Term \rightarrow S$ is defined recursively as follows:

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We define $\tau : Term \rightarrow V^+$. For a term $t \in Term$ we form the list of variables $\tau(t) = \begin{cases} () & \text{if } t = c \\ v & \text{if } t = v \\ \tau(t_1) \cdots \tau(t_n) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$
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Monoidal Meta-Theorem
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Let $\sigma = (S, M, V)$ be a signature. An element $v$ of $V^+$ is called a context if no variable repeats in $v$. Furthermore, we define for a term $t$:

- $v$ is a cartesian monoidal context or just a context for $t$ if all variables expressed in $t$ are expressed in $v$. 
- $v$ is a symmetric monoidal context for $t$, if $\tau(t)$ is a permutation of $v$.
- $v$ is a monoidal context for $t$, if $\tau(t) = v$.

The term $t$ is called a monoidal term, if $\tau(t)$ is a context.

David Forsman  david.forsman@uclouvain.be

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Monoidal terms can be constructed recursively.
Terms of Universal Algebra

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Monoidal terms can be constructed recursively.
Theories of Universal Algebra

Definition (Equation & Theory)

Let \( \sigma \) be a signature. For \( \sigma \)-terms \( t_1, t_2 : s \), we call \( t_1 \approx t_2 \) a \( \sigma \)-equation.

A set of \( \sigma \)-equations \( E \) is called a \( \sigma \)-theory.

A \( \sigma \)-equation \( t_1 \approx t_2 \) is called (symmetric) monoidal if \( t_1 \) and \( t_2 \) have a common (symmetric) monoidal context.

A set of (symmetric) monoidal \( \sigma \)-equations is called (symmetric) monoidal \( \sigma \)-theory.
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- A $\sigma$-equation $t_1 \approx t_2$ is called (symmetric) monoidal if $t_1$ and $t_2$ have a common (symmetric) monoidal context.
- A set of (symmetric) monoidal $\sigma$-equation is called (symmetric) monoidal $\sigma$-theory.
Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$. We attain a strict functor $m : \text{Free}(V) \to C$ from $V$ typing $\cdots \to S \xrightarrow{m} \text{Obj}(C)$. For a (cartesian/symmetric) context $v \in V^{\ast}$ of $w \in V^{\ast}$, we define $m(v),w = m(!) : m(v) \to m(w)$, where $!$ is the unique morphism $v \to w$ in $\text{Free}(V)$. We call $m(v),w : m(v) \to m(w)$ a canonical morphism.
Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$. 
Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$.

- We attain a strict functor $\overline{m}: \text{Free}(V) \to C$ from $V \xrightarrow{\text{typing}} S \xrightarrow{m} \text{Obj}(C)$. 

David Forsman  david.forsman@uclouvain.be
Canonical Morphisms

Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$.

1. We attain a strict functor $\overline{m}: \text{Free}(V) \to C$ from $V \xrightarrow{\text{typing}} S \xrightarrow{m} \text{Obj}(C)$.
2. For a (cartesian/symmetric) context $v \in V^*$ of $w \in V^*$, we define

$$m_{v,w} = \overline{m}(!): \overline{m}(v) \to \overline{m}(w),$$

where $!$ is the unique morphism $v \to w$ in $\text{Free}(V)$.
Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$.

- We attain a strict functor $\bar{m}: \text{Free}(V) \to C$ from $V \xrightarrow{\text{typing}} S \xrightarrow{m} \text{Obj}(C)$.
- For a (cartesian/symmetric) context $v \in V^*$ of $w \in V^*$, we define

$$m_{v,w} = \bar{m}(!) : \bar{m}(v) \to \bar{m}(w),$$

where $!$ is the unique morphism $v \to w$ in Free($V$).
- We call $m_{v,w} : m_v \to m_w$ a canonical morphism.
Let \( m \) be a \( \sigma \)-model in a (cartesian/symmetric) monoidal category \( C \). Let \( v \in V^* \) be a (cartesian/symmetric) monoidal context for a term \( t \).

We define the term morphism \( m^v(t) : m^v \rightarrow m^b \) of \( t \) in context \( v \) as follows:

- For \( t = c \), \( m^v \xrightarrow{e} I \xrightarrow{m^b} \).
- For \( t = x \), \( m^v \xrightarrow{x} m^x \).
- For \( t = f(t_1, \ldots, t_n) \) and \( v_i = \tau(t_i) \), \( i \leq n \).

David Forsman  
david.forsman@uclouvain.be

Monoidal Meta-Theorem
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Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$. Let $\nu \in V^*$ be a (cartesian/symmetric) monoidal context for a term $t$. We define the term morphism $m_\nu(t): m_\nu \to m(b)$ of $t$ in context $\nu$ as follows:
Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$. Let $\nu \in V^*$ be a (cartesian/symmetric) monoidal context for a term $t$. We define the term morphism $m_\nu(t) : m_\nu \to m(b)$ of $t$ in context $\nu$ as follows:

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Let $m$ be a $\sigma$–model in a (cartesian/symmetric) monoidal category $C$. Let $\nu \in \mathcal{V}^*$ be a (cartesian/symmetric) monoidal context for a term $t$. We define the term morphism $m_{\nu}(t) : m_{\nu} \to m(b)$ of $t$ in context $\nu$ as follows:

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- $m_\nu \xrightarrow{m_{\nu,x}} m_x$ for $t = x$
- $m_\nu \xrightarrow{m_{\nu,v_1 \cdots v_n}} m_{v_1} \otimes \cdots \otimes m_{v_n}$
- $m_\nu(t) \xrightarrow{m_{\nu}} m(b)$ for $t = f(t_1, \ldots, t_n)$ and $m_{v_1}(t_1) \otimes \cdots \otimes m_{v_n}(t_n)$
- $\nu_i = rb(\tau(t_i)), i \leq n.$

David Forsman  david.forsman@uclouvain.be
Logical Entailment

Lemma (Partial Context Independence)

Let $m$ be a $\sigma$-model in a (cartesian/symmetric) monoidal category $C$. Let $v, w \in V^*$ be (cartesian/symmetric) monoidal contexts for terms $t, t_1, t_2$: $b$ and $v$ is a context for $w$. Then the following assertions hold:

The diagram $m \xrightarrow{v} m \xrightarrow{w} m (b)$ commutes.

If the equation $m \xrightarrow{w} (t_1) = m \xrightarrow{w} (t_2)$ holds, so does $m \xrightarrow{v} (t_1) = m \xrightarrow{v} (t_2)$.

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Lemma (Partial Context Independence)

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- The diagram

\[
\begin{array}{ccc}
  m_v & \xrightarrow{m_v(t)} & m(b) \\
  m_{v,w} & \downarrow & \downarrow & \downarrow \\
  m_w & \xrightarrow{m_w(t)} & m(b)
\end{array}
\]

commutes.
Lemma (Partial Context Independence)

Let \( m \) be a \( \sigma \)-model in a (cartesian/symmetric) monoidal category \( C \). Let \( v, w \in V^* \) be (cartesian/symmetric) monoidal contexts for terms \( t, t_1, t_2 : b \) and \( v \) is a context for \( w \). Then the following assertions hold:

- The diagram

\[
\begin{array}{ccc}
m_v & \longrightarrow & m_v(t) \\
m_v, w \downarrow & & \downarrow \\
m_w & \longrightarrow & m(b) \\
m_w(t) & \longrightarrow & \\
\end{array}
\]

commutes.

- If the equation \( m_w(t_1) = m_w(t_2) \) holds, so does \( m_v(t_1) = m_v(t_2) \).
Definition (Satisfiability and Entailment)

Let $m$ be a $\sigma$-model in a (symmetric) monoidal category $C$, we define:

A $m$ satisfies (symmetric) monoidal equation $t_1 \approx t_2$, iff $m^v(t_1) = m^v(t_2)$ for some (symmetric) monoidal context $v$.

We then denote $m \models t_1 \approx t_2$.

(Symmetric) monoidal theory $E$ entails $\phi$ in $C$, if $m \models E$ implies $m \models \phi$ for all $\sigma$-models $m$ in $C$. This is denoted $E \models_C \phi$. 

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Definition (Satisfiability and Entailment)

Let $m$ be a $\sigma$-model in a (symmetric) monoidal category $C$, we define:

- Model $m$ satisfies (symmetric) monoidal equation $t_1 \approx t_2$, iff $m_v(t_1) = m_v(t_2)$ for some (symmetric) monoidal context $v$. We then denote $m \models t_1 \approx t_2$. 

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### Definition (Satisfiability and Entailment)

Let $m$ be a $\sigma$-model in a (symmetric) monoidal category $C$, we define:

- Model $m$ satisfies (symmetric) monoidal equation $t_1 \approx t_2$, iff $m_\nu(t_1) = m_\nu(t_2)$ for some (symmetric) monoidal context $\nu$. We then denote $m \vDash t_1 \approx t_2$.

- (Symmetric) monoidal theory $E$ entails $\phi$ in $C$, if $m \vDash E$ implies $m \vDash \phi$ for all $\sigma$-models $m$ in $C$. This is denoted $E \vDash_C \phi$. 

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**David Forsman**  
david.forsman@uclouvain.be

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**Monoidal Meta-Theorem**
Example (Enriched Category)

Let $S$ be a set. Consider a signature $\sigma = (S \times S, M = \{ \circ_{a,b,c} : (b, c)(a, b) \to (a, c), id_a : () \to (a, a)|a, b, c \in S\})$. 
Example (Enriched Category)

Let $S$ be a set. Consider a signature $\sigma = (S \times S, M = \{ \circ_{a,b,c}: (b, c)(a, b) \rightarrow (a, c), id_a: () \rightarrow (a, a) | a, b, c \in S \})$. Fix the theory $E$ consisting of

\[
(h \circ_{b,c,d} g) \circ_{a,b,d} f \approx h \circ_{a,c,d} (g \circ_{a,b,c} f) \\
f \circ_{a,a,b} id_a \approx f \\
id_b \circ_{a,b,b} f \approx f
\]

for all $a, b, c, d \in S$ and distinct variable symbols $f: (a, b), g: (b, c)$ and $h: (c, d)$. 
Example (Enriched Category)

Let $S$ be a set. Consider a signature $\sigma = (S \times S, M = \{\circ_{a,b,c}: (b, c)(a, b) \to (a, c), id_a: () \to (a, a) | a, b, c \in S\})$. Fix the theory $E$ consisting of

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(h \circ_{b,c,d} g) \circ_{a,b,d} f \approx h \circ_{a,c,d} (g \circ_{a,b,c} f)
\]

\[
f \circ_{a,a,b} id_a \approx f
\]

\[
id_b \circ_{a,b,b} f \approx f
\]

for all $a, b, c, d \in S$ and distinct variable symbols $f: (a, b), g: (b, c)$ and $h: (c, d)$.

The $\sigma$-models satisfying $E$ in a monoidal category $C$ are exactly $C$-enriched categories with $S$ being the set of objects.
Examples of Theories

Example (Enriched Multi-Category)

Let $S$ be a set. Consider the signature $\sigma = (S^+ \times S, M)$.
Examples of Theories

Example (Enriched Multi-Category)

Let $S$ be a set. Consider the signature $\sigma = (S^+ \times S, M)$, where $M$ consists of morphism symbols

$\circ_{a^1, \ldots, a^n, b, c} : (b, c)(a^1, b_1) \cdots (a^n, b_n) \rightarrow (a^1 \cdots a^n, c)$ and $id_d : (d, d)$

for $a^1, \ldots, a^n, b = b_1 \cdots b_n \in S^+$ and $c, d \in S$. 
Example (Enriched Multi-Category)

Let $S$ be a set. Consider the signature $\sigma = (S^+ \times S, M)$, where $M$ consists of morphism symbols

$$\circ_{a_1, \ldots, a^n, b, c} : (b, c)(a_1, b_1) \cdots (a^n, b_n) \to (a^1 \cdots a^n, c)$$

and $id_d : (d, d)$ for $a_1, \ldots, a^n, b = b_1 \cdots b_n \in S^+$ and $c, d \in S$. Let $E$ consist of the following equations

- $h \circ (g_n \circ (f^n_1, \ldots, f^n_{m_n}), \ldots, g_1 \circ (f^1_1, \ldots, f^1_{m_1})) \approx (h \circ (g_n, \ldots, g_1)) \circ (f^n_1, \ldots, f^1_{m_1})$
- $f \circ (id, \ldots, id) \approx f$
- $id \circ f \approx f$

for a suitable choice of sorts and distinct variable symbols $h, g_i, f^{i}_j$ for $i \leq n$ and $j \leq m_i$. 

David Forsman  david.forsman@uclouvain.be

Monoidal Meta-Theorem
Example (Enriched Multi-Category)

Let $S$ be a set. Consider the signature $\sigma = (S^+ \times S, M)$, where $M$ consists of morphism symbols

$$\circ a^1, \ldots, a^n, b, c : (b, c)(a^1, b_1) \cdots (a^n, b_n) \rightarrow (a^1 \cdots a^n, c)$$

and $id_d : (d, d)$

for $a^1, \ldots, a^n, b = b_1 \cdots b_n \in S^+$ and $c, d \in S$. Let $E$ consist of the following equations

1. $h \circ (g_n \circ (f_1^n, \ldots, f_{m_1}^n), \ldots, g_1 \circ (f_1^1, \ldots, f_{m_1}^1)) \approx (h \circ (g_n, \ldots, g_1)) \circ (f_1^n, \ldots, f_{m_1}^1)$
2. $f \circ (id, \ldots, id) \approx f$
3. $id \circ f \approx f$

for a suitable choice of sorts and distinct variable symbols $h, g_i, f_j^i$ for $i \leq n$ and $j \leq m_i$. Models for $E$ in a symmetric monoidal category $V$ are exactly the $V$-enriched multi-categories with $S$ as the set of objects. A single object $V$-enriched multi-category is then an $V$-enriched operad.
Definition (Renaming)

Let $\sigma = (S, M, V)$ be a signature and let $v = v_1 \cdots v_n$ be a context, where $v_1, \ldots, v_n \in V$. A function $s: \text{Var}(v) = \{v_1, \ldots, v_n\} \to \text{Term}$ that preserves the typing is called a renaming of variables in $v$.

The renaming $s$ is called monoidal renaming, if the terms $s(v_i), i \leq n$, are monoidal and the variable sets $\text{Var}(\tau s(v_i)), i \leq n$, are pairwise disjoint.

Any renaming $s: \text{Var}(v) \to \text{Term}$ extends uniquely to a typing preserving function $s: \text{Term} v \to \text{Term}$, where $s(t) = \begin{cases} c, & \text{if } t = c \\ s(x) & \text{if } t = x f(s(t_1), \ldots, s(t_n)) \\ f(t_1, \ldots, t_n), & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$, for $t \in \text{Term} v$.

Furthermore, if $s$ is a monoidal renaming, then $s$ maps monoidal terms to monoidal terms.
Definition (Renaming)

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Any renaming $s : \text{Var}(v) \to \text{Term}$ extends uniquely to a typing preserving function $s : \text{Term}^v \to \text{Term}$, where $s(t) = \begin{cases} c, & \text{if } t = c \\ s(x) & \text{if } t = x \\ f(s(t_1), \ldots, s(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$ for $t \in \text{Term}^v$.

Furthermore, if $s$ is a monoidal renaming, then $s$ maps monoidal terms to monoidal terms.
Substitution

**Definition (Renaming)**

Let $\sigma = (S, M, V)$ be a signature and let $\nu = \nu_1 \cdots \nu_n$ be a context, where $\nu_1, \ldots, \nu_n \in V$. A function $s : \text{Var}(\nu) = \{\nu_1, \ldots, \nu_n\} \to \text{Term}$ that preserves the typing is called a renaming of variables in $\nu$. The renaming $s$ is called monoidal renaming, if the terms $s(\nu_i), i \leq n$, are monoidal and the variable sets $\text{Var}(\tau s(\nu_i)), i \leq n$, are pairwise disjoint.
Substitution

**Definition (Renaming)**

Let $\sigma = (S, M, V)$ be a signature and let $\nu = \nu_1 \cdots \nu_n$ be a context, where $\nu_1, \ldots, \nu_n \in V$. A function $s: \text{Var}(\nu) = \{\nu_1, \ldots, \nu_n\} \rightarrow \text{Term}$ that preserves the typing is called a renaming of variables in $\nu$. The renaming $s$ is called monoidal renaming, if the terms $s(\nu_i), i \leq n$, are monoidal and the variable sets $\text{Var}(\tau s(\nu_i)), i \leq n$, are pairwise disjoint.

Any renaming $s: \text{Var}(\nu) \rightarrow \text{Term}$ extends uniquely to a typing preserving function $\bar{s}: \text{Term}_\nu \rightarrow \text{Term}$, where

$$\bar{s}(t) = \begin{cases} 
c, & \text{if } t = c \\
s(x), & \text{if } t = x \\
f(\bar{s}(t_1), \ldots, \bar{s}(t_n)), & \text{if } t = f(t_1, \ldots, t_n),
\end{cases}$$

for $t \in \text{Term}_\nu$.

Furthermore, if $s$ is a monoidal renaming, then $\bar{s}$ maps monoidal terms to monoidal terms.
Synatactic deduction

Definition (Deduction)

Let $\sigma$ be a signature and $E$ a (symmetric) monoidal theory. We define the set $D_E$ of all deduced equations from $E$ as the smallest set satisfying the following conditions:

1. $E \subset D_E$.
2. $t \approx t \in D_E$ for all monoidal terms $t$.
3. If $t_1 \approx t_2 \in D_E$, then $t_2 \approx t_1 \in D_E$.
4. If $t_1 \approx t_2, t_2 \approx t_3 \in D_E$, then $t_1 \approx t_3 \in D_E$.
5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2 : \text{Var}(t_1) \to \text{Term}$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in \text{Var}(t_1)$. Then $s_1(t_1) \approx s_2(t_2) \in D_E$.

If $\phi \in D_E$, we write $E \vdash \phi$ and say that $\phi$ is syntactically deduced from $E$. 

David Forsman  david.forsman@uclouvain.be

Monoidal Meta-Theorem
Definition (Deduction)

Let $\sigma$ be a signature and $E$ a (symmetric) monoidal theory. We define the set $D_E$ of all deduced equations from $E$ as the smallest set satisfying the following conditions:

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4. If $t_1 \approx t_2, t_2 \approx t_3 \in D_E$, then $t_1 \approx t_3 \in D_E$.
5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2 : \text{Var}(t_1) \to \text{Term}$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in \text{Var}(t_1)$. Then $s(t_1) \approx s(t_2) \in D_E$.

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David Forsman  
david.forsman@uclouvain.be
Synatactic deduction

Definition (Deduction)

Let $\sigma$ be a signature and $E$ a (symmetric) monoidal theory. We define the set $D_E$ of all deduced equations from $E$ as the smallest set satisfying the following conditions:

1. $E \subseteq D_E$.
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3. If $t_1 \approx t_2 \in D_E$, then $t_2 \approx t_1 \in D_E$.
4. If $t_1 \approx t_2, t_2 \approx t_3 \in D_E$, then $t_1 \approx t_3 \in D_E$. 

If $\varphi \in D_E$, we write $E \vdash \varphi$ and say that $\varphi$ is syntactically deduced from $E$. 

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Definition (Deduction)

Let $\sigma$ be a signature and $E$ a (symmetric) monoidal theory. We define the set $D_E$ of all deduced equations from $E$ as the smallest set satisfying the following conditions:

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5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2: \text{Var}(t_1) \to \text{Term}$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in \text{Var}(t_1)$. Then $s(t_1) \approx s(t_2) \in D_E$. 
Definition (Deduction)

Let $\sigma$ be a signature and $E$ a (symmetric) monoidal theory. We define the set $D_E$ of all deduced equations from $E$ as the smallest set satisfying the following conditions:

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4. If $t_1 \approx t_2, t_2 \approx t_3 \in D_E$, then $t_1 \approx t_3 \in D_E$.
5. Let $t_1 \approx t_2 \in D_E$. Let $s_1, s_2 : \text{Var}(t_1) \rightarrow \text{Term}$ be a monoidal renamings, where $s_1(x) \approx s_2(x) \in D_E$ for all $x \in \text{Var}(t_1)$. Then $s(t_1) \approx s(t_2) \in D_E$.

If $\phi \in D_E$, we write $E \vdash \phi$ and say that $\phi$ is syntactically deduced from $E$. 
Soundness

### Lemma (Substitution Lemma)
Let $m$ be a $\sigma$-model in (cartesian/symmetric) monoidal category $C$. Assume that $v = v_1 \ldots v_n$ is a (cartesian/symmetric) monoidal context for a term $t$.

Assume that $s : \text{Var}(v) \to \text{Term}$ is a (cartesian) monoidal renaming.

Then for any (cartesian/symmetric) monoidal contexts $w, w_1, \ldots, w_n$ for $s(t), s(v_1), \ldots, s(v_n)$, respectively, where $w_i$ has its variables expressed in $w$ for $i \leq n$, the equation holds $m_w(s(t)) = m_v(t) \circ m_{w_1}(s(v_1)) \otimes \ldots \otimes m_{w_n}(s(v_n)) \circ m_{w_1} \ldots w_n$.

### Theorem (Soundness)
Let $E \cup \{\phi\}$ be a (symmetric) monoidal $\sigma$-theory. Let $C$ be a (symmetric) monoidal category. Then $E \vdash \phi$ implies $E \models C \phi$.

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David Forsman  david.forsman@uclouvain.be
Lemma (Substitution Lemma)

Let $m$ be a $\sigma$-model in (cartesian/symmetric) monoidal category $C$. Assume that $v = v_1 \ldots v_n$ is a (cartesian/symmetric) monoidal context for a term $t : b$. Assume that $s : \text{Var}(v) \to \text{Term}$ is a (cartesian) monoidal renaming.


**Lemma (Substitution Lemma)**

Let $m$ be a $\sigma$-model in (cartesian/symmetric) monoidal category $C$. Assume that $v = v_1 \ldots v_n$ is a (cartesian/symmetric) monoidal context for a term $t : b$. Assume that $s : \text{Var}(v) \rightarrow \text{Term}$ is a (cartesian) monoidal renaming.

Then for any (cartesian/symmetric) monoidal contexts $w, w_1, \ldots, w_n$ for $s(t), s(v_1), \ldots, s(v_n)$, respectively, where $w_i$ has its variables expressed in $w$ for $i \leq n$, the equation holds

$$m_w(s(t)) = m_v(t) \circ m_{w_1}(s(v_1)) \otimes \cdots \otimes m_{w_n}(s(v_n)) \circ m_{w,w_1\ldots w_n}.$$
Soundness

Lemma (Substitution Lemma)

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Theorem (Soundness)

Let $E \cup \{\phi\}$ be a (symmetric) monoidal $\sigma$-theory. Let $C$ be a (symmetric) monoidal category. Then $E \vdash \phi$ implies $E \models_C \phi$. 

David Forsman david.forsman@uclouvain.be
Soundness & Completeness

Proof.

Let $m \models E$ in $C$ and denote by $T = \{ \phi \mid m \models \phi \}$. For $D E \subset T$, it suffices to show the substitution condition for $T$:

Assume that $v$ is a (symmetric) monoidal context for terms $t_1, t_2$: $b$ and $t_1 \approx t_2 \in T$ and $w$ is a (symmetric) monoidal context for $s_1(t_1), s_2(t_2)$ for monoidal renamings $s_1, s_2$: $\text{Var}(v) \rightarrow \text{Term}$ where $s_1(x) \approx s_2(x) \in T$ for all $x \in \text{Var}(v)$.

By the Substitution Lemma, we have $m w(s_1(t_1)) = m v(t_1) \circ m w_1(s_1(v_1)) \otimes \ldots \otimes m w_n(s_1(v_n)) \circ m w_1 \cdot \ldots \cdot m w_n$ for $w_i = \tau(s_1(v_i)), i \leq n$. Hence $s_1(t_1) \approx s_2(t_2) \in T$.
Proof.

Let \( m \models E \) in \( C \) and denote by \( T = \{ \phi \mid m \models \phi \} \). For \( D_E \subset T \), it suffices to show the substitution condition for \( T \):
Proof.

Let $m \models E$ in $C$ and denote by $T = \{ \phi | m \models \phi \}$. For $D_E \subset T$, it suffices to show the substitution condition for $T$: Assume that $v$ is a (symmetric) monoidal context for terms $t_1, t_2 : b$ and $t_1 \approx t_2 \in T$ and $w$ is a (symmetric) monoidal context for $s_1(t), s_2(t)$ for monoidal renamings $s_1, s_2 : Var(v) \rightarrow Term$ where $s_1(x) \approx s_2(x) \in T$ for all $x \in Var(v)$. 

David Forsman  
david.forsman@uclouvain.be
Proof.

Let $m \models E$ in $C$ and denote by $T = \{ \phi | m \models \phi \}$. For $D_E \subset T$, it suffices to show the substitution condition for $T$: Assume that $v$ is a (symmetric) monoidal context for terms $t_1, t_2 : b$ and $t_1 \approx t_2 \in T$ and $w$ is a (symmetric) monoidal context for $s_1(t), s_2(t)$ for monoidal renamings $s_1, s_2 : \text{Var}(v) \to \text{Term}$ where $s_1(x) \approx s_2(x) \in T$ for all $x \in \text{Var}(v)$.

By the Substitution Lemma, we have

$$m_w(s_1(t_1)) = m_v(t_1) \circ m_{w_1}(s_1(v_1)) \otimes \ldots \otimes m_{w_n}(s_1(v_n)) \circ m_{w, w_1 \ldots w_n}$$

$$= m_v(t_2) \circ m_{w_1}(s_2(v_1)) \otimes \ldots \otimes m_{w_n}(s_2(v_n)) \circ m_{w, w_1 \ldots w_n}$$

$$= m_w(s_2(t_2))$$

for $w_i = \tau(s_1(v_i)), i \leq n$. Hence $s_1(t_1) \approx s_2(t_2) \in T$. 

David Forsman  david.forsman@uclouvain.be  Monoidal Meta-Theorem
Modified Lindebaum-Tarski-algebras

Definition (Modified Term-algebras)

Let $E$ be a (symmetric) monoidal $\sigma$-theory. We define the monoidal $\sigma$-term model $n$ and the monoidal $E$-model $m$ in $\mathcal{Set}$ as follows:

$n(s) = \{ \ast, t \mid t: s \text{ is a monoidal term} \}$

and

$m(s) = n(s) / \sim_s$, where $\sim_s = \{ (t_1, t_2) \mid E \vdash t_1 \approx t_2, t_1: s \}$ for sorts $s$.

We denote the quotient map by $q_s: n(s) \rightarrow m(s)$ for sorts $s$.

$n(\alpha): n(a) \rightarrow n(b), (u_1, \ldots, u_n) \mapsto (f(u_1, \ldots, u_n), \text{if } f(u_1, \ldots, u_n) \in n(b) \ast, \text{else})$ and

$m(\alpha): m(a) \rightarrow m(b)$ is the unique map making the commutative diagram $n(a) \rightarrow n(b)$ $m(a) \rightarrow m(b)$ $q_a \leftarrow^n \leftarrow^b q_b$.

David Forsman  
david.forsman@uclouvain.be

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Let $E$ be a (symmetric) monoidal $\sigma$-theory. We define the monoidal $\sigma$-term model $n$ and the monoidal $E$-model $m$ in $\textbf{Set}$ as follows:

\[ n(s) = \{*, t \mid t: s \text{ is a monoidal term} \} \]
Modified Lindebaum-Tarski-algebras

**Definition (Modified Term-algebras)**

Let $E$ be a (symmetric) monoidal $\sigma$-theory. We define the monoidal $\sigma$-term model $n$ and the monoidal $E$-model $m$ in $\textbf{Set}$ as follows:

- $n(s) = \{*, t | t: s \text{ is a monoidal term}\}$ and $m(s) = n(s)/ \sim_s$, where $\sim_s = \{(t_1, t_2), (*, *) | E \vdash t_1 \approx t_2, t_1: s\}$ for sorts $s$. We denote the quotient map by $q_s: n(s) \rightarrow m(s)$ for sorts $s$. 

David Forsman  
david.forsman@uclouvain.be  
Monoidal Meta-Theorem
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Let $E$ be a (symmetric) monoidal $\sigma$-theory. We define the monoidal $\sigma$-term model $n$ and the monoidal $E$-model $m$ in $\textbf{Set}$ as follows:

- $n(s) = \{*, t \mid t: s \text{ is a monoidal term}\}$ and $m(s) = n(s)/\sim_s$, where $\sim_s = \{(t_1, t_2), (\ast, \ast) \mid E \vdash t_1 \approx t_2, t_1: s\}$ for sorts $s$. We denote the quotient map by $q_s: n(s) \to m(s)$ for sorts $s$.

- $n(\alpha): n(a) \to n(b)$,
  \[(u_1, \ldots, u_n) \mapsto \begin{cases} f(u_1, \ldots, u_n), & \text{if } f(u_1, \ldots, u_n) \in n(b) \\ \ast, & \text{else} \end{cases} \]

David Forsman  david.forsman@uclouvain.be  Monoidal Meta-Theorem
Definition (Modified Term-algebras)

Let $E$ be a (symmetric) monoidal $\sigma$-theory. We define the monoidal $\sigma$ term model $n$ and the monoidal $E$-model $m$ in $\textbf{Set}$ as follows:

- $n(s) = \{*, t \mid t : s \text{ is a monoidal term}\}$ and $m(s) = n(s)/\sim_s$, where $\sim_s = \{(t_1, t_2), (*, *) \mid E \vdash t_1 \approx t_2, t_1 : s\}$ for sorts $s$. We denote the quotient map by $q_s : n(s) \rightarrow m(s)$ for sorts $s$.

- $n(\alpha) : n(a) \rightarrow n(b)$,

\[
(u_1, \ldots, u_n) \mapsto \begin{cases} 
    f(u_1, \ldots, u_n), & \text{if } f(u_1, \ldots, u_n) \in n(b) \\
    *, & \text{else}
\end{cases}
\]

and

$m(\alpha) : m(a) \rightarrow m(b)$ is the unique map making the commutative diagram

\[
\begin{array}{c}
  n(a) \xrightarrow{n(\alpha)} n(b) \\
  q_a \downarrow \quad \quad \quad \downarrow q_b \\
  m(a) \xrightarrow{m(\alpha)} m(b)
\end{array}
\]
Let $f: m \rightarrow n$ be a morphism of $\sigma$-models in a (cartesian/symmetric) monoidal category $C$. Let $v$ be a (cartesian/symmetric) monoidal context for a term $t: b$. Then the diagram $m^v (b) \rightarrow n^v (b)$ commutes.
Lemma (Term-Naturality of Model Morphisms)

Let $f : m \rightarrow n$ be a morphism of $\sigma$-models in a (cartesian/symmetric) monoidal category $C$. Let $\nu$ be a (cartesian/symmetric) monoidal context for a term $t : b$. Then the diagram

$$
\begin{array}{ccc}
  m_{\nu} & \xrightarrow{m_{\nu}(t)} & m(b) \\
  \downarrow f_{\nu} & & \downarrow f_{b} \\
  n_{\nu} & \xrightarrow{n_{\nu}(t)} & n(b)
\end{array}
$$

commutes.
Lemma (Evaluation Lemma)

Let $E$ be a (symmetric) monoidal $\sigma$-theory. Let $m$ be the monoidal $E$-model. Let $v = v_1 \cdots v_n$ be a context for a term $t: b$. Denote the variables expressed in $t$ by $v_{i_1}, \ldots, v_{i_k}$. Then

$$n(v(t)(u_1, \ldots, u_n)) = \begin{cases} \ast, & \text{if } \ast \text{ or a variable twice in } (u_{i_1}, \ldots, u_{i_k}) \\ s(t), & \text{else} \end{cases}$$

where $s(v_{i_j}) = u_{i_j}, j \leq k$. for $(u_1, \ldots, u_n) \in n(v)$. 

David Forsman  
david.forsman@uclouvain.be 

Monoidal Meta-Theorem
**Lemma (Evaluation Lemma)**

Let $E$ be a (symmetric) monoidal $\sigma$-theory. Let $m$ be the monoidal $E$-model. Let $v = v_1 \cdots v_n$ be a context for a term $t : b$. Denote the variables expressed in $t$ by $v_{i_1}, \ldots, v_{i_k}$. Then

$$n_v(t)(u_1, \ldots, u_n) = \begin{cases} *, & \text{if } * \text{ or a variable twice in } (u_{i_1}, \ldots, u_{i_k}), \\ s(t), & \text{else where } s(v_{i_j}) = u_{i_j}, j \leq k. \end{cases}$$

for $(u_1, \ldots, u_n) \in n_v$. 

David Forsman  david.forsman@uclouvain.be
Theorem (Completeness)

Let $E \cup \{ \phi \}$ be a (symmetric) monoidal theory and let $m$ be the monoidal $E$-model. Then $m \models \phi$ if and only if $E \vdash \phi$. Especially, $E \vdash \phi$ if and only if $E \models_{\text{Set}} \phi$. 

Proof.

$\Rightarrow$: Let $v = v_1 \ldots v_n$ be a (symmetric) monoidal context for terms $t_1, t_2 : b$. Assume that $m(v(t_1)) = m(v(t_2))$. By the Evaluation Lemma and the term naturality of the quotient $q : n \to m$, it follows that $[t_1] = m(v(t_1)([v_1], \ldots, [v_n])) = m(v(t_2)([v_1], \ldots, [v_n])) = [t_2]$ and hence $E \vdash t_1 \approx t_2$. 

$\Leftarrow$: 

David Forsman  david.forsman@uclouvain.be

Monoidal Meta-Theorem
Theorem (Completeness)

Let $E \cup \{\phi\}$ be a (symmetric) monoidal theory and let $m$ be the monoidal $E$-model. Then $m \models \phi$ if and only if $E \vdash \phi$. Especially, $E \vdash \phi$ if and only if $E \models_{\text{Set}} \phi$.

Proof.

$\Rightarrow$: Let $\nu = \nu_1 \ldots \nu_n$ be a (symmetric) monoidal context for terms $t_1, t_2 : b$. Assume that $m_{\nu}(t_1) = m_{\nu}(t_2)$. By the Evaluation Lemma and the term naturality of the quotient $q : n \to m$, it follows that

\[
[t_1] = m_{\nu}(t_1)([\nu_1], \ldots, [\nu_n])
\]

\[
= m_{\nu}(t_2)([\nu_1], \ldots, [\nu_n])
\]

\[
= [t_2]
\]

and hence $E \vdash t_1 \approx t_2$. 
Assume then that $E \vdash t_1 \approx t_2$. We show that $m_v(t_1) = m_v(t_2)$. Let $([u_1], \ldots, [u_n]) \in m_v$. Now again by the previous lemmas again

\[
m_v(t_1)([u_1], \ldots, [u_n]) = n_v(t_1)(u_1, \ldots, u_n) = \begin{cases} [\ast], & \text{if } \ast \text{ or a variable twice in } (u_1, \ldots, u_n) \\ [s(t_1)], & \text{else where } s(v_i) = u_i, i \leq n \end{cases}
\]

\[
= \begin{cases} [\ast], & \text{if } \ast \text{ or a variable twice in } (u_1, \ldots, u_n) \\ [s(t_2)], & \text{else where } s(v_i) = u_i, i \leq n \end{cases}
\]

\[= n_v(t_2)(u_1, \ldots, u_n) = m_v(t_2)([u_1], \ldots, [u_n])
\]

Thus $m_v(t_1) = m_v(t_2)$. 

\[\blacksquare\]
Monoidal Meta-Theorem

Theorem (Monoidal Meta-Theorem)

Let $E \cup \{\phi\}$ (cartesian/symmetric) monoidal theory. Then $E \models Set \phi$ implies $E \models C\phi$ for all (cartesian/symmetric) monoidal categories $C$.

Proof.
If $E \models Set \phi$, then by completeness $E \vdash \phi$ and hence by soundness $E \models C\phi$ for all (cartesian/symmetric) monoidal categories $C$. 
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Thank you for your attention!