Flow-preserving rewriting in the ZX-calculus

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SYCO 12, University of Birmingham, 15 April 2024

Outline

Quantum computing

The ZX-calculus

The one-way model of measurement-based quantum computing

Flow-preserving rewriting

Conclusions

Quantum computing in a nutshell

Describing the (pure state) quantum mechanics of qubits:

states: vectors (in a complex Hilbert space of dimension 2^n)

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so represent everything in terms of a chosen set of generators.

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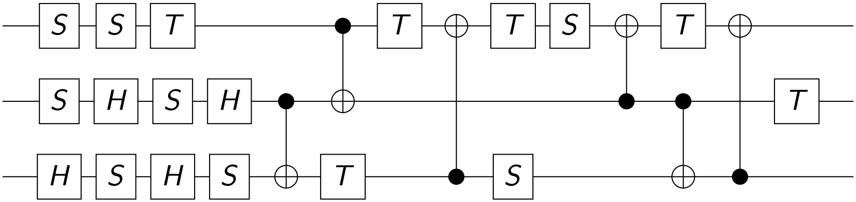

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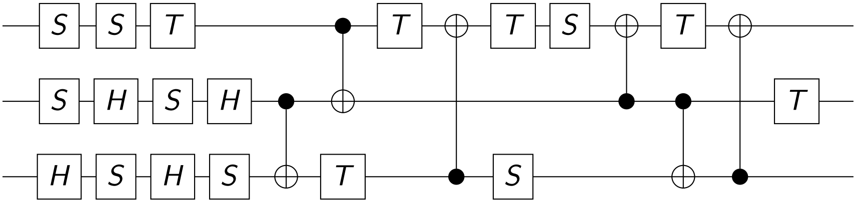

Quantum circuits act as a sort of 'machine language' for quantum computers.

Questions arising from limited quantum computing resources

Currently, and near-term future: noisy intermediate-scale quantum (NISQ) era

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Questions: compilation, optimisation, routing

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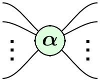
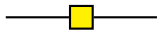
ZX-calculus generators



$$\rightsquigarrow \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & e^{i\alpha} \end{pmatrix}$$



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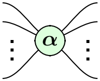

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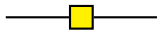

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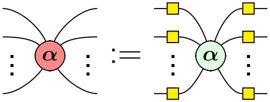
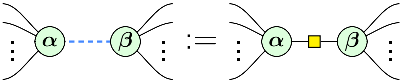
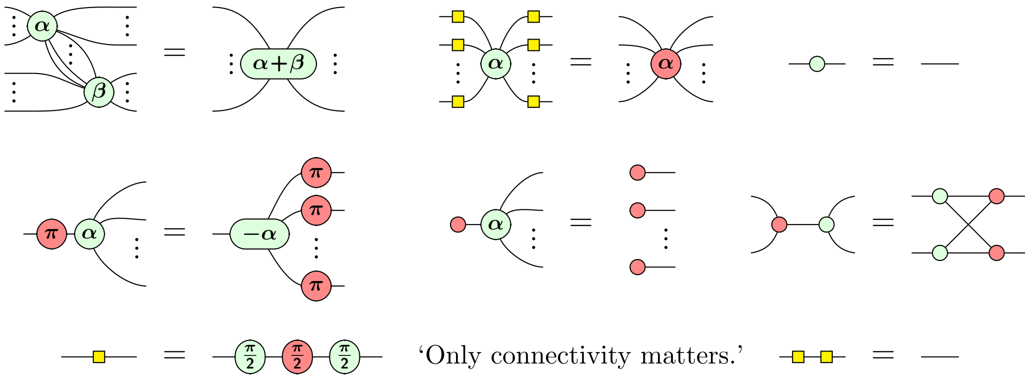
Common syntactic sugars:



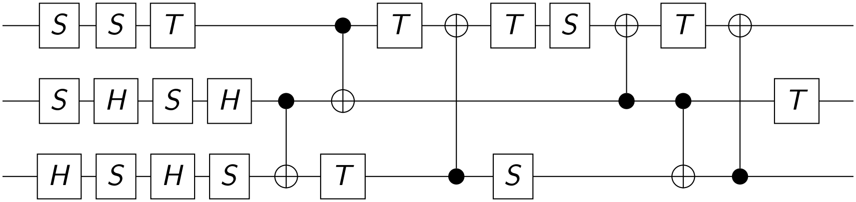


(Some) ZX-calculus rewrite rules



These rules are complete for the 'stabiliser fragment' (ignoring scalars) [B. 2014].
The universal calculus can be made complete [Jeandel et al. 2017; Ng & Wang 2017].

Optimising quantum computations using the ZX-calculus



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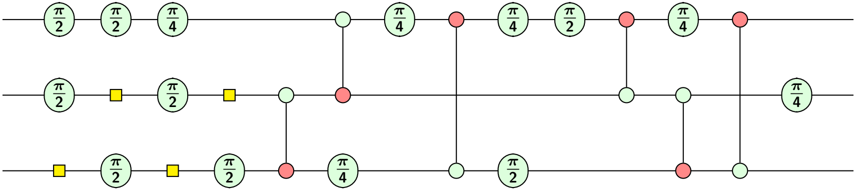

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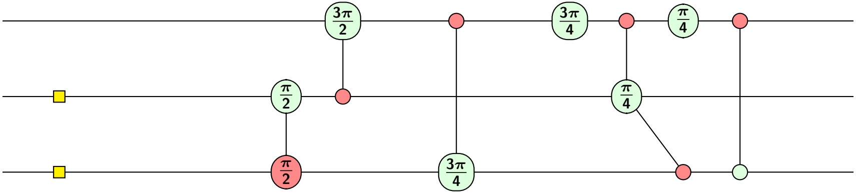

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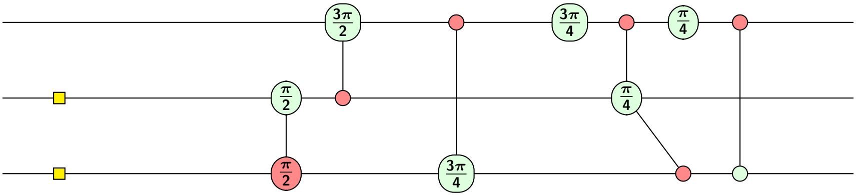

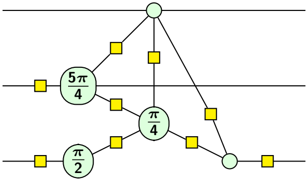

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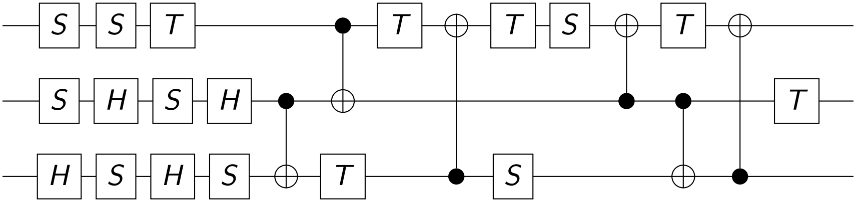
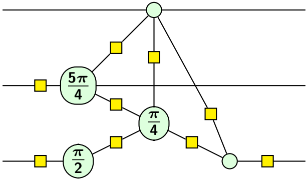
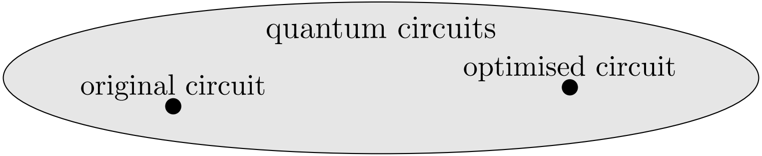




Problem: translating ZX-diagrams to circuits is #P-hard [de Beaudrap et al. 2022]

Flow permits efficient circuit extraction



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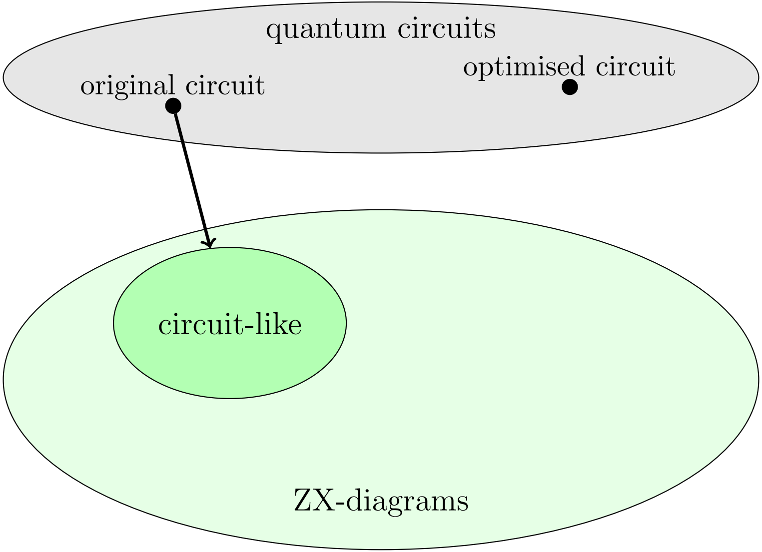

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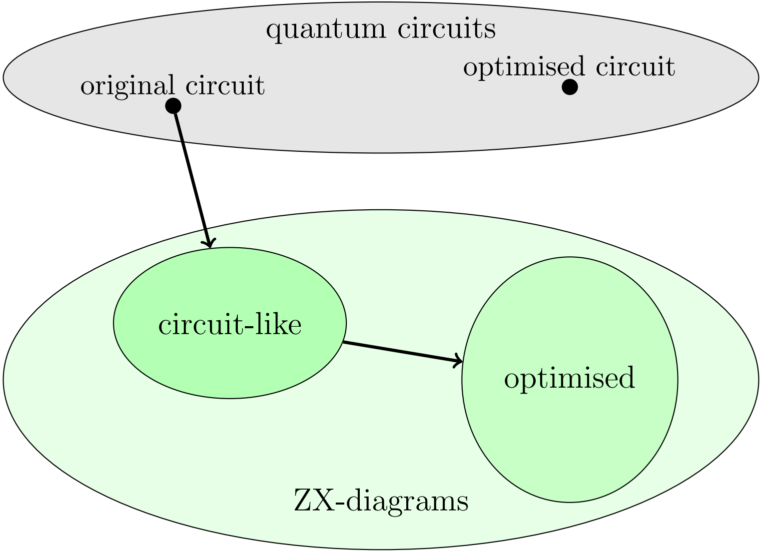

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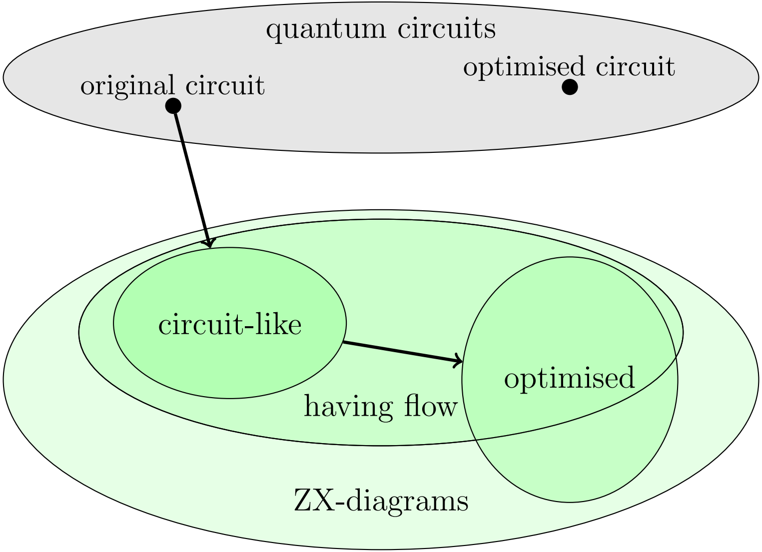

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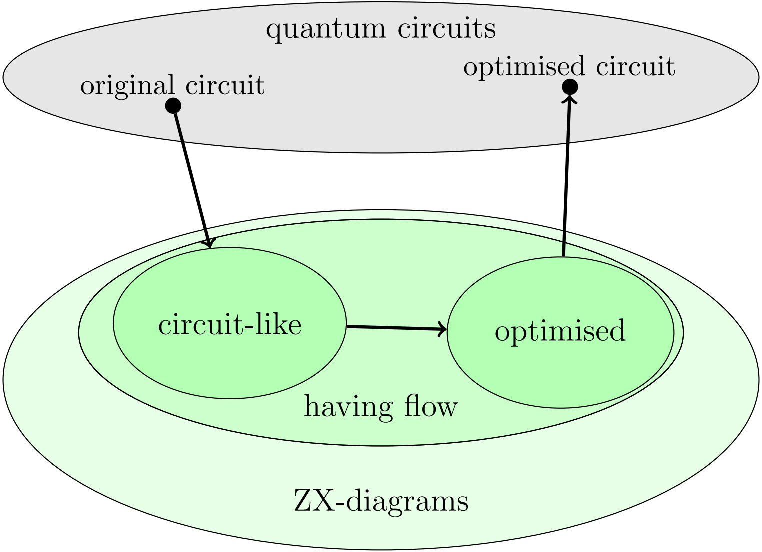

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- initialise entangled 'graph state' (can be made independent of computation)
- computation driven by successive adaptive single-qubit measurements
- if goal is state preparation, may need very simple unitary gates as correction at the end

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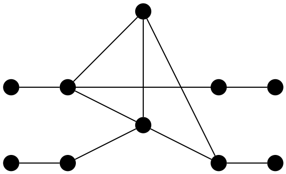

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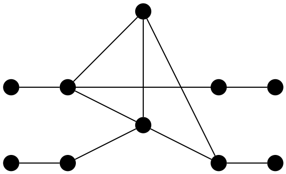

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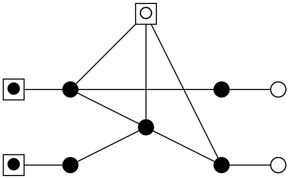

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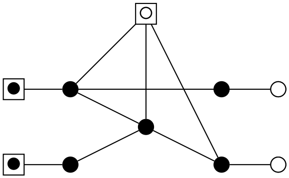

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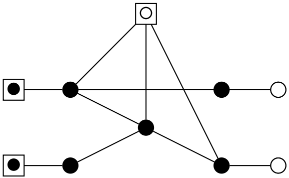

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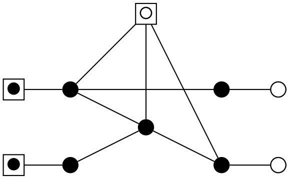

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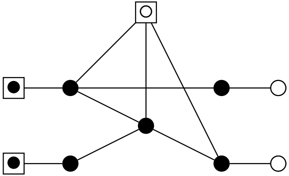

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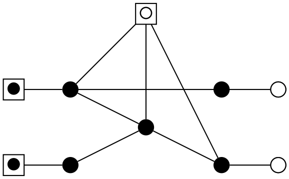

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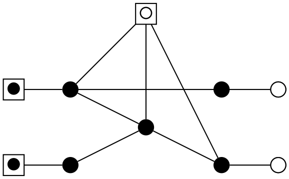

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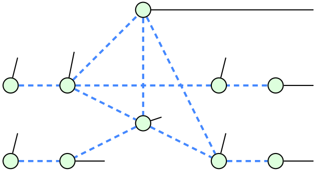
Theorem [Browne et al. 2007, Mhalla et al. 2022]

A one-way-model computation has a robustly deterministic implementation if and only if the underlying labelled open graph has flow.

MBQC-form ZX-diagrams

An MBQC-form ZX-diagram consists of:

- a graph state diagram



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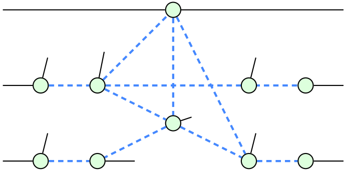

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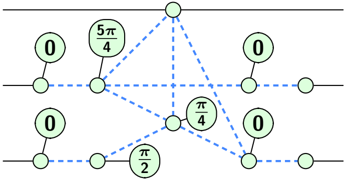

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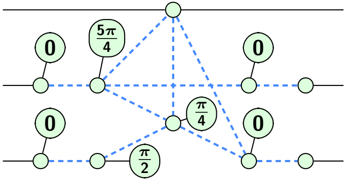

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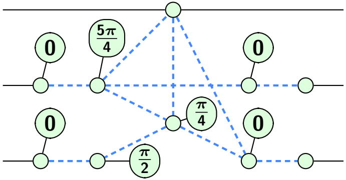

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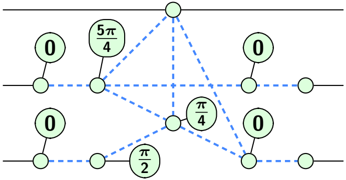

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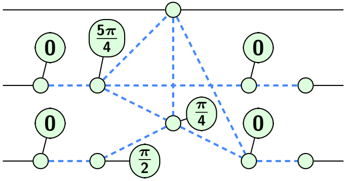

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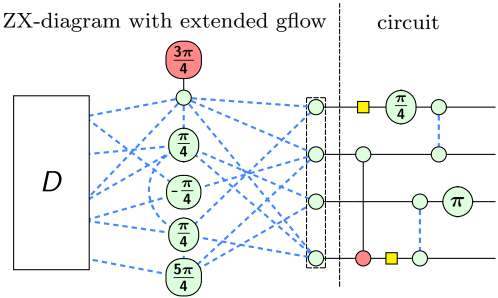
Definition

An MBQC-form ZX-diagram has flow if its underlying labelled open graph has flow.

Efficient circuit extraction from MBQC-form diagrams with flow



[Duncan et al. 2020; B., Miller-Bakewell, Felice, Lobski, van de Wetering 2021; Simmons 2021]

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The one-way model of measurement-based quantum computing

Flow-preserving rewriting

Conclusions

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(all phase labels are integer multiples of $\pi/2$)

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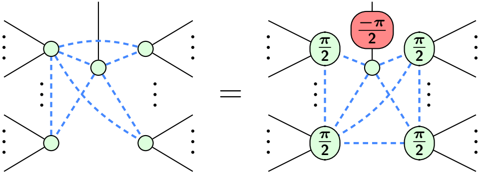

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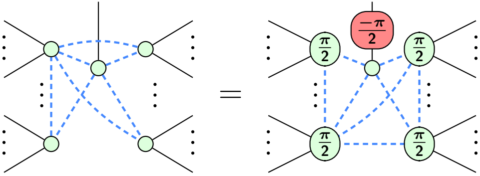

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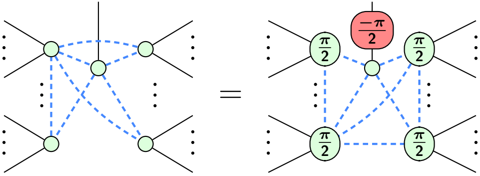
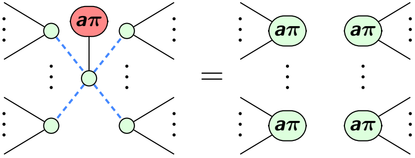


Z-deletion/insertion



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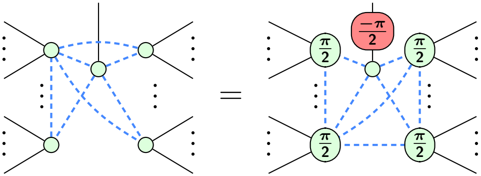

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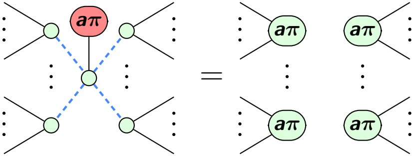

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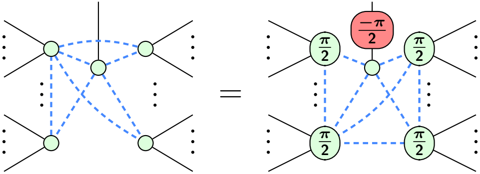

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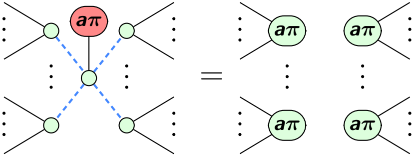
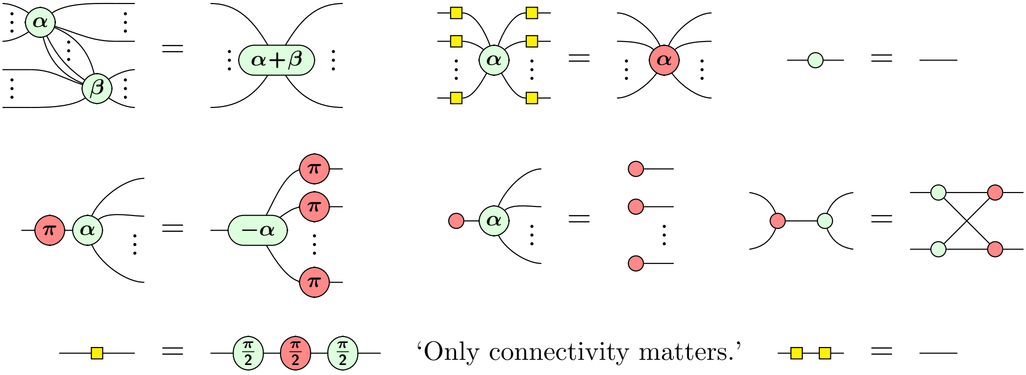


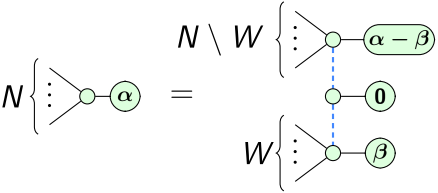
Theorem [McElvanney & B. 2023]

Suppose D and D' are two stabiliser ZX-diagrams with flow that both represent the same linear map. Then one can be rewritten into the other using local complementation, Z-insertion, and Z-deletion.

The standard stabiliser ZX-calculus rewrite rules again



A further flow-preserving rewrite rule: vertex splitting



[McElvanney & B. 2023]

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- Traditional schemes use a lot of redundant resources, can reduce that using flow-preserving ZX-calculus rewriting [Cao 2023]

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Thank you!