The Flower Calculus

Pablo Donato

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SYCO 12, Birmingham

Based on arXiv:2402.15174

Context

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Disclaimer: no *category theory* in this talk!

Outline of this talk

1. **Classical Logic**: Existential Graphs
2. **Intuitionistic Logic**: Flowers
3. **Reasoning** with Flowers
4. **Metatheory**: Nature vs. Culture
5. **The Flower Prover**

Classical Logic: Existential Graphs

Existential Graphs (Peirce, 1896)

Three diagrammatic proof systems for **classical** logic:

- `Alpha`: *propositional* logic
- `Beta`: *first-order* logic
- `Gamma`: *higher-order* and *modal* logics

Existential Graphs (Peirce, 1896)

Three diagrammatic proof systems for **classical** logic:

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The three icons of `Alpha`

- Sheet of assertion

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$$a$$

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$$G \quad H$$

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$$G \quad H \quad \mapsto \quad G \text{ and } H \text{ are true}$$

- Cut

The three icons of Alpha

- Sheet of assertion

$a \quad \mapsto \quad a$ is true

$\mapsto \quad \top$ (no assertion)

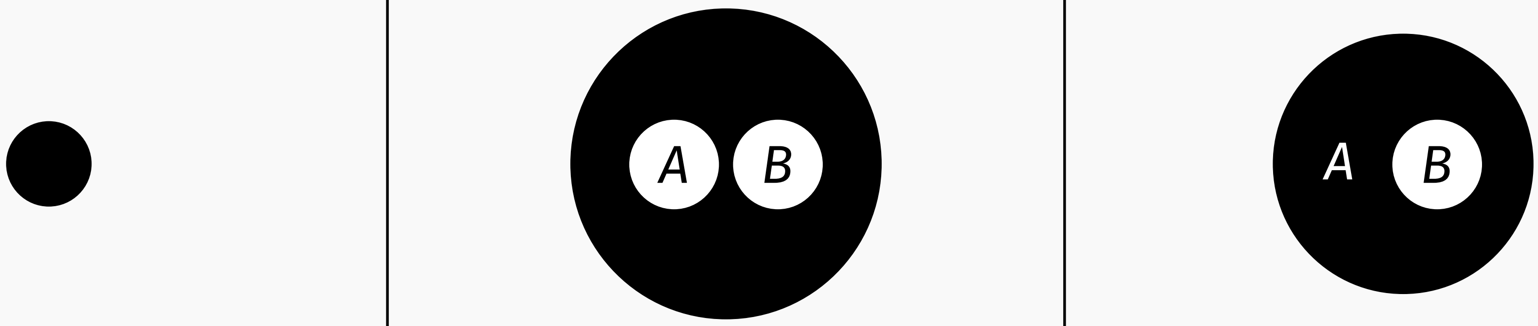
- Juxtaposition

$G \quad H \quad \mapsto \quad G$ and H are true

- Cut

$\mapsto \quad G$ is not true

Relationship with formulas



$\bot$	$A \vee B$	$A \Rightarrow B$

Illative transformations

Only 4 edition principles!

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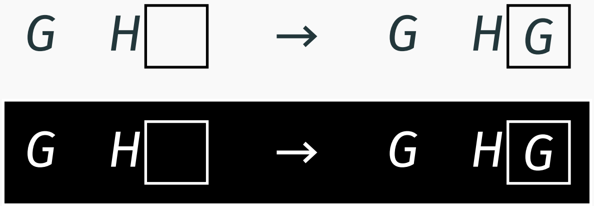
Iteration (copy-paste)			



Only 4 edition principles!

Iteration (copy-paste)	**Deiteration** (unpaste)		
$G \quad H\boxed{} \quad \rightarrow \quad G \quad H\boxed{G}$	$G \quad H\boxed{G} \quad \rightarrow \quad G \quad H\boxed{}$		
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Iteration (copy-paste)	Deiteration (unpaste)	Insertion	

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Iteration (copy-paste)	**Deiteration** (unpaste)	**Insertion**	**Deletion**
G *H*[ ] → *G* *H*[*G*]	*G* *H*[*G*] → *G* *H*[ ]	(on black ground) → *G*	*G* →
(on black ground) *G* *H*[ ] → *G* *H*[*G*]	(on black ground) *G* *H*[*G*] → *G* *H*[ ]		

Illative transformations

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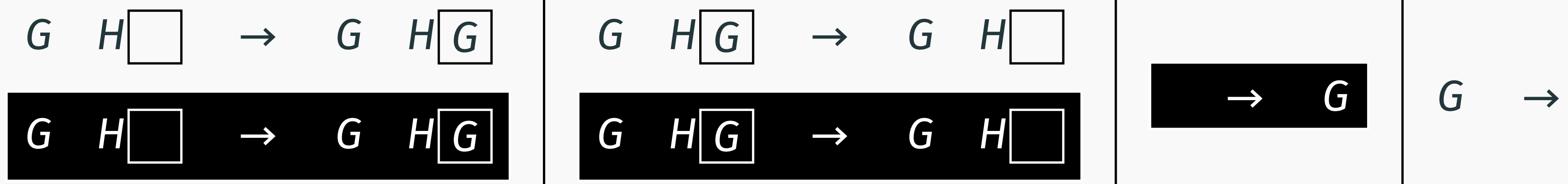
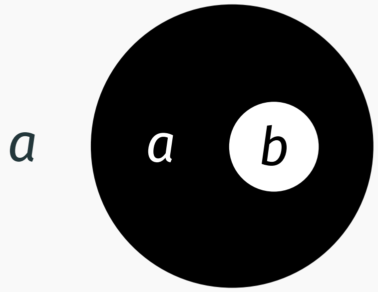


and a space principle, the **Double-cut** law:

Example: *modus ponens*



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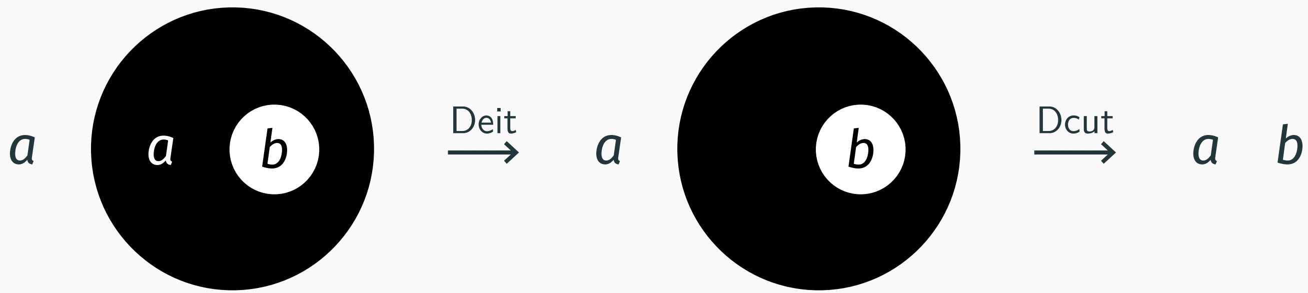

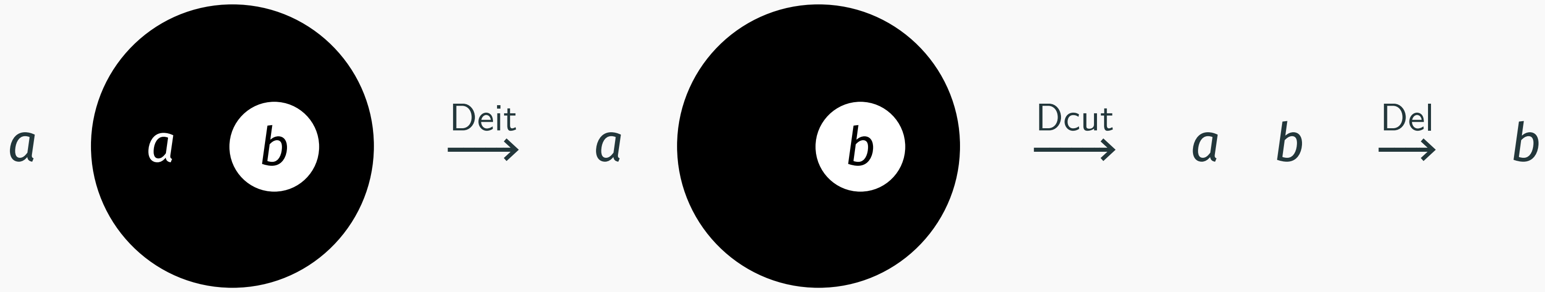
a
a
b
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a
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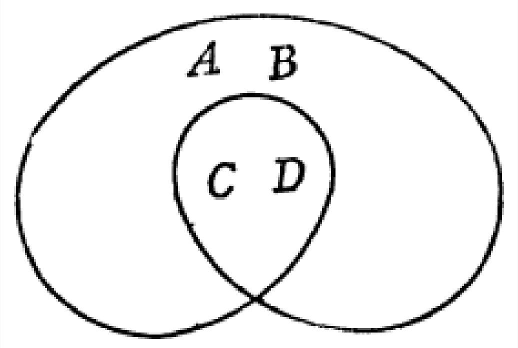
a
b
Del
b

Intuitionistic Logic: Flowers

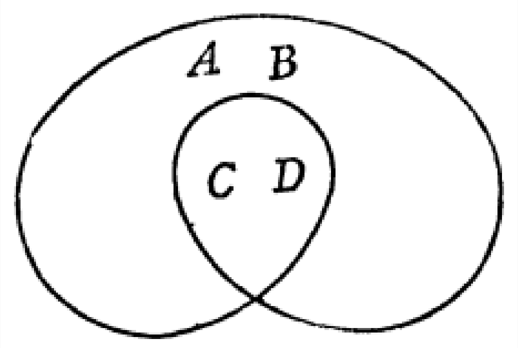
The scroll



I thought I ought to take the general form of argument as the basal form of composition of signs in my diagrammatization; and this necessarily took the form of a "scroll", that is [...] a curved line without contrary flexure and returning into itself after once crossing itself.

— (Peirce 1906, pp. 533-534)

The scroll



$A \wedge B \Rightarrow C \wedge D$

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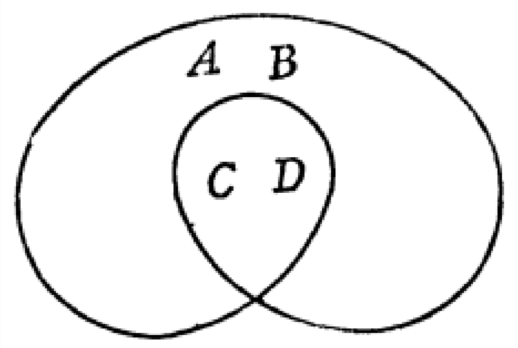

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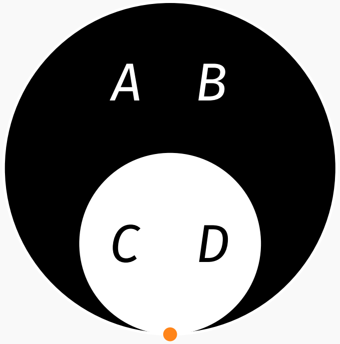

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↳ scroll = two *nested cuts*

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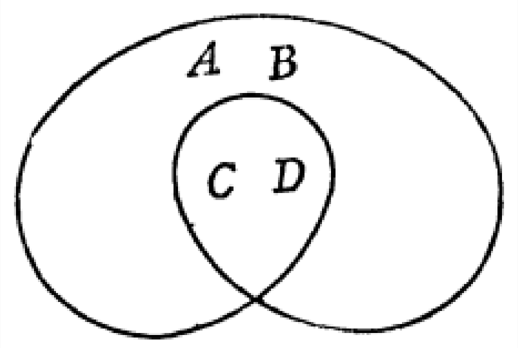

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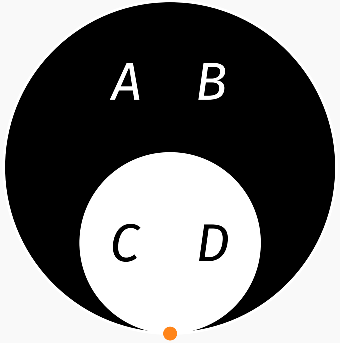

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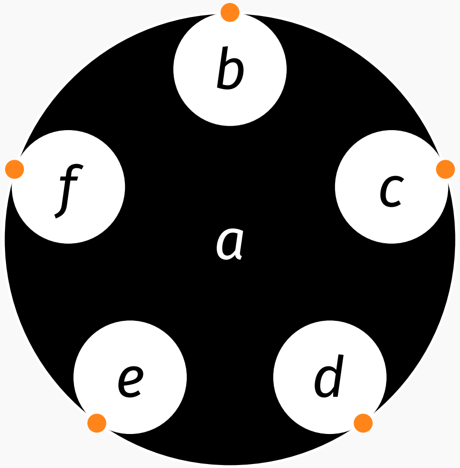
↳ scroll = two *nested cuts*

- Peirce also introduced ⇒ in logic! (Lewis 1920, p. 79)

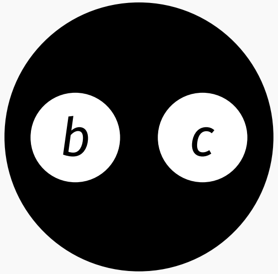
The n-ary scroll (Oostra 2010)



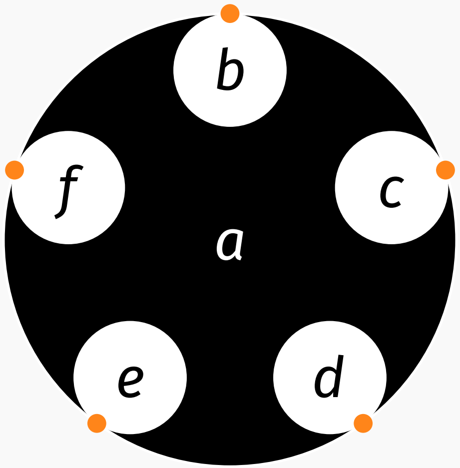
$n = 5$

The n-ary scroll (Oostra 2010)

Classical



$b \vee c$



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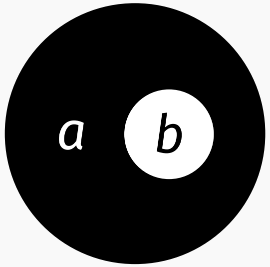

$a \Rightarrow b$

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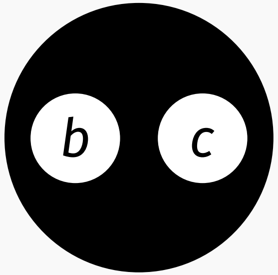

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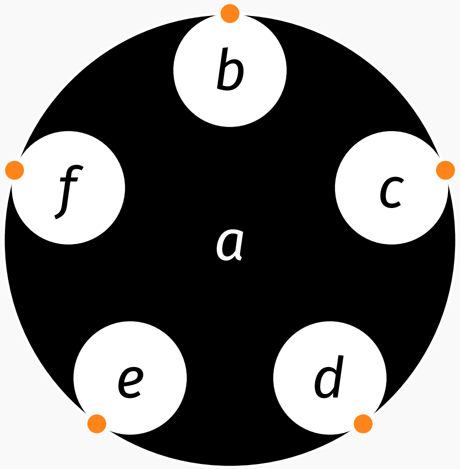
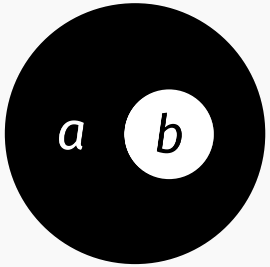


$a \Rightarrow b \vee c \vee d \vee e \vee f$

$n = 5$

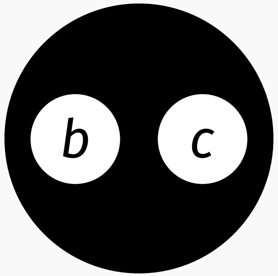
Classical



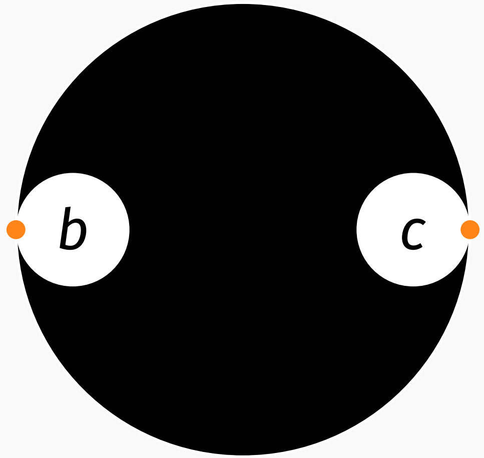
$a \Rightarrow b$

Continuity!

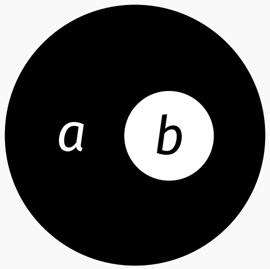
Intuitionistic



$\neq$



Classical



$\neg(\neg b \wedge \neg c)$

$b \vee c$

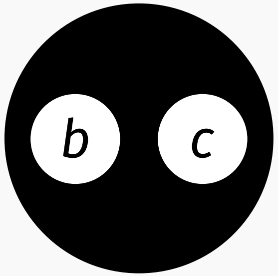
$n = 2$

$a \Rightarrow b$

The n-ary scroll (Oostra 2010)

Continuity! *Generalizes* Peirce's **scroll**

Intuitionistic



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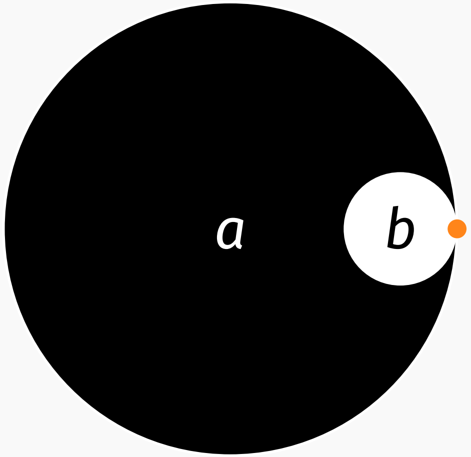
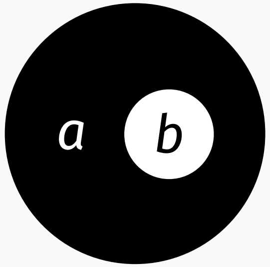


$a \Rightarrow b$

$n = 1$

$\neq$

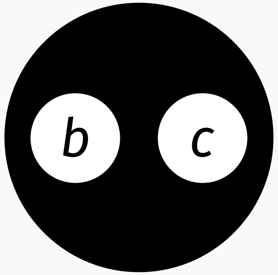
Intuitionistic



$\neg(a \wedge \neg b)$

Continuity! *Generalizes* Peirce's **scroll** and **cut**

Intuitionistic



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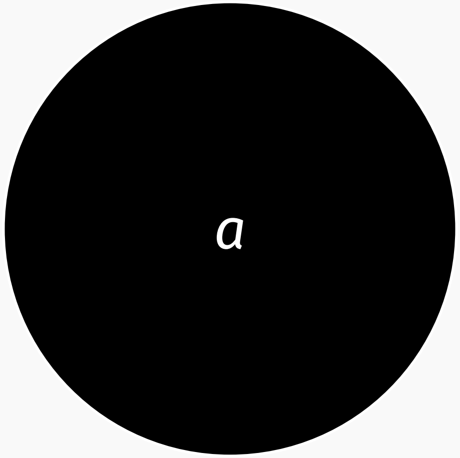
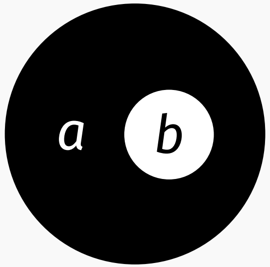


$\neg a \triangleq a \Rightarrow \bot$

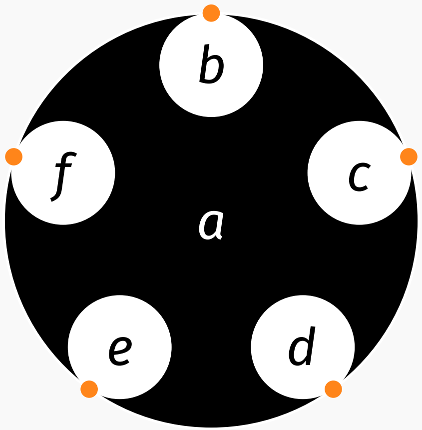
$n = 0$

Intuitionistic



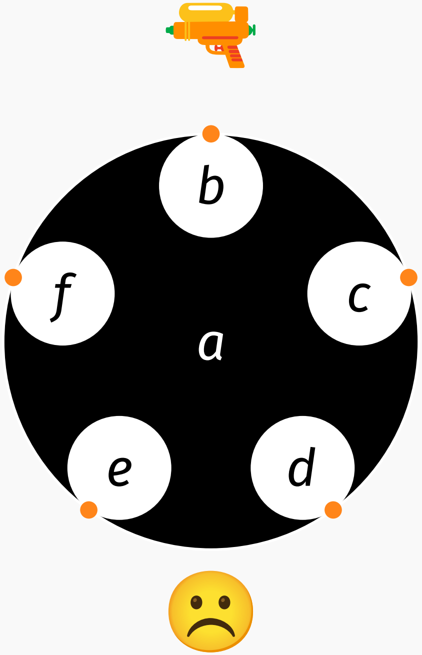
$\neg(a \wedge \neg b)$

Blooming



Blooming

(Me, 2022)



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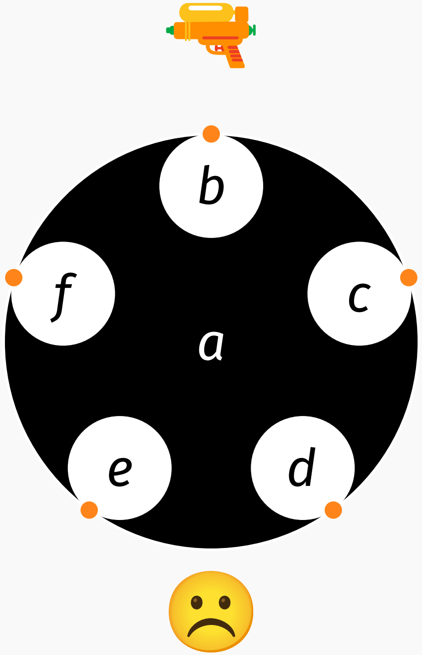
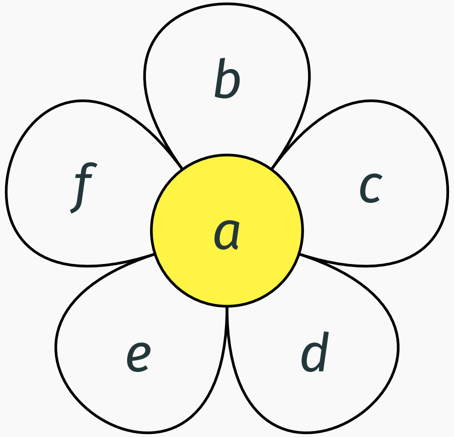
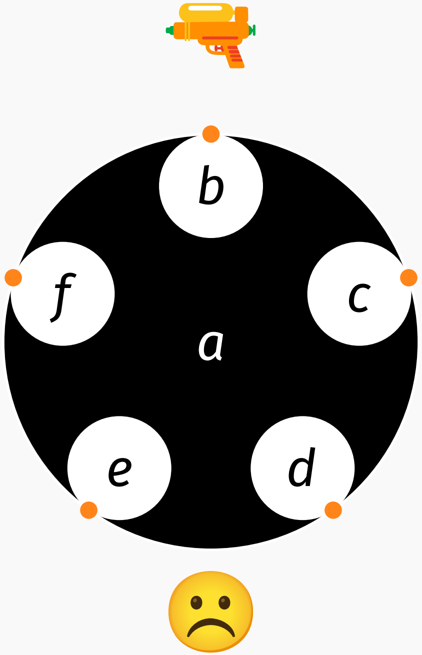


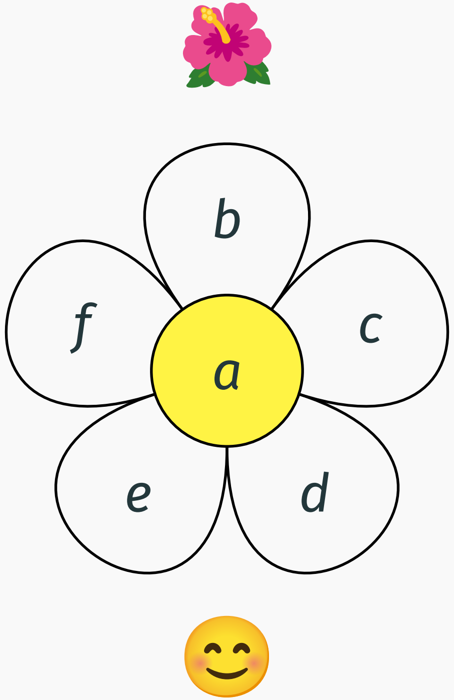


Turn **inloops** into petals.

Blooming

(Me, 2022)





"Make love, not war"

Corollaries

> The original "theorems" of geometry were those propositions that Euclid proved, while the **corollaries** were simple deductions from the theorems inserted by Euclid's commentators and editors. They are said to have been marked the figure of a little garland (or corolla), in the origin.
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> — Peirce, MS 514 (1909) (Peirce 1976)

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Petals = (possible) **corolla**-ries of pistil!

$\exists/\forall$ = binder in petal/pistil

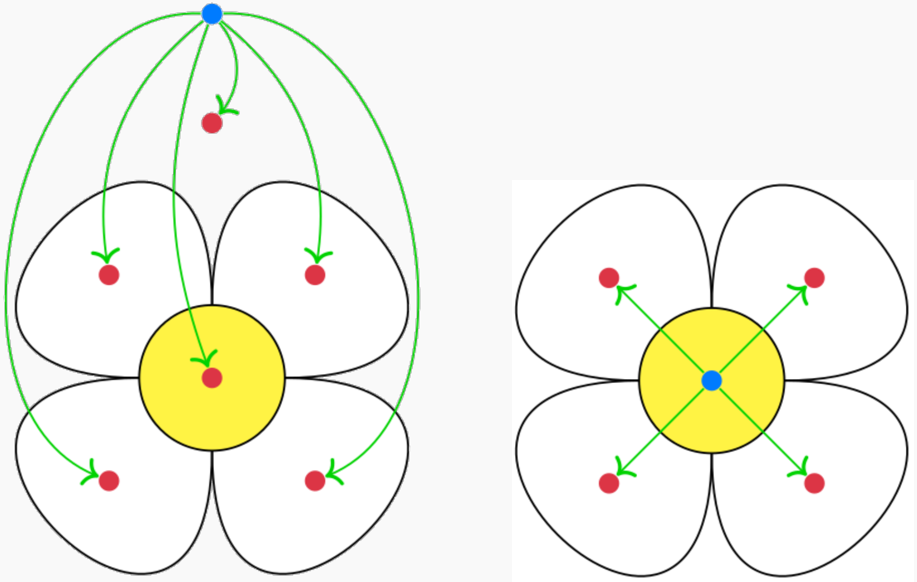
$\exists x.P(x) \land Q(x)$

$\forall x.R(x) \Rightarrow S(x)$

garden = content of an area (binders + flowers)

Reasoning with Flowers

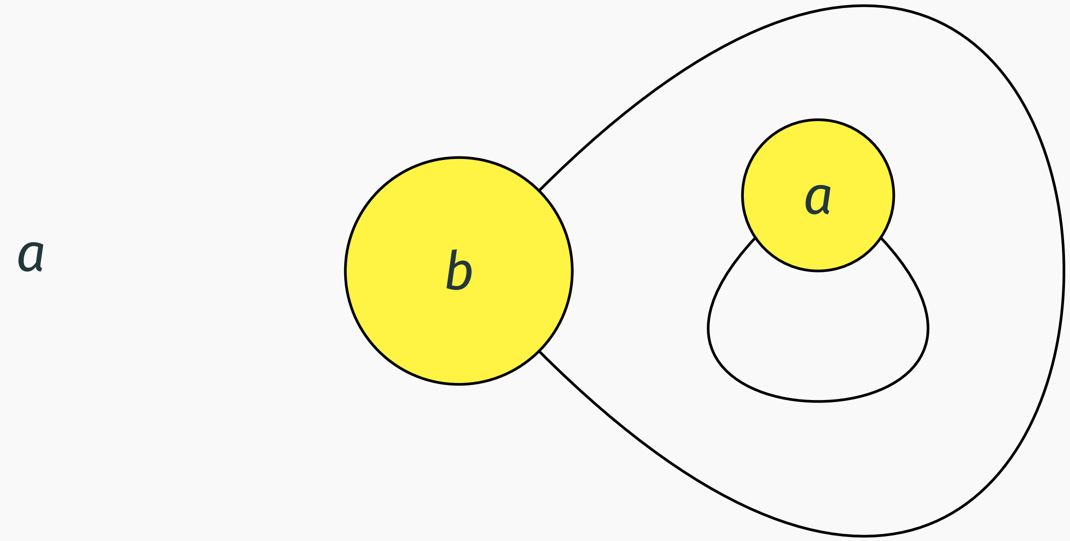
Justify a target flower by a source flower



cross-pollination self-pollination

Iteration and Deiteration

Works at **arbitrary** depth!



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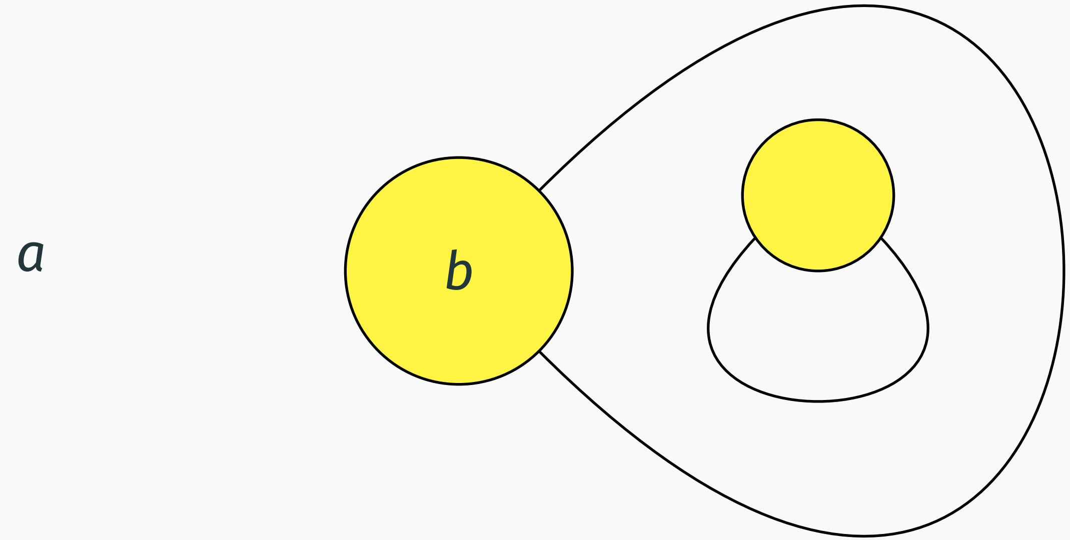

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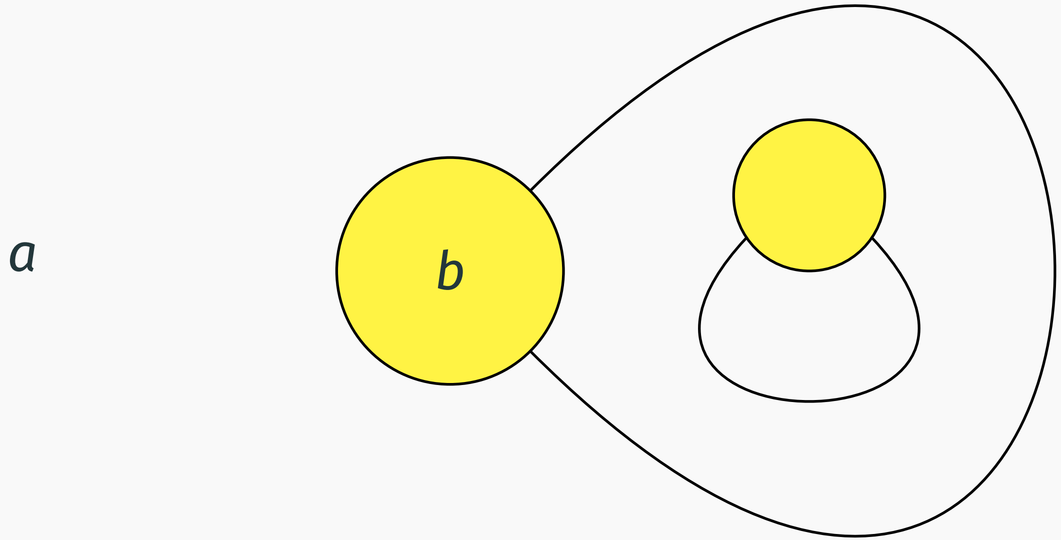
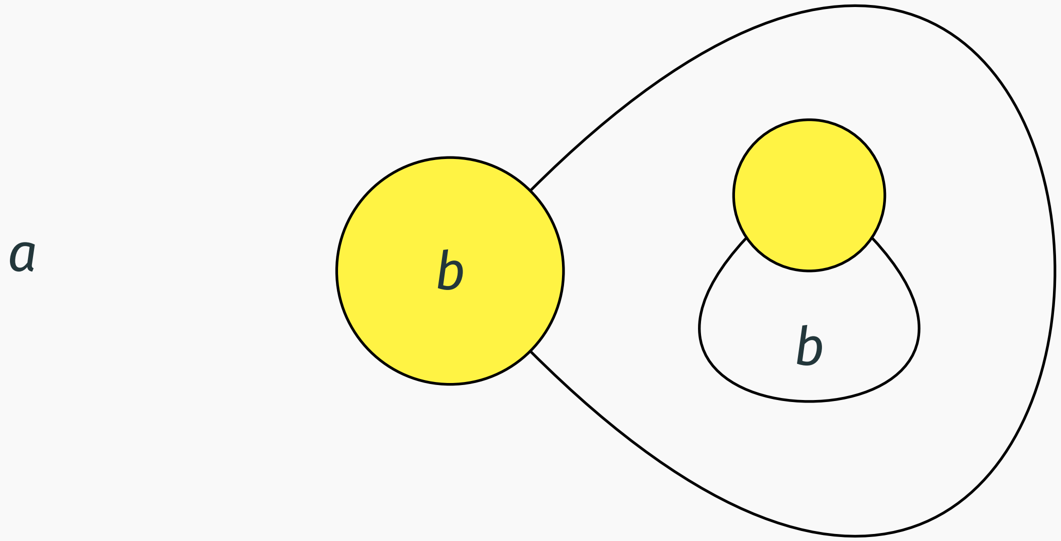


Self-pollination

Iteration and Deiteration

Works at **arbitrary** depth!

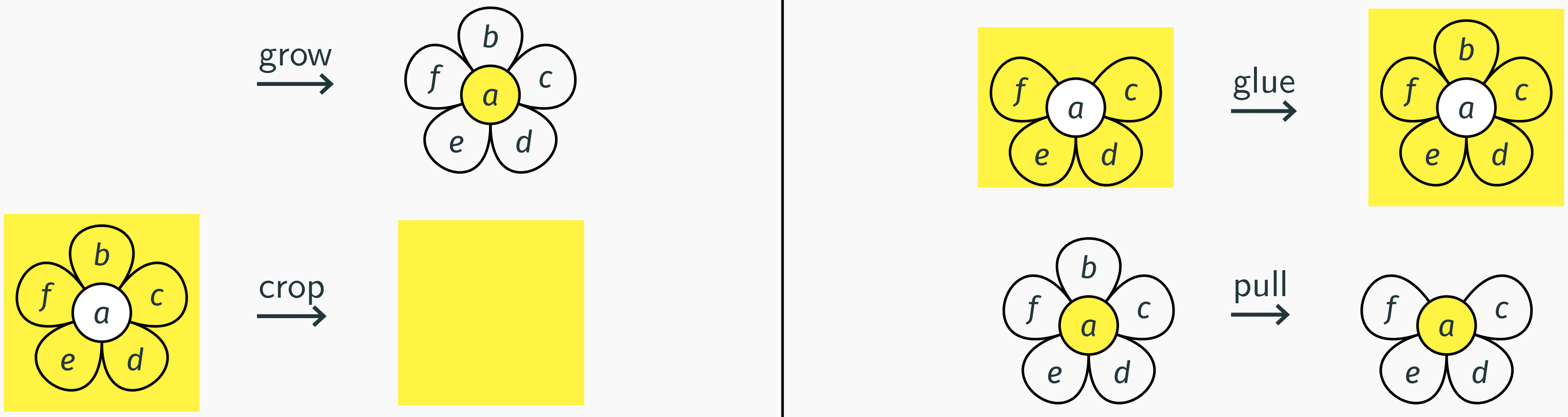


Self-pollination

Insertion and Deletion

Split in two:

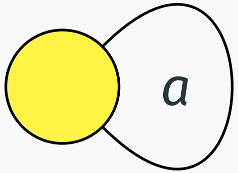
Flower	Petal



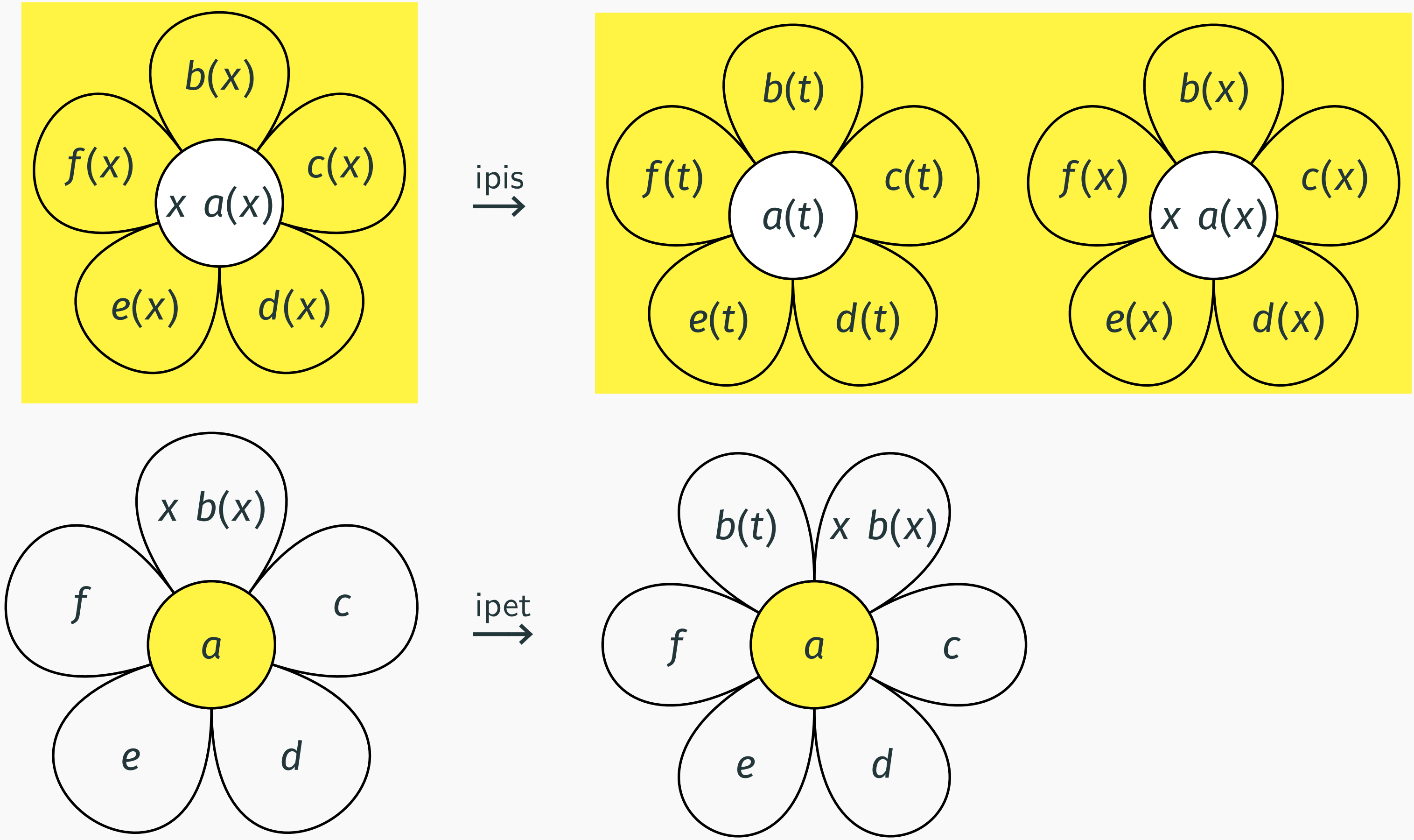
Backward reading: **conclusion** $\longrightarrow$ **premiss**

Scrolling

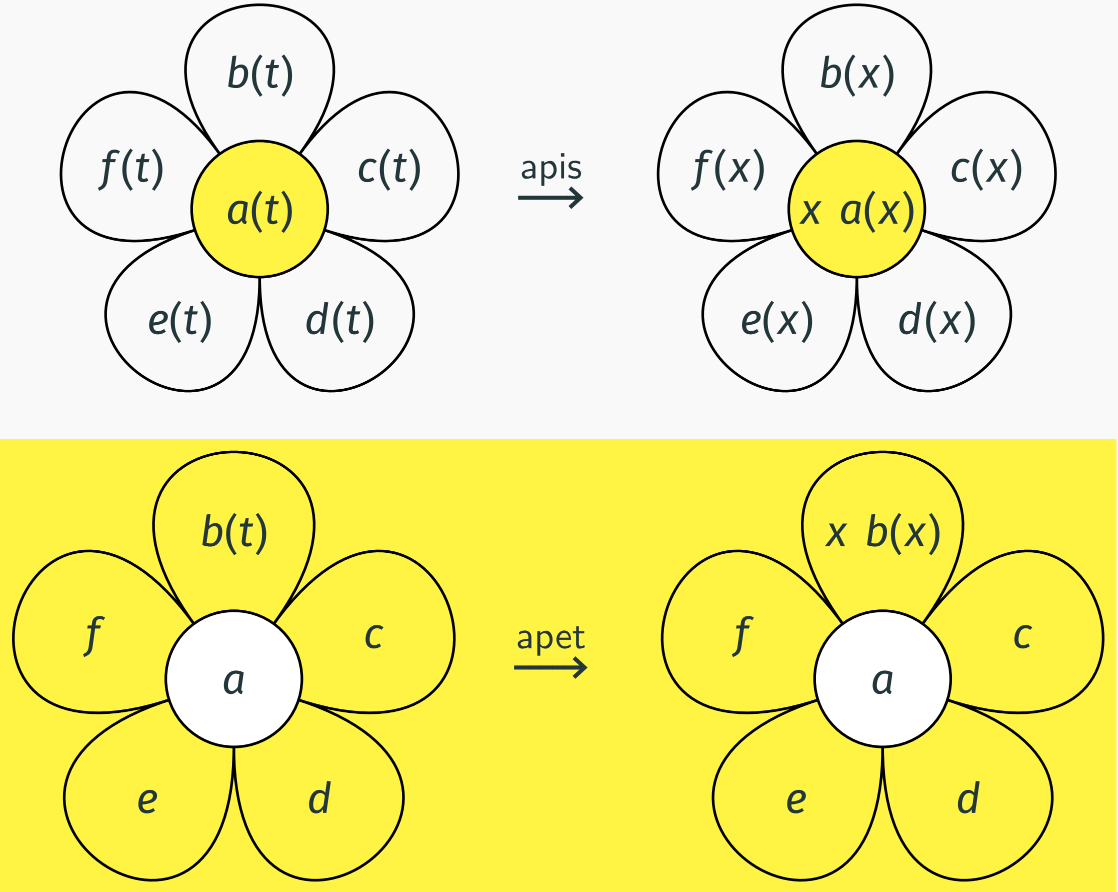
Intuitionistic restriction of **double-cut** principle:

$a \xrightarrow{\text{epis}}$ 

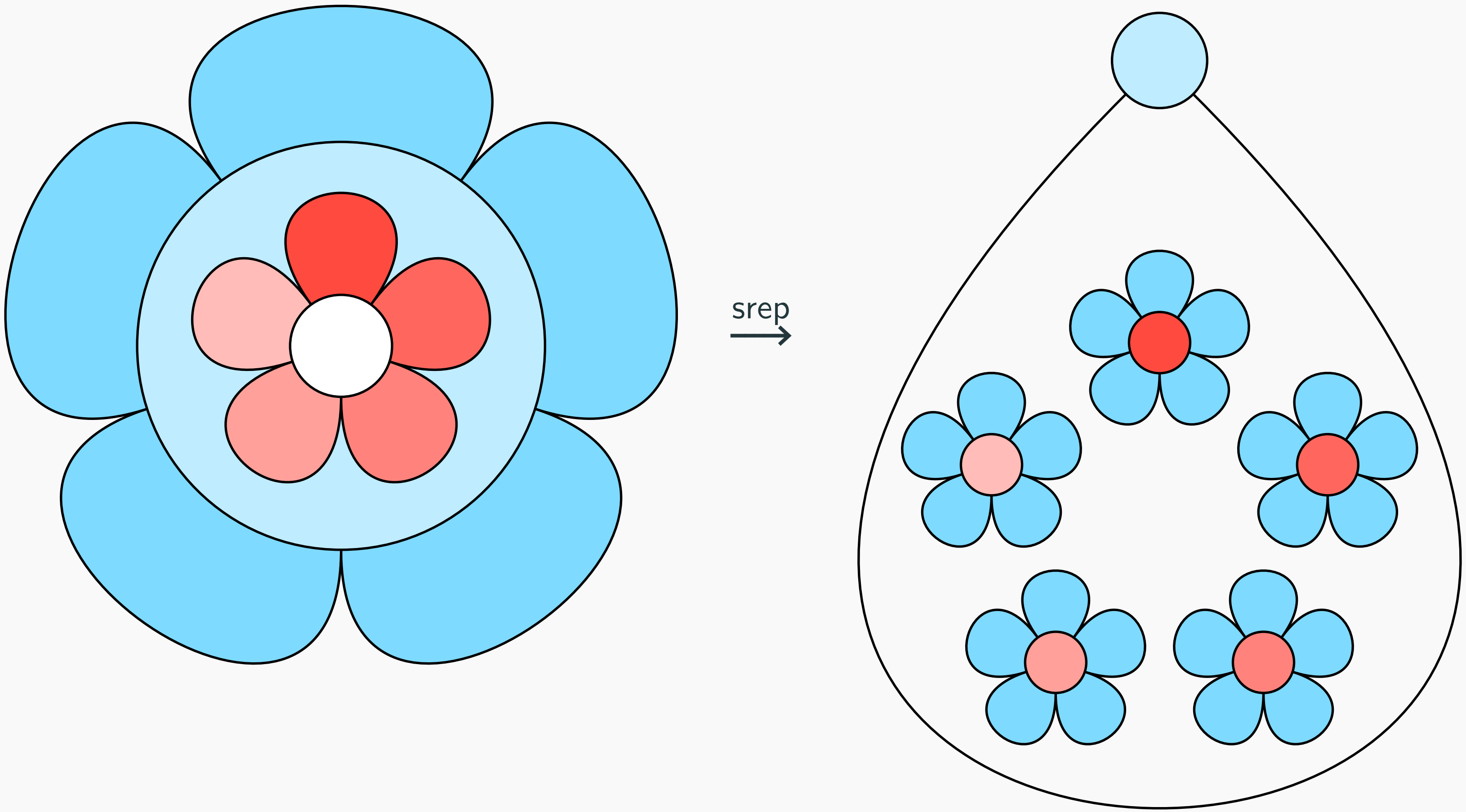
Instantiation



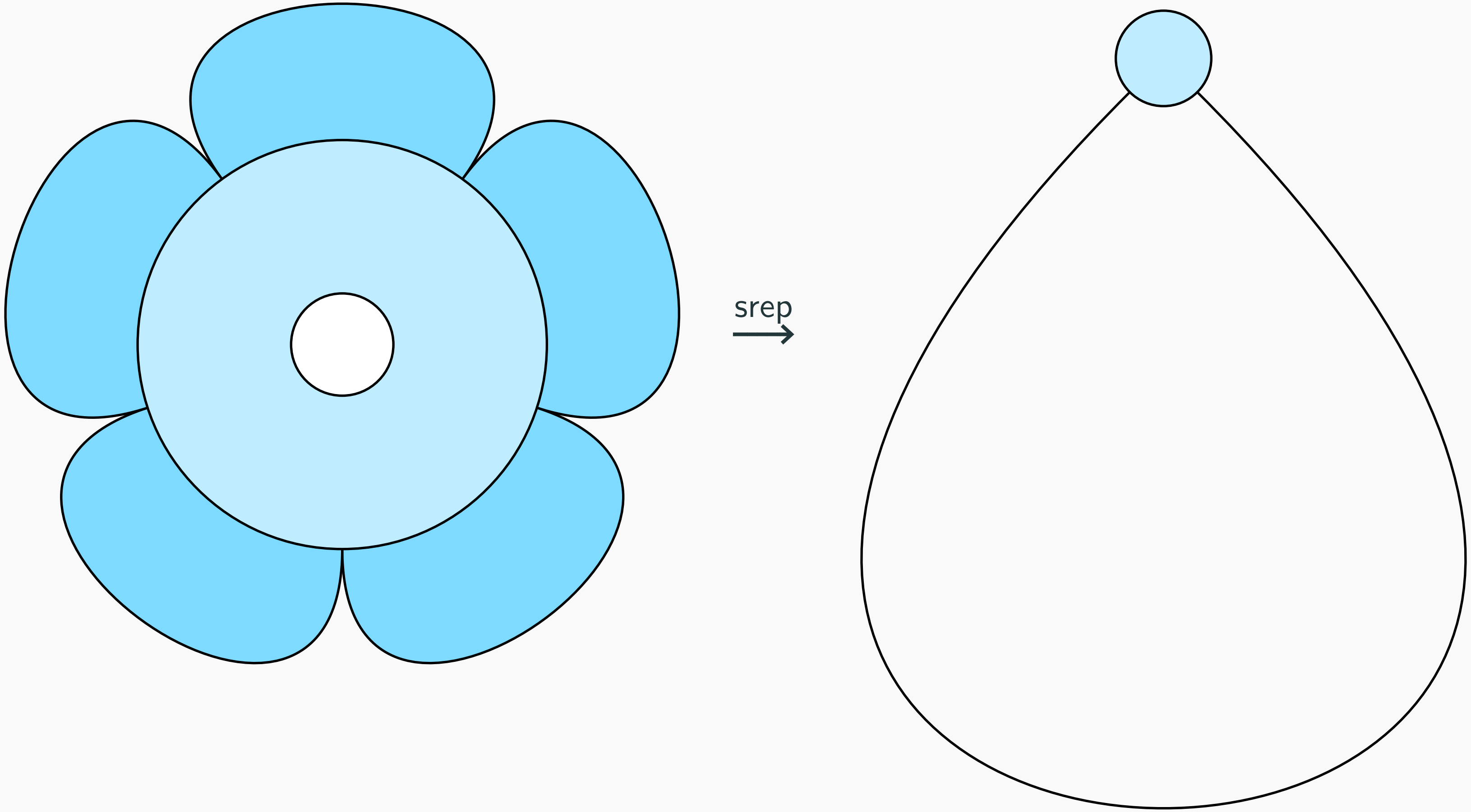
Abstraction



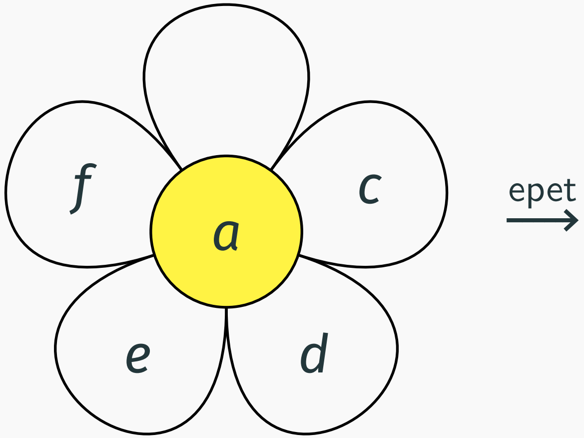
Case reasoning



Ex falso quodlibet



QED



Metatheory: Nature vs. Culture

Natural rules ❀

$$\text{❀} = \underbrace{\text{(De)iteration}}_{\{\text{poll}\downarrow,\text{poll}\uparrow\}} \cup \underbrace{\text{Instantiation}}_{\{\text{ipis},\text{ipet}\}} \cup \underbrace{\text{Scrolling}}_{\{\text{epis}\}} \cup \underbrace{\text{QED}}_{\{\text{epet}\}} \cup \underbrace{\text{Case reasoning}}_{\{\text{srep}\}}$$

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Let Φ, Ψ be *bouquets*, i.e. multisets of flowers.

All rules are:

- **Invertible**: if $\Phi \longrightarrow \Psi$ then Ψ equivalent to Φ

Natural rules ❀

$$❀ = \underbrace{\text{(De)iteration}}_{\{\text{poll}\downarrow,\text{poll}\uparrow\}} \cup \underbrace{\text{Instantiation}}_{\{\text{ipis},\text{ipet}\}} \cup \underbrace{\text{Scrolling}}_{\{\text{epis}\}} \cup \underbrace{\text{QED}}_{\{\text{epet}\}} \cup \underbrace{\text{Case reasoning}}_{\{\text{srep}\}}$$

Let Φ, Ψ be *bouquets*, i.e. multisets of flowers.

All rules are:

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↳ "Equational" reasoning

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Let $\mathbf{\Phi}, \mathbf{\Psi}$ be *bouquets*, i.e. multisets of flowers.

All rules are:

- **Invertible**: if $\mathbf{\Phi} \longrightarrow \mathbf{\Psi}$ then $\mathbf{\Psi}$ equivalent to $\mathbf{\Phi}$

↳ "Equational" reasoning

- **Analytic**: if $\mathbf{\Phi} \longrightarrow \mathbf{\Psi}$ and $\boldsymbol{a}$ occurs in $\mathbf{\Psi}$ then $\boldsymbol{a}$ occurs in $\mathbf{\Phi}$

Natural rules ❀

$$❀ = \underbrace{\text{(De)iteration}}_{\{\text{poll}\downarrow,\text{poll}\uparrow\}} \cup \underbrace{\text{Instantiation}}_{\{\text{ipis},\text{ipet}\}} \cup \underbrace{\text{Scrolling}}_{\{\text{epis}\}} \cup \underbrace{\text{QED}}_{\{\text{epet}\}} \cup \underbrace{\text{Case reasoning}}_{\{\text{srep}\}}$$

Let Φ, Ψ be *bouquets*, i.e. multisets of flowers.

All rules are:

- **Invertible**: if $\Phi \longrightarrow \Psi$ then Ψ equivalent to Φ

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- **Analytic**: if $\Phi \longrightarrow \Psi$ and a occurs in Ψ then a occurs in Φ

↳ Reduces proof-search space

Cultural rules ✂

$$✂ = \underbrace{\text{Insertion}}_{\{\text{grow},\text{glue}\}} \cup \underbrace{\text{Deletion}}_{\{\text{crop},\text{pull}\}} \cup \underbrace{\text{Abstraction}}_{\{\text{apis},\text{apet}\}}$$

Cultural rules ✂

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- All rules are **non-invertible**
- Some rules are **non-analytic**

Hypothetical provability

- Remember our paradigm:

$$\text{proving} = \text{erasing}$$

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- This works in arbitrary contexts X (i.e. one-holed bouquets)
- Formally:

Definition: For any bouquets Φ and Ψ, Ψ is *provable* from Φ, written $\Phi \vdash \Psi$, if for any context X in which Φ occurs and *pollinates* the hole of X, we have

$$X\boxed{\Psi} \longrightarrow X\boxed{}$$

Cult-elimination

Theorem (Soundness): If $\Phi \longrightarrow \Psi$ then $\Psi \overset{\mathscr{K}}{\models} \Phi$ in every Kripke structure $\mathscr{K}$.

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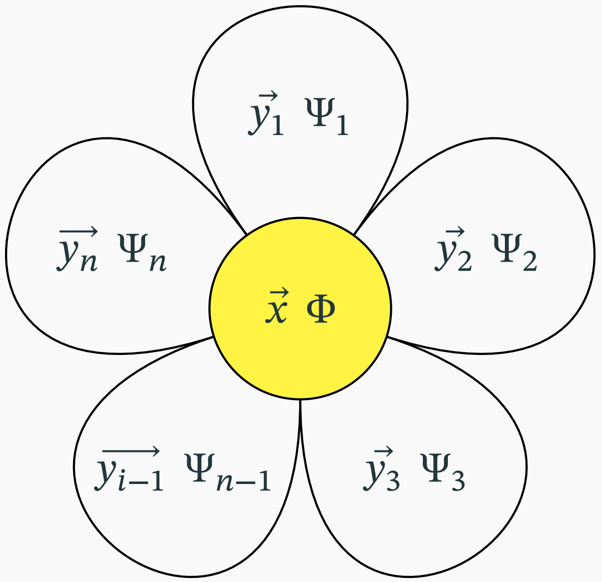
Completeness of analytic fragment ❀!

The Flower Prover

*A **demo** is worth a thousand pictures!*

Related works (non-exhaustive)

- **Structural proof theory:**
  - (Guenot 2013): rewriting-based **nested sequent** calculi
  - (Lyon 2021; Girlando et al. 2023): **fully invertible** labelled sequent calculi
- **Proof assistants:**
  - (Ayers 2021): `Box` datastructure similar to flowers
- **Categorical logic:**
  - (Johnstone 2002): coherent/geometric formulas in **topos theory**
  - (Bonchi et al. 2024): algebra of `Beta` (previous talk!)



$$\forall \vec{x}.\left(\bigwedge \Phi \Rightarrow \bigvee_i \exists \vec{y_i}.\Psi_i\right)$$

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