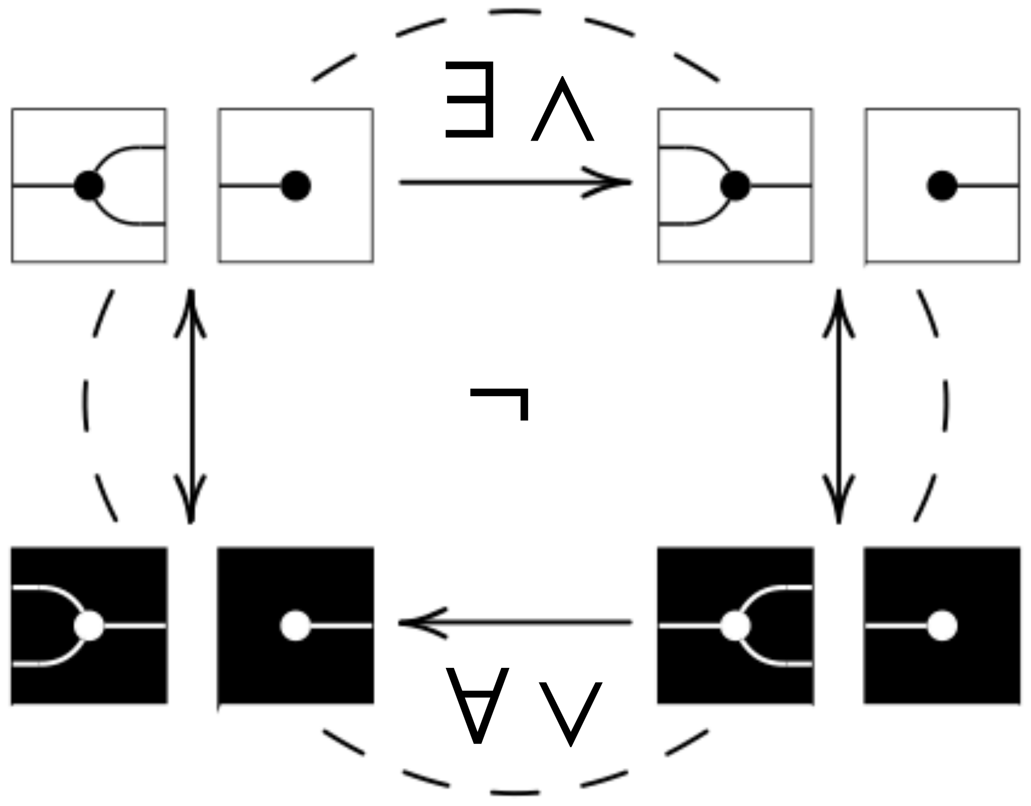


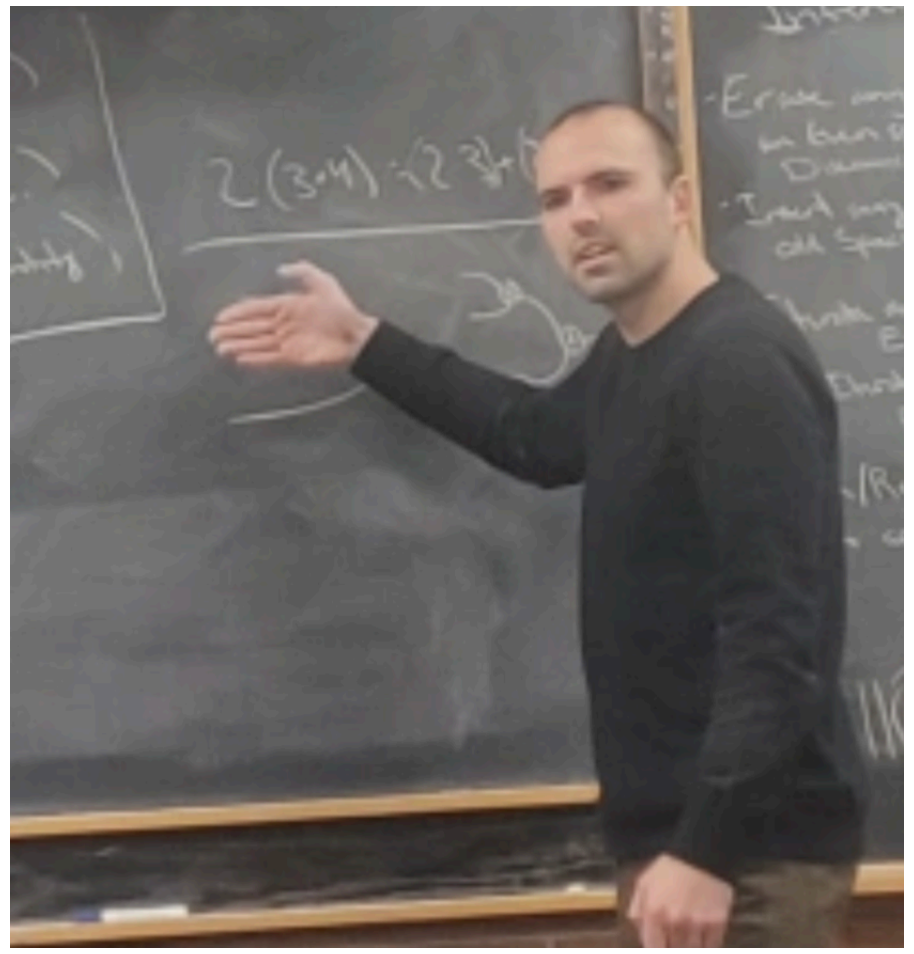
Diagrammatic Algebra of First Order Logic

Alessandro Di Giorgio
University College London

SYCO 12
Birmingham, UK

Collaborators

Filippo Bonchi
University of Pisa



Nathan Haydon
TalTech

Paweł Sobociński
TalTech

Motivation

“Logic in his adolescent phase was algebraic. There was Boole’s algebra of classes and Peirce’s algebra of relations. But in 1879 logic come of age, with Frege’s quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic.”

William Quine, 1971

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William Quine, 1971

Logic of Relatives

$R, S, T, \ldots, \cap, \cup, ;, (\cdot)^{\dagger}, \neg$

1870
Peirce

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Logic of Relatives	Quantification Theory
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1870 Peirce ---------------------- 1879 Frege

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Quantification Theory

$x, y, z, \forall, \exists$

Binary relations are less expressive than FOL

1870 Peirce ------------------------ 1879 Frege ----------------- 1915 Lowenheim

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Logic of Relatives	Quantification Theory	Binary relations are less expressive than FOL	Calculus of Relations
$R, S, T, \ldots, \cap, \cup, ;, (\cdot)^{\dagger}, \neg$	$x, y, z, \forall, \exists$		$R;(S \cup Q) = (R;S) \cup (R;Q)$
1870 Peirce	1879 Frege	1915 Lowenheim	1941 Tarski

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Motivation

The Calculus of Relations in computer science

- Theory of Databases **Codd, 1970**
- Proof Assistants **Pous, 2013**
- Rewriting **Gavazzo, 2023**
- Program logic **Hoare&He, 1986** **O'Hearn, 2019**

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FOL is a major specification language

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FOL is a major specification language

But CR is strictly less expressive :(

In this talk

- We develop a *categorical algebra of relations*
- As *expressive as FOL*
- With a *complete equational axiomatization*
- Axioms arise from well-known *algebraic structures*

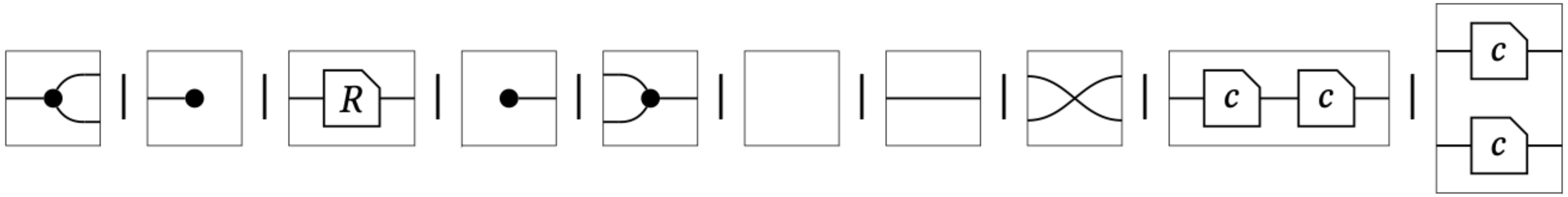
Cartesian Bicategories

Carboni & Walters, 1987

Cartesian Bicategories

Carboni & Walters, 1987

Syntax



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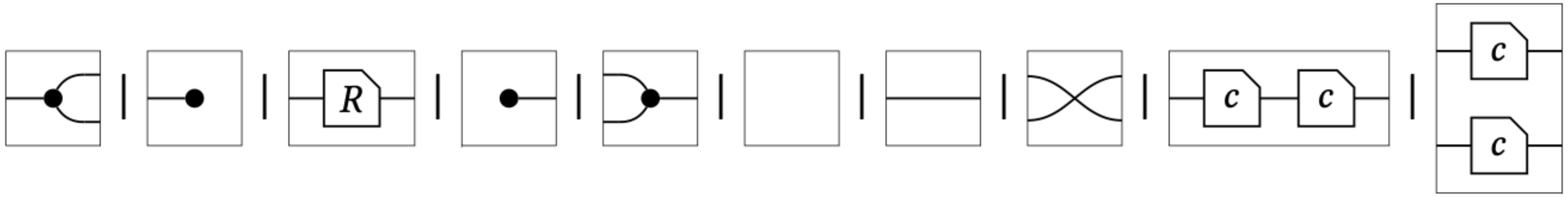
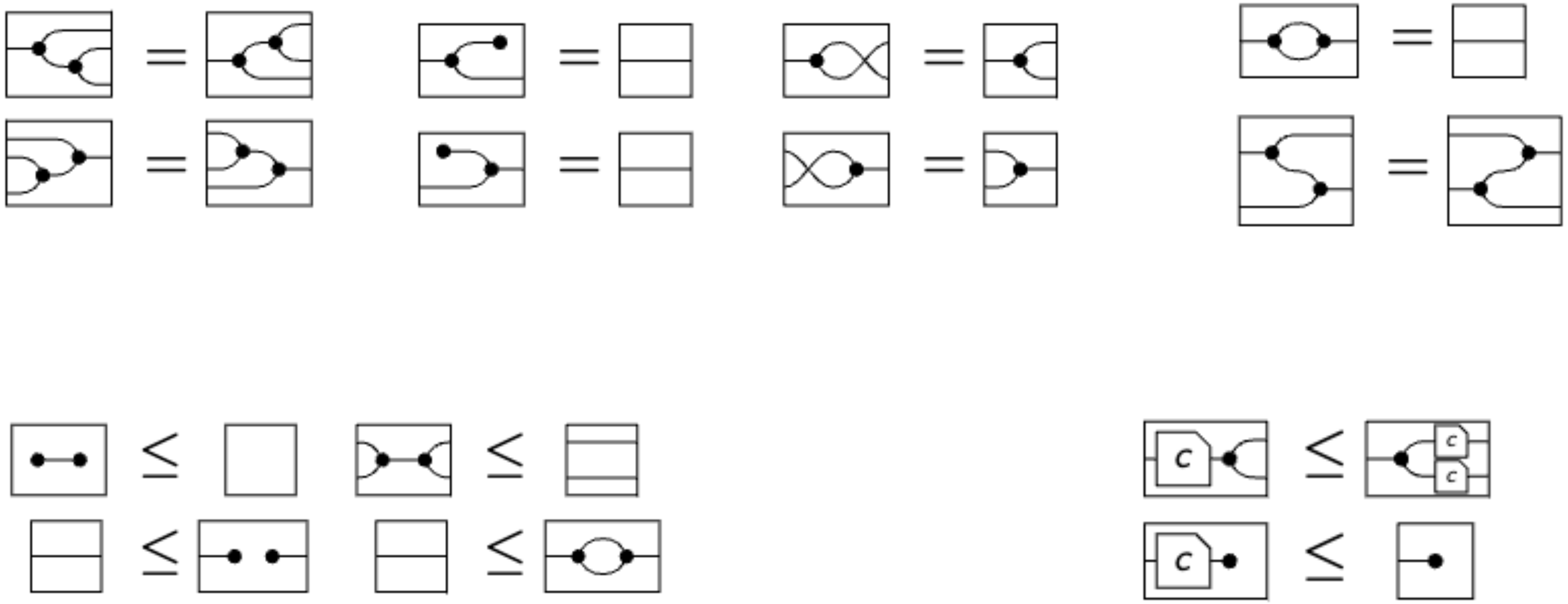


Axioms



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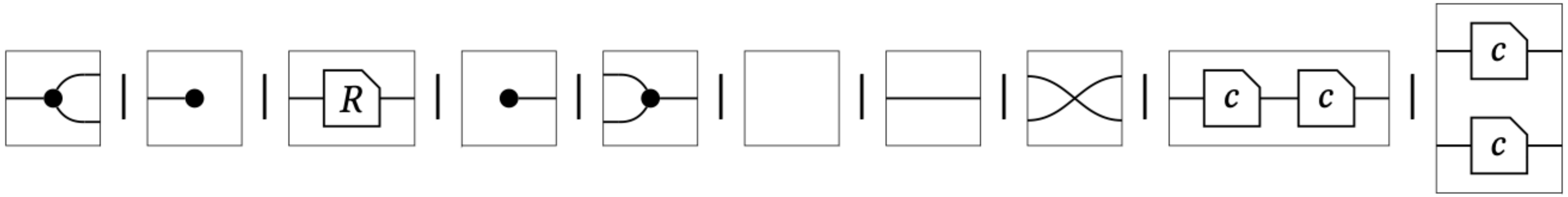

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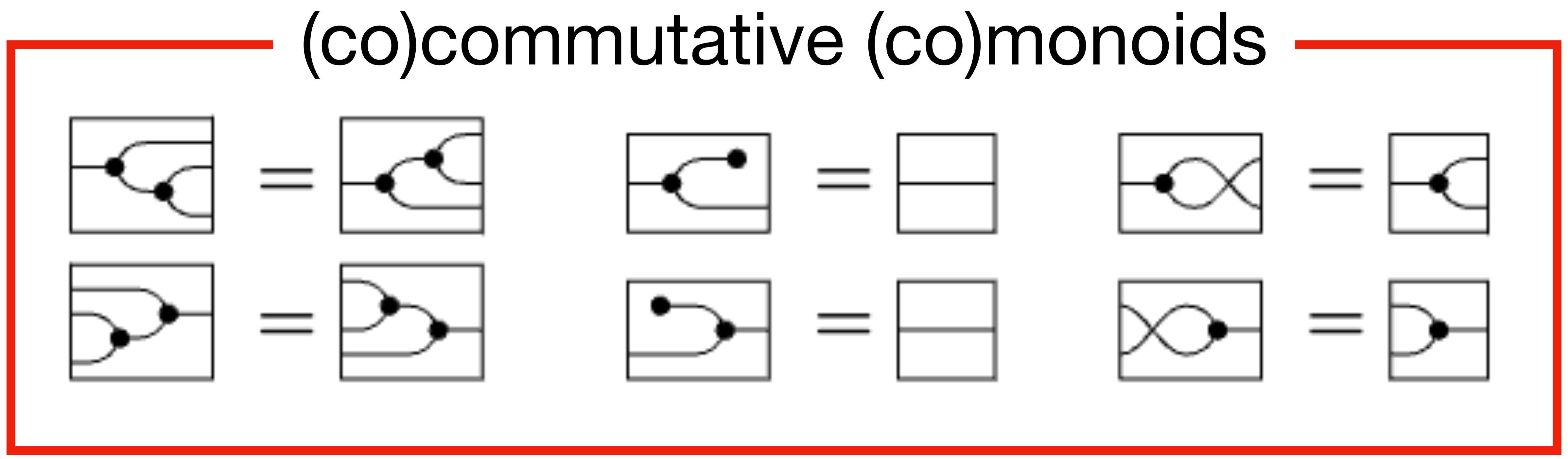

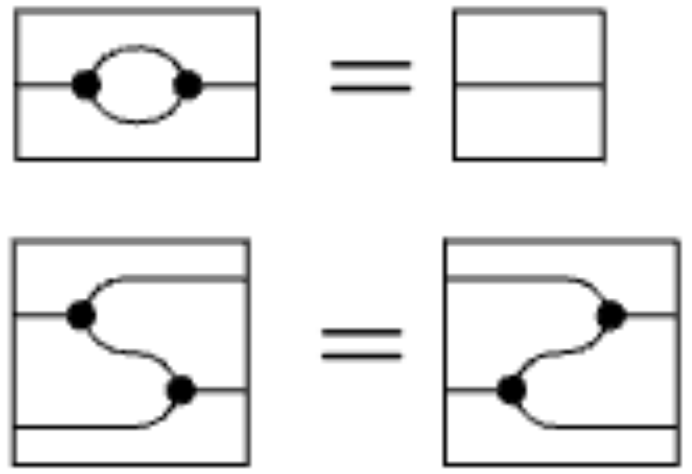

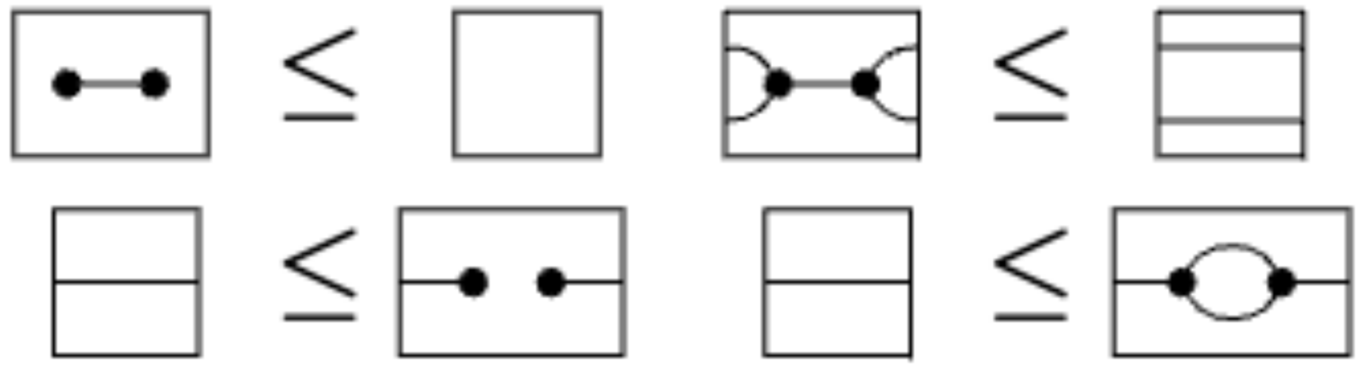

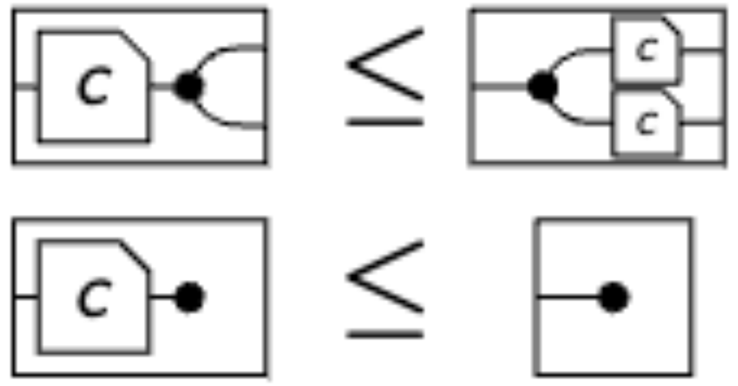

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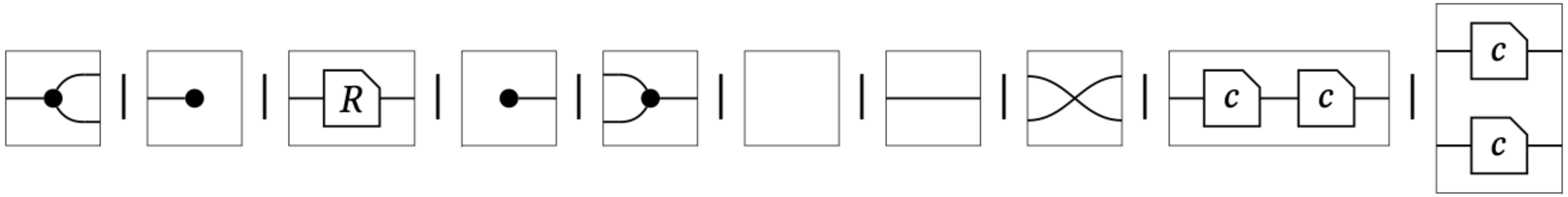

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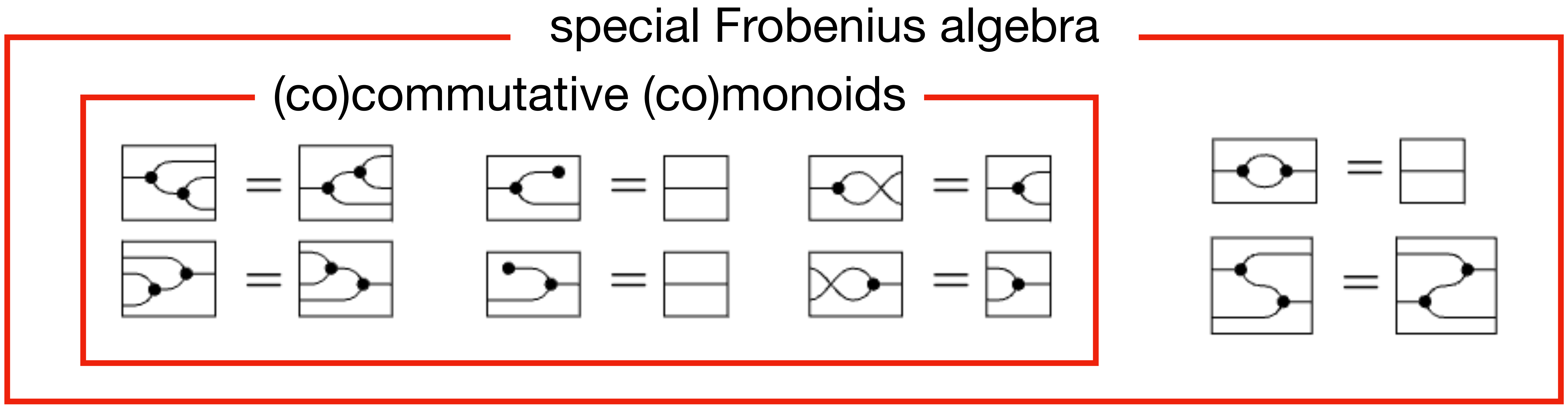

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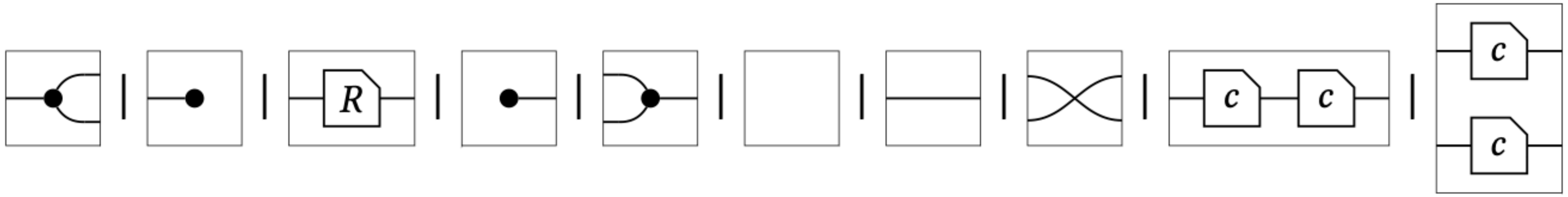

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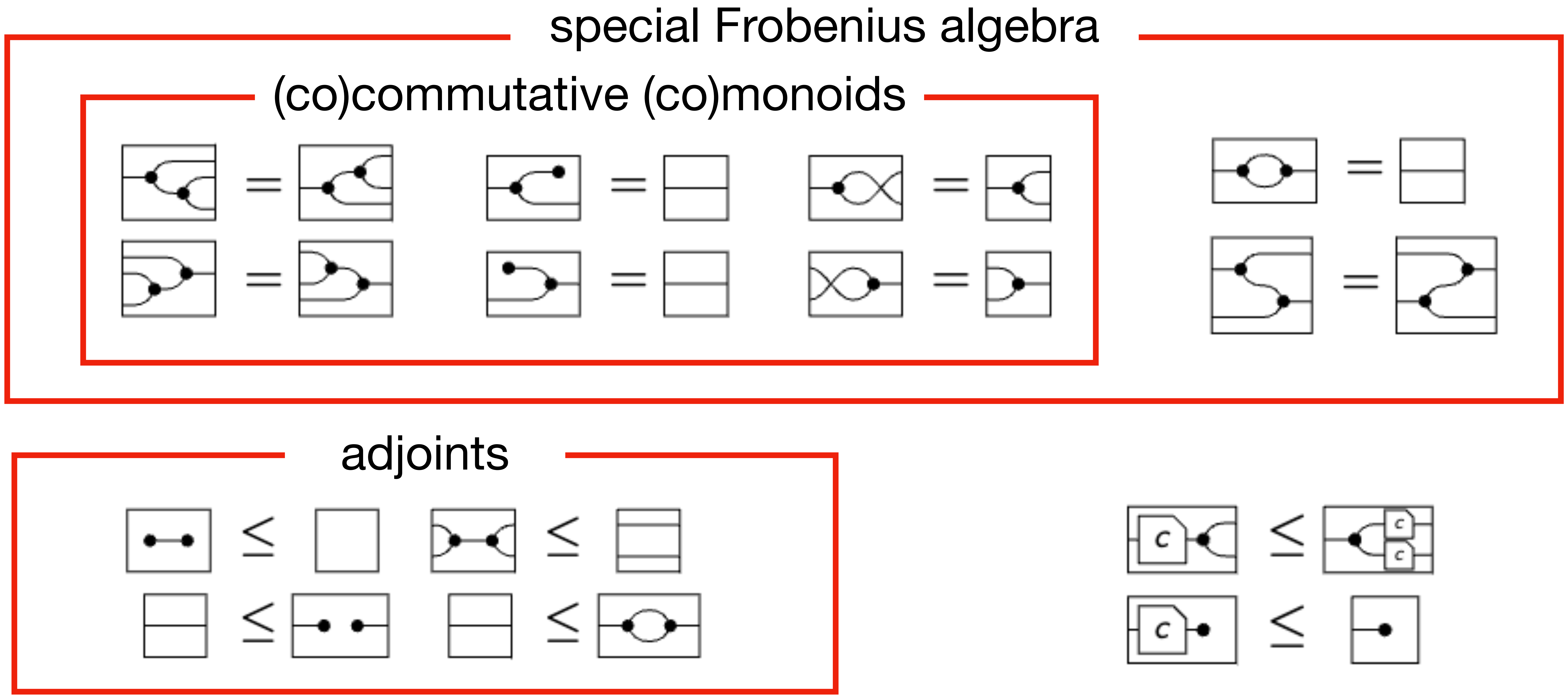

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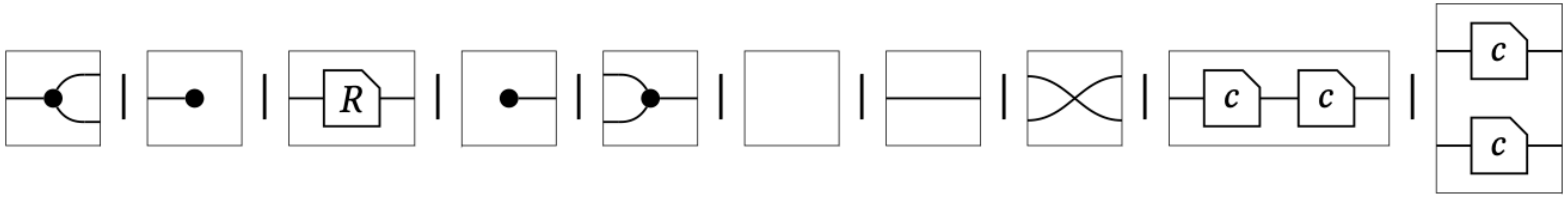
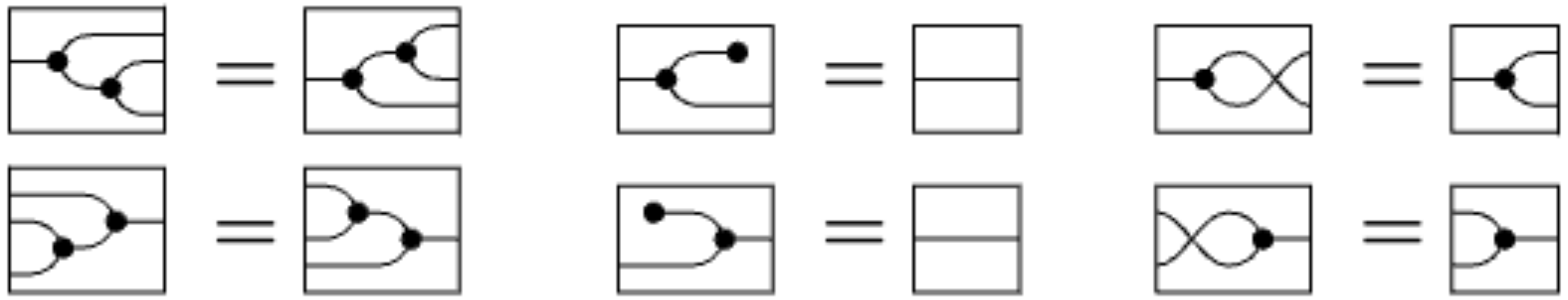
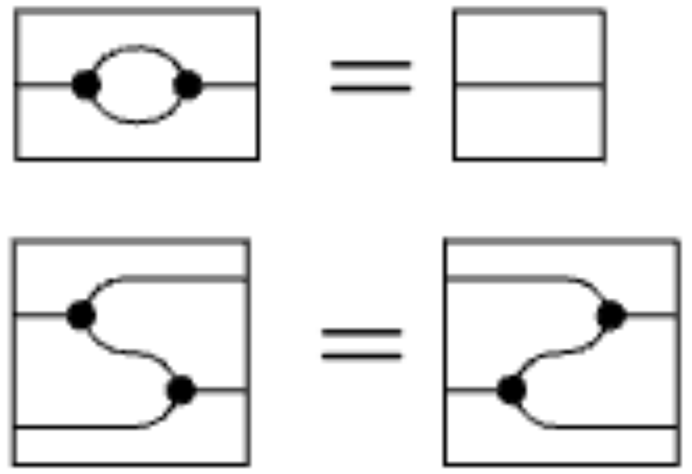


Axioms

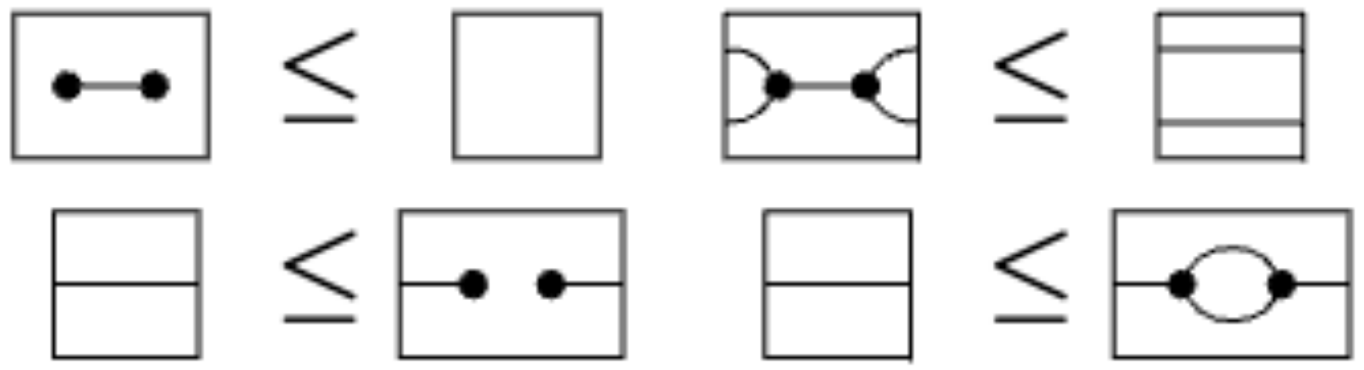
special Frobenius algebra

(co)commutative (co)monoids

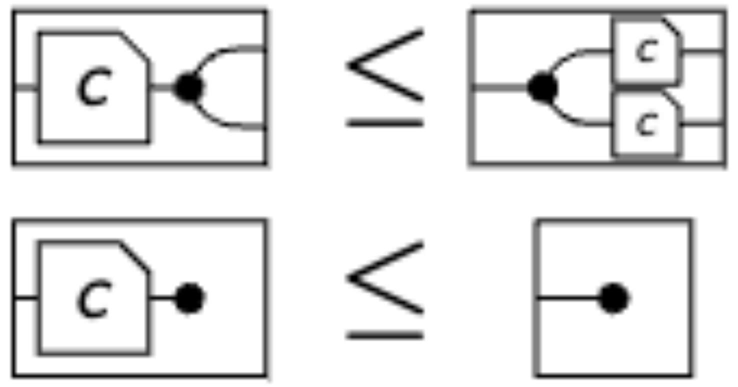




adjoints



lax naturality

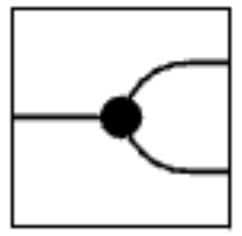


Cartesian Bicategories

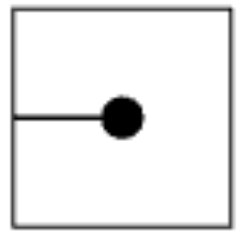
Example Rel$^\circ$ the category of sets and relations

$$= \{(x,y) \mid \exists z.(x,z) \in R \wedge (z,y) \in S\}$$

$$= \{(x,y) \mid x = y\} \subseteq X \times X$$



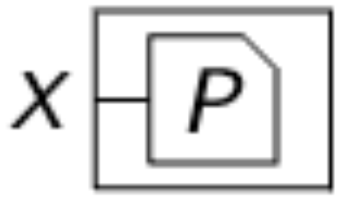
$$= \{(x, \binom{x}{x})\} \subseteq X \times (X \times X)$$

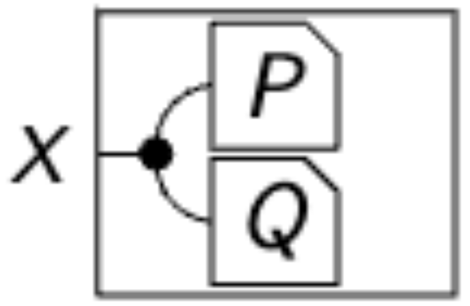


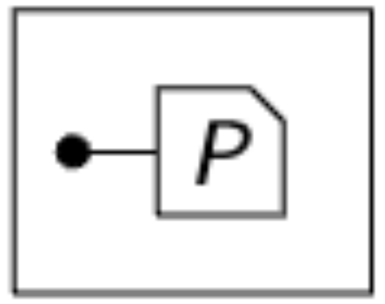
$$= \{(x, \star)\} \subseteq X \times \{\star\}$$

Cartesian Bicategories

Internal language *regular logic* (∃ ∧ fragment of FOL)

$P(x) \rightsquigarrow$ 

$P(x) \wedge Q(x) \rightsquigarrow$ 

$\exists x . P(x) \rightsquigarrow$ 

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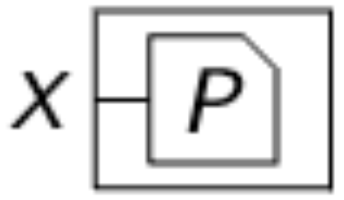
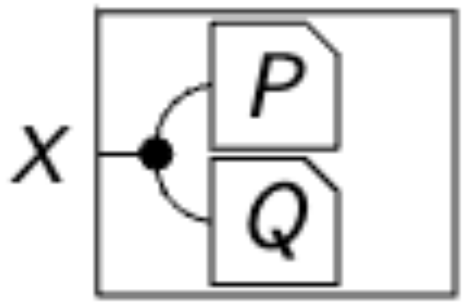
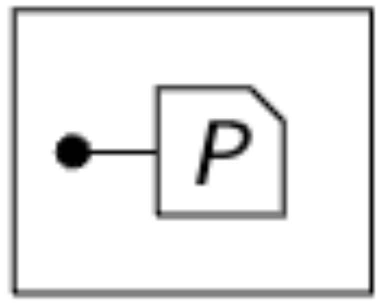
$P(x) \rightsquigarrow$ 

$P(x) \wedge Q(x) \rightsquigarrow$ 

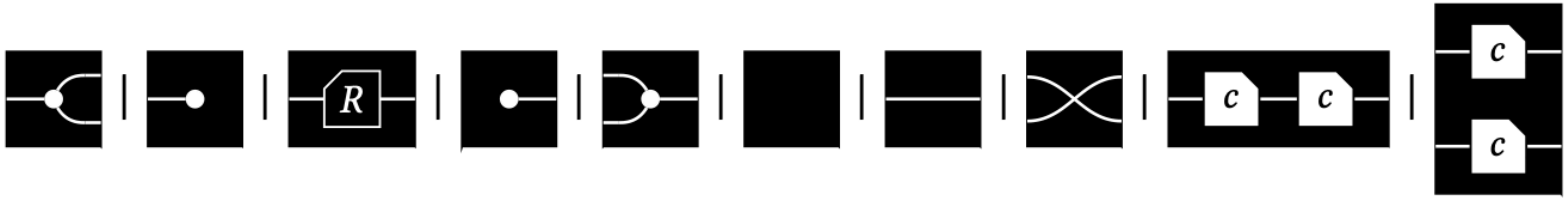
$\exists x. P(x) \rightsquigarrow$ 

Completeness Every theorem of regular logic can be proved with the axioms of cartesian bicategories

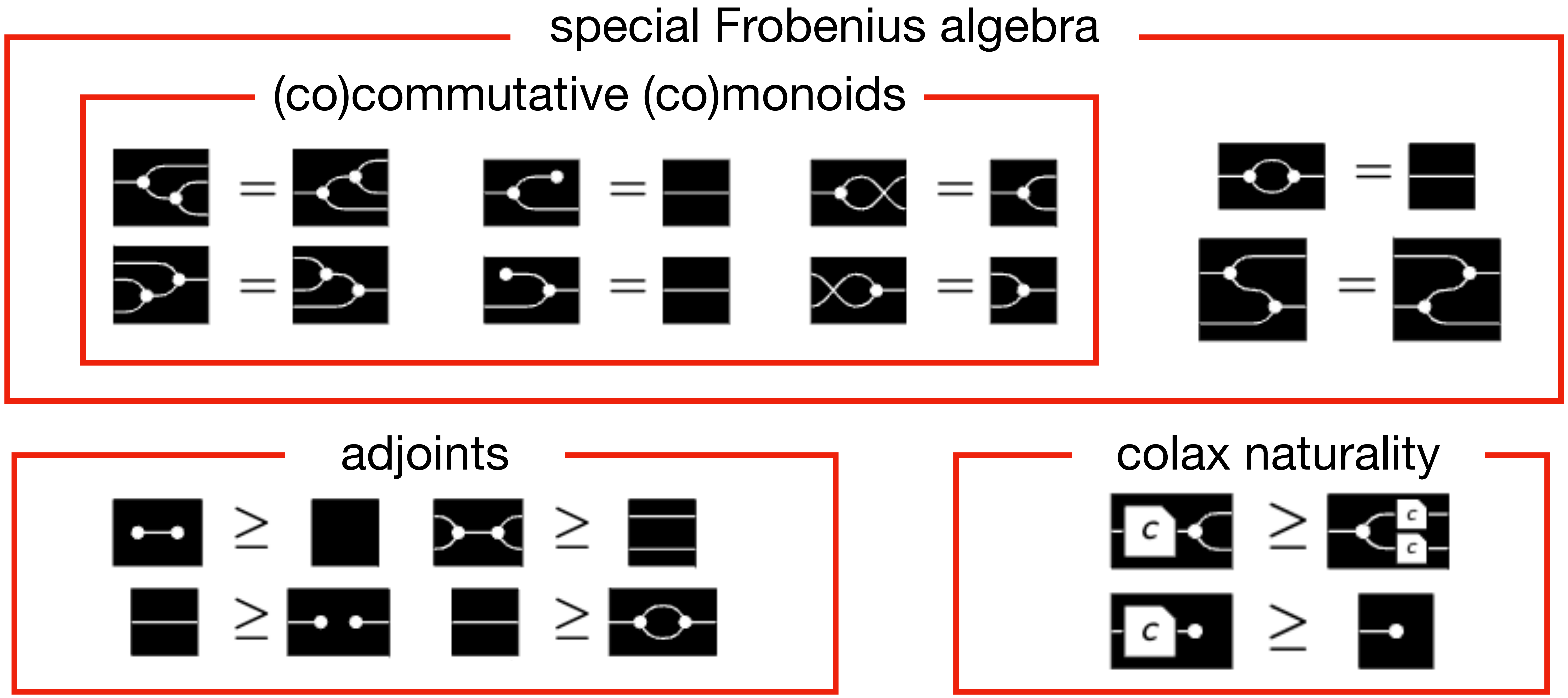
Bonchi, Seeber, Sobocinski, 2018

CoCartesian Bicategories

Syntax



Axioms



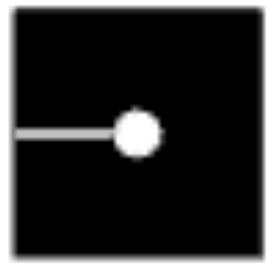
CoCartesian Bicategories

Example Rel$^\bullet$ the other category of sets and relations

$$= \{(x,y) \mid \forall z.(x,z) \in R \vee (z,y) \in S\}$$

$$= \{(x,y) \mid x \neq y\} \subseteq X \times X$$

$$= \{(x, \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)) \mid x \neq y \vee x \neq z\}$$

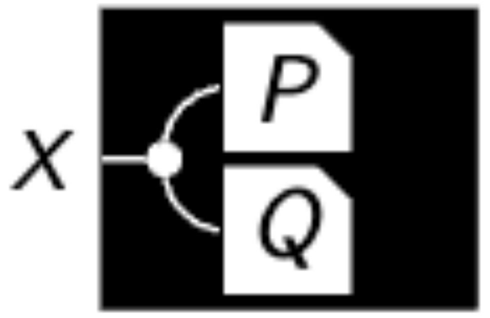


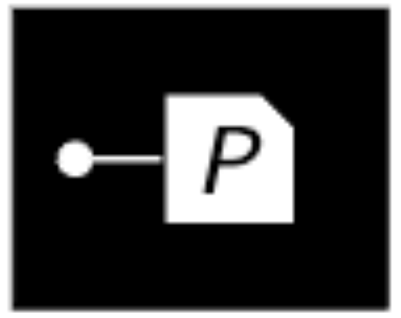
$$= \{\}$$

CoCartesian Bicategories

Internal language   co*regular logic* (∀ ∨ fragment of FOL)

$P(x) \rightsquigarrow$ 

$P(x) \vee Q(x) \rightsquigarrow$ 

$\forall x. P(x) \rightsquigarrow$ 

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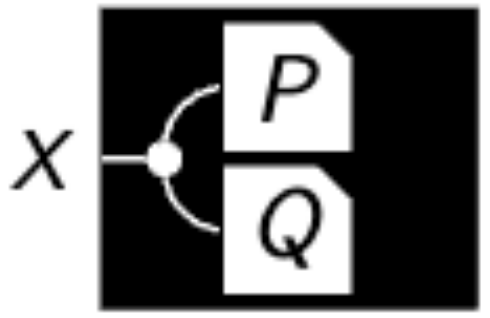
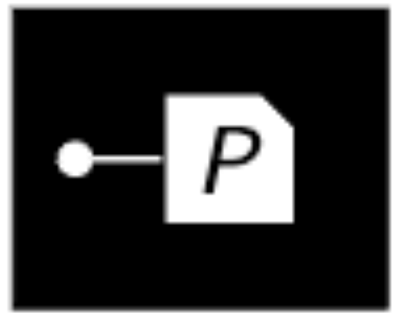
$P(x) \vee Q(x) \rightsquigarrow$ 

$\forall x.P(x) \rightsquigarrow$ 

Completeness Every theorem of coregular logic can be proved with the axioms of cocartesian bicategories

Two categories of relations

Sharing the same objects and arrows but with different compositions….

Rel°	$\mathsf{Rel}^{\bullet}$
Cartesian bicategory	CoCartesian bicategory
$\exists \wedge$ -FOL	$\forall \wedge$ -FOL

Two categories of relations

Sharing the same objects and arrows but with different compositions….

Rel°

Cartesian bicategory

$\exists \wedge$ -FOL

$\mathsf{Rel}^{\bullet}$

CoCartesian bicategory

$\forall \wedge$ -FOL

?

FOL

How do they interact?

Peirce, 1897

Perice knew it since 1897

“Two formulae so constantly used that hardly anything can be done without them”

$$R \mathbin{\mathring{;}} (S \mathbin{\boldsymbol{;}} Q) \subseteq (R \mathbin{\mathring{;}} S) \mathbin{\boldsymbol{;}} Q \qquad\qquad (R \mathbin{\boldsymbol{;}} S) \mathbin{\mathring{;}} Q \subseteq R \mathbin{\boldsymbol{;}} (S \mathbin{\mathring{;}} Q)$$

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Peirce, 1897

"Two formulae so constantly used that hardly anything can be done without them"

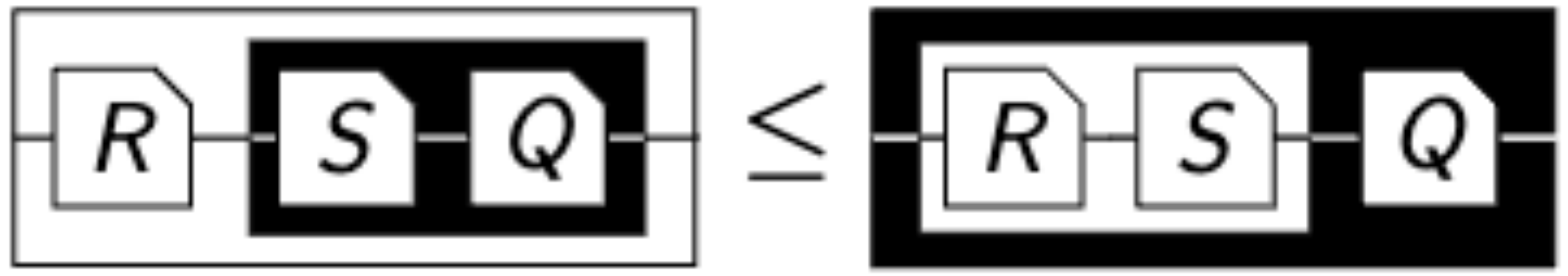
$$R \fatsemi (S \mathbin{;} Q) \subseteq (R \fatsemi S) \mathbin{;} Q \qquad\qquad (R \mathbin{;} S) \fatsemi Q \subseteq R \mathbin{;} (S \fatsemi Q)$$

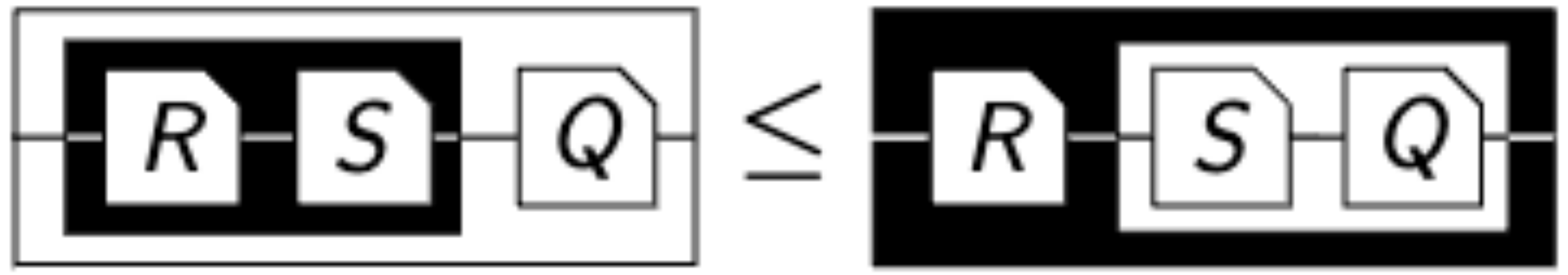
Cockett et al. categorified it 100 years later

Cockett, Koslowski, Seely, 2000

A *linear bicategory* consists of

- two bicategory structures
- sharing the same objects and arrows but with different compositions
- such that one linearly distributes over the other





First Order Bicategories

A *first order bicategory* is a linear bicategory, such that

- one bicategory is cartesian
- the other is cocartesian
- they interact via linear adjunctions

- and linear versions of the Frobenius axioms

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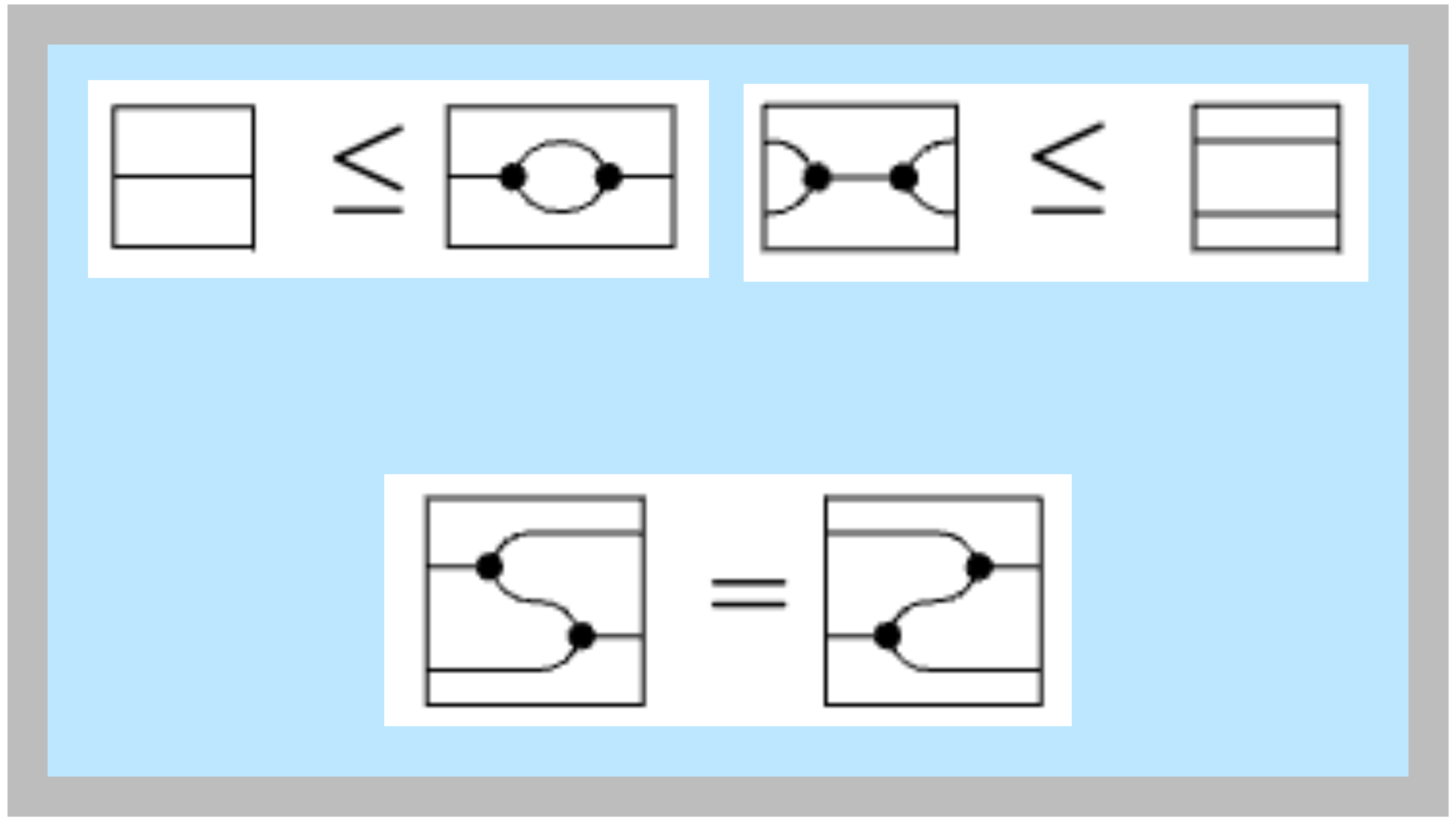
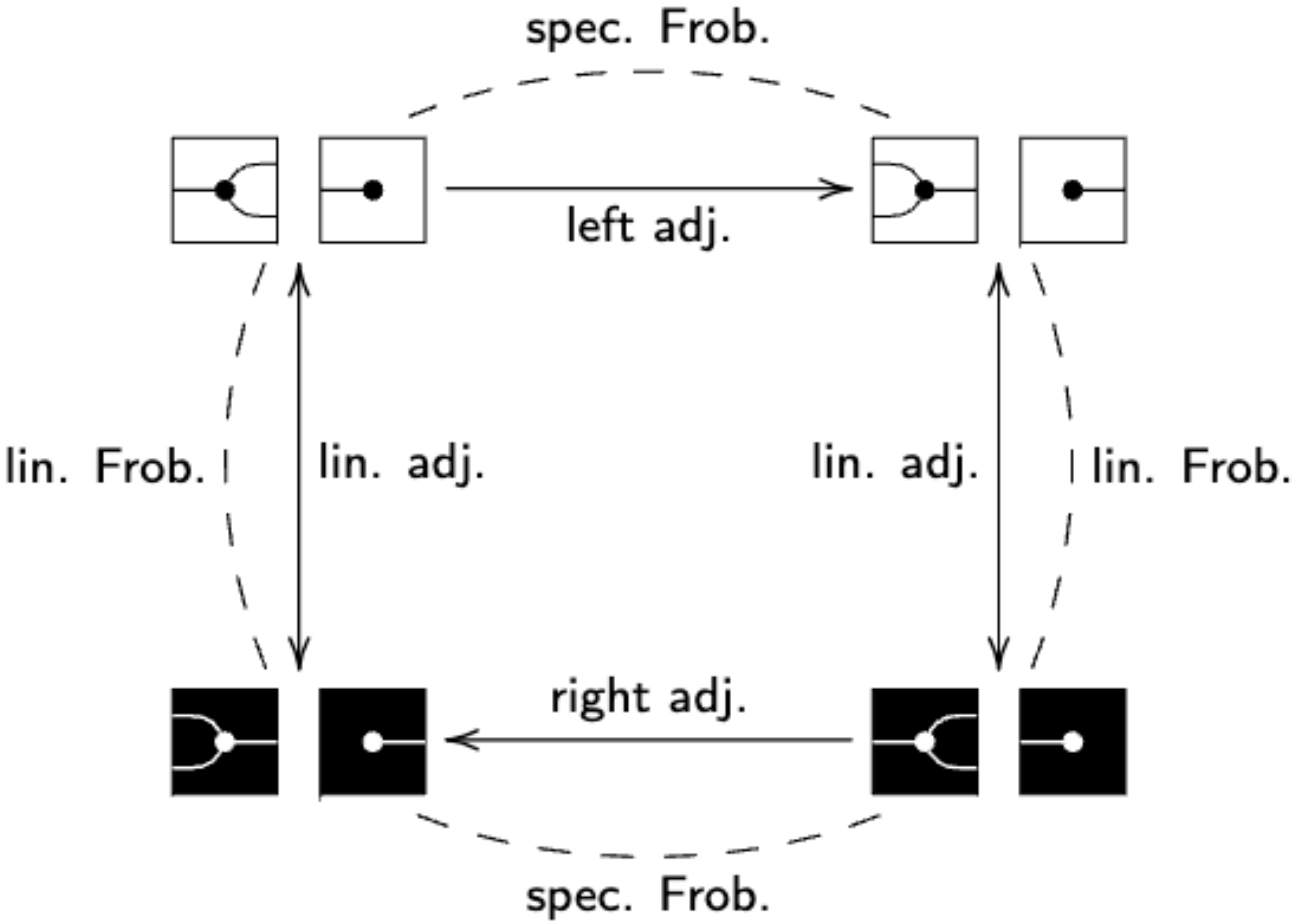


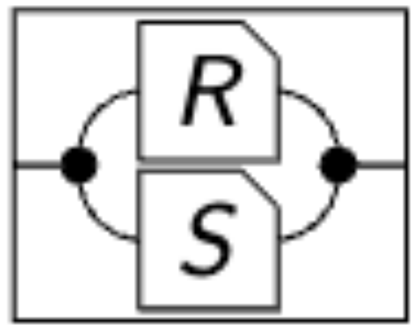
First Order Bicategories



Properties of First Order Bicategories

Every homset carries a Boolean algebra

$\top \stackrel{\text{def}}{=}$ 

$R \wedge S \stackrel{\text{def}}{=}$ 

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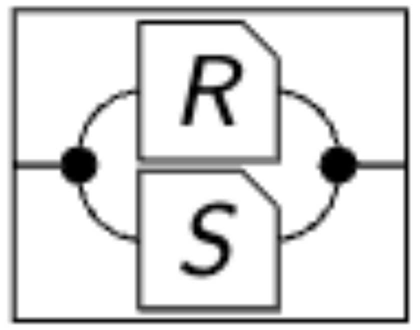
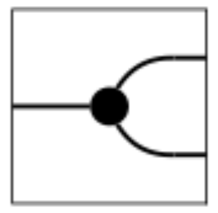
$R \wedge S \stackrel{\text{def}}{=}$ 

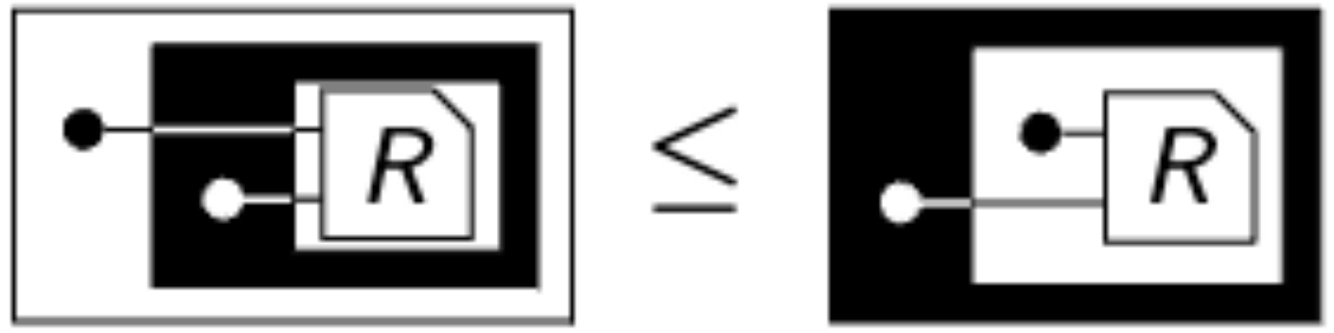
$\bot \stackrel{\text{def}}{=}$ 

$R \vee S \stackrel{\text{def}}{=}$ 

$\neg : C \to C^{co}$ is an isomorphism that swaps colors, e.g. $\neg$  $=$ 

Proofs as diagram rewrites

$$\exists x.\forall y.R(x,y) \implies \forall y.\exists x.R(x,y)$$



Proofs as diagram rewrites

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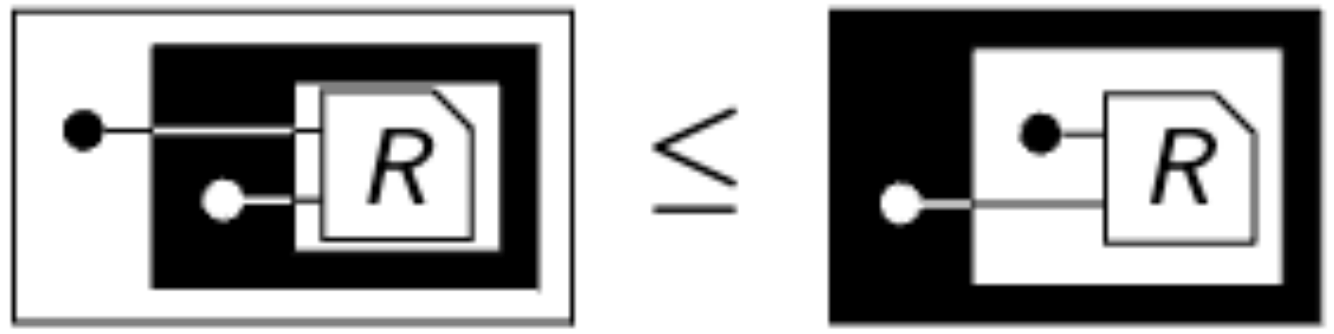
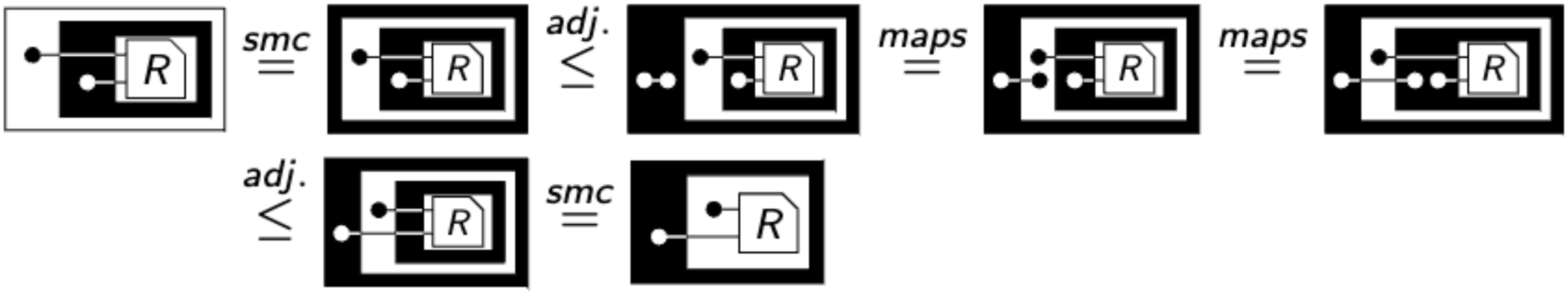


Proof



First Order Diagrammatic Theories

A theory is a pair $\mathbb{T} = (\Sigma, \mathbb{I})$

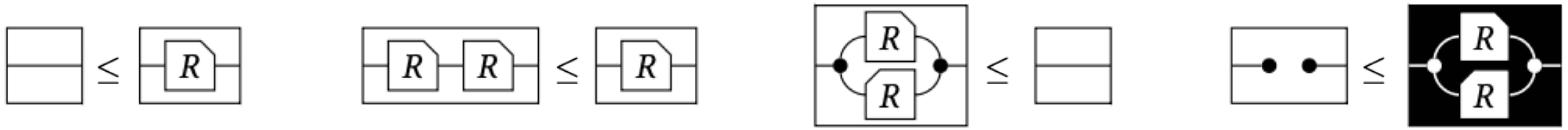
First Order Diagrammatic Theories

A theory is a pair $\mathbb{T} = (\Sigma, \mathbb{I})$

Example (linear orders)

$\Sigma = \{R \colon 1 \to 1\}$

$\mathbb{I} = \{$



$\}$

Relfexive Transitive Antisymmetric Total

First Order Diagrammatic Theories

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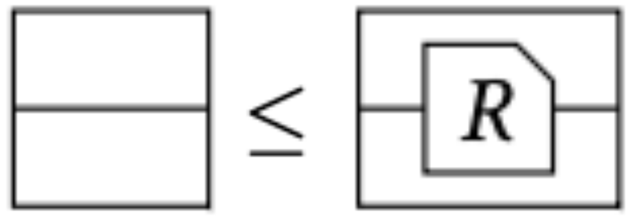
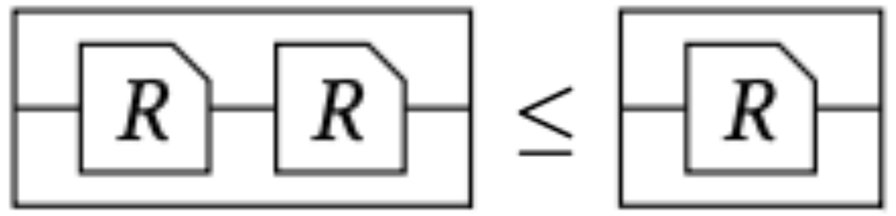
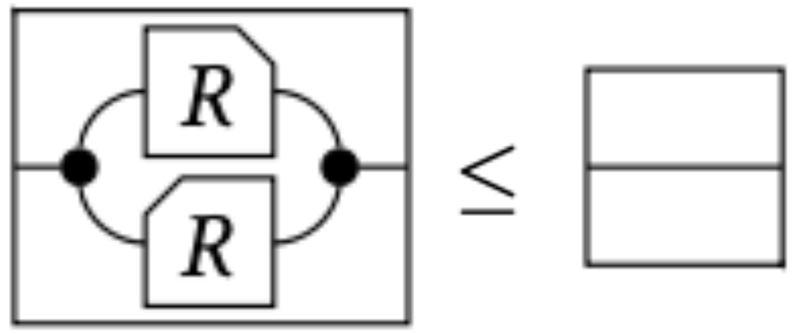


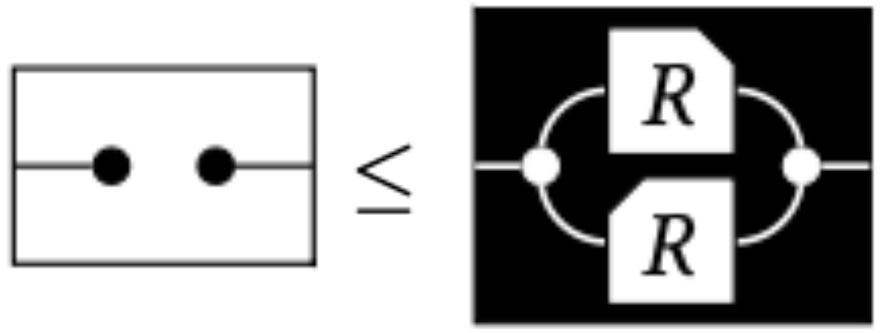
Relfexive



Transitive



Antisymmetric



Total

$\}$

Models are functors $F \colon \mathsf{FOB}_{\mathbb{T}} \to \mathsf{Rel}$

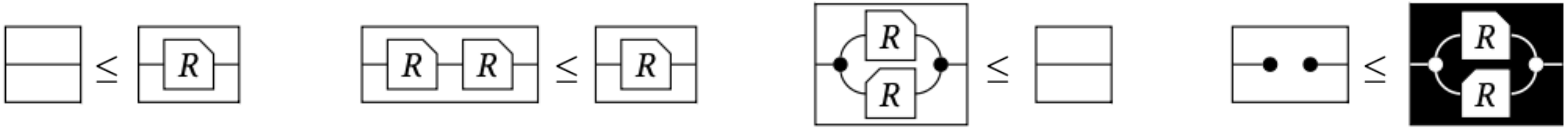
First Order Diagrammatic Theories

A theory is a pair $\mathbb{T} = (\Sigma, \mathbb{I})$

Example (linear orders)

$\Sigma = \{R\colon 1 \to 1\}$

$\mathbb{I} = \{$



$\}$

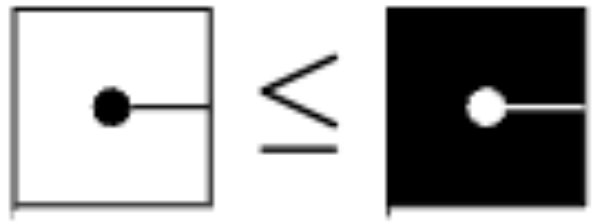
Relfexive | Transitive | Antisymmetric | Total

Models are functors $F\colon \mathsf{FOB}_{\mathbb{T}} \to \mathsf{Rel}$

Completeness If $\forall F\colon \mathsf{FOB}_{\mathbb{T}} \to \mathsf{Rel} \,.\, F(c) = F(d)$ then $c =_{\mathbb{T}} d$

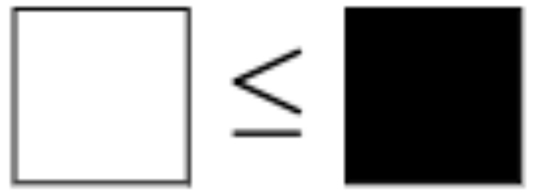
First Order Diagrammatic Theories

Trivial



⇒ all models have empty domain

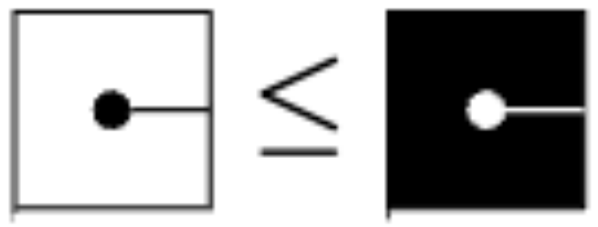
Contradictory



⇒ there are no models

First Order Diagrammatic Theories

Trivial



$\Rightarrow$ all models have empty domain

Contradictory

$\Rightarrow$ there are no models

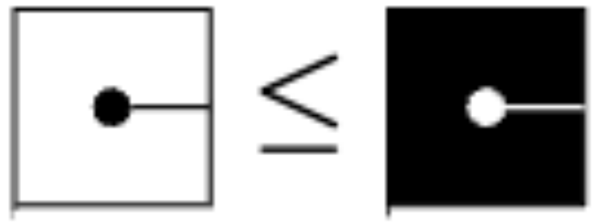
Trivial theories correspond to propositional theories

and the axioms collapse to the deep inference system SKSg

Brünnler, 2003

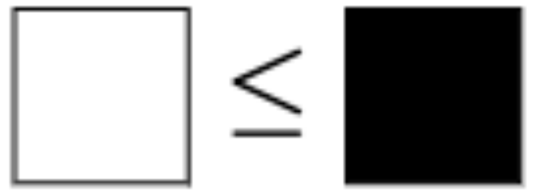
First Order Diagrammatic Theories

Trivial



$\Rightarrow$ all models have empty domain

Contradictory



$\Rightarrow$ there are no models

Trivial theories correspond to propositional theories

and the axioms collapse to the deep inference system SKSg

Brünnler, 2003

Conclusions

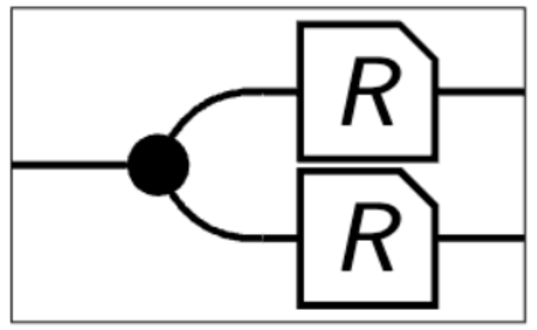
- Categorical Algebra of Relations as expressive as FOL
- Complete equational axiomatization
- Axioms arise from the interaction of algebraic structures
- No variables, No quantifiers
- It encodes other variable free approaches (see the paper)
- We recently showed* $\mathbb{FOB} \underset{\perp}{\rightleftarrows} \mathbb{BHD}$

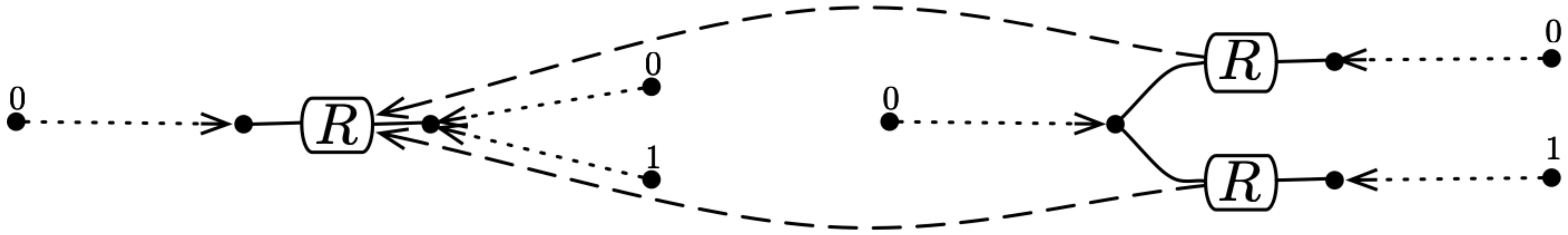
*joint work with Davide Trotta

Future work

- Beyond classical FOL: Intuitionistic? Higher-order?

- Combinatorial characterization by means of hypergraphs?





- Diagrammatic interactive proof assistant?

- Investigate the connection with deep inference