A Mixed Linear and Graded Logic

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University of Kent

Joint work with some really cool people

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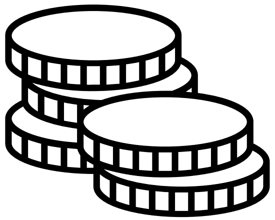
1 ⊙ Augusta University, 3 ⊙ University of Kent ⊢

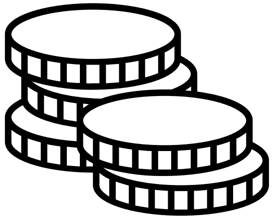
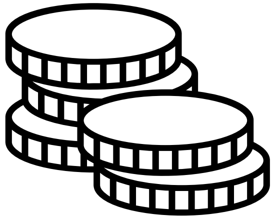
Intuitionistic Linear logic

 : Money

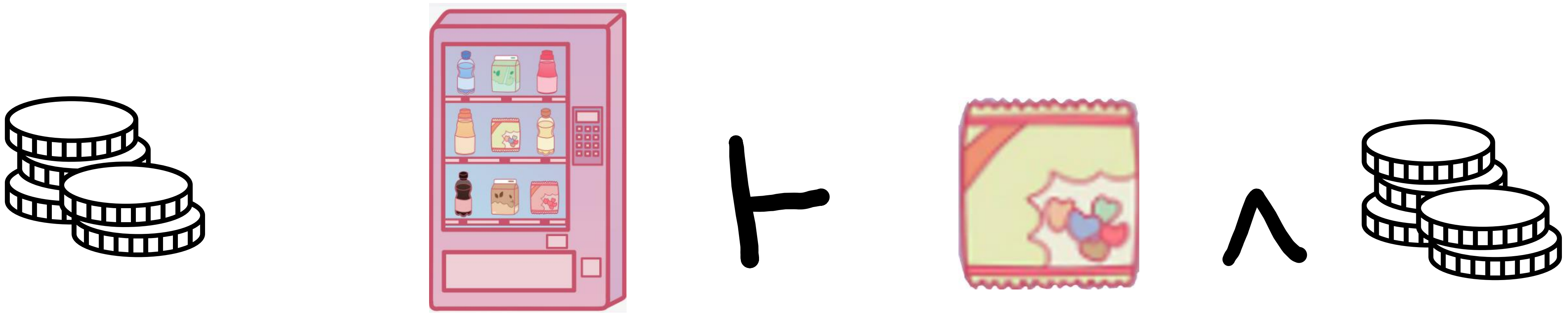
 : Money → food

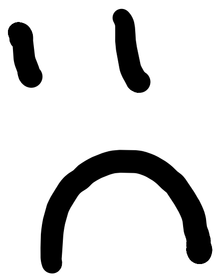
 : food

Intuitionistic Linear Logic

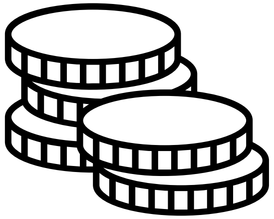
  $\vdash$  $\wedge$ 

Intuitionistic Linear Logic





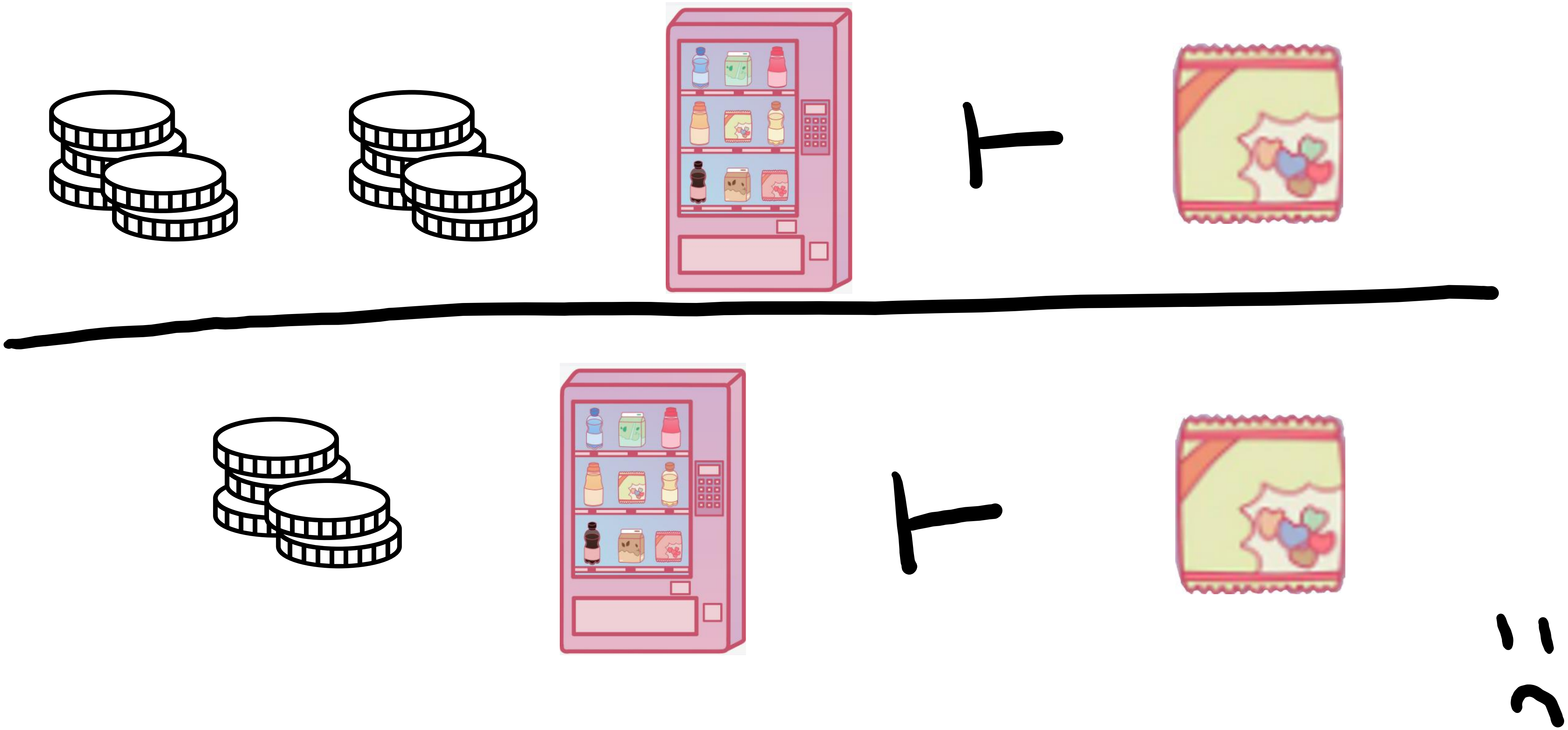
Intuitionistic Linear Logic



$\vdash$

Intuitionistic Linear logic

Don't have :



Intuitionistic Linear logic

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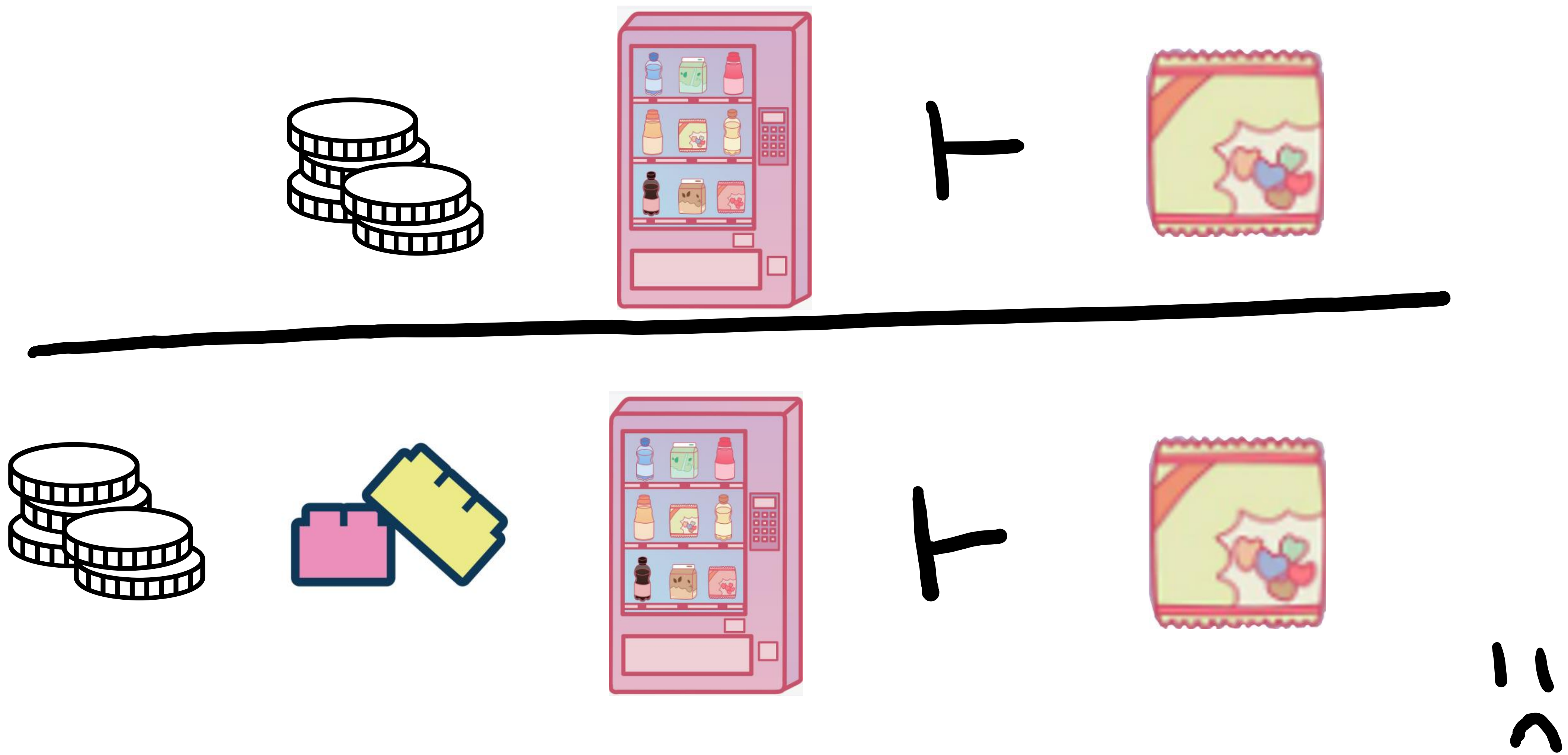
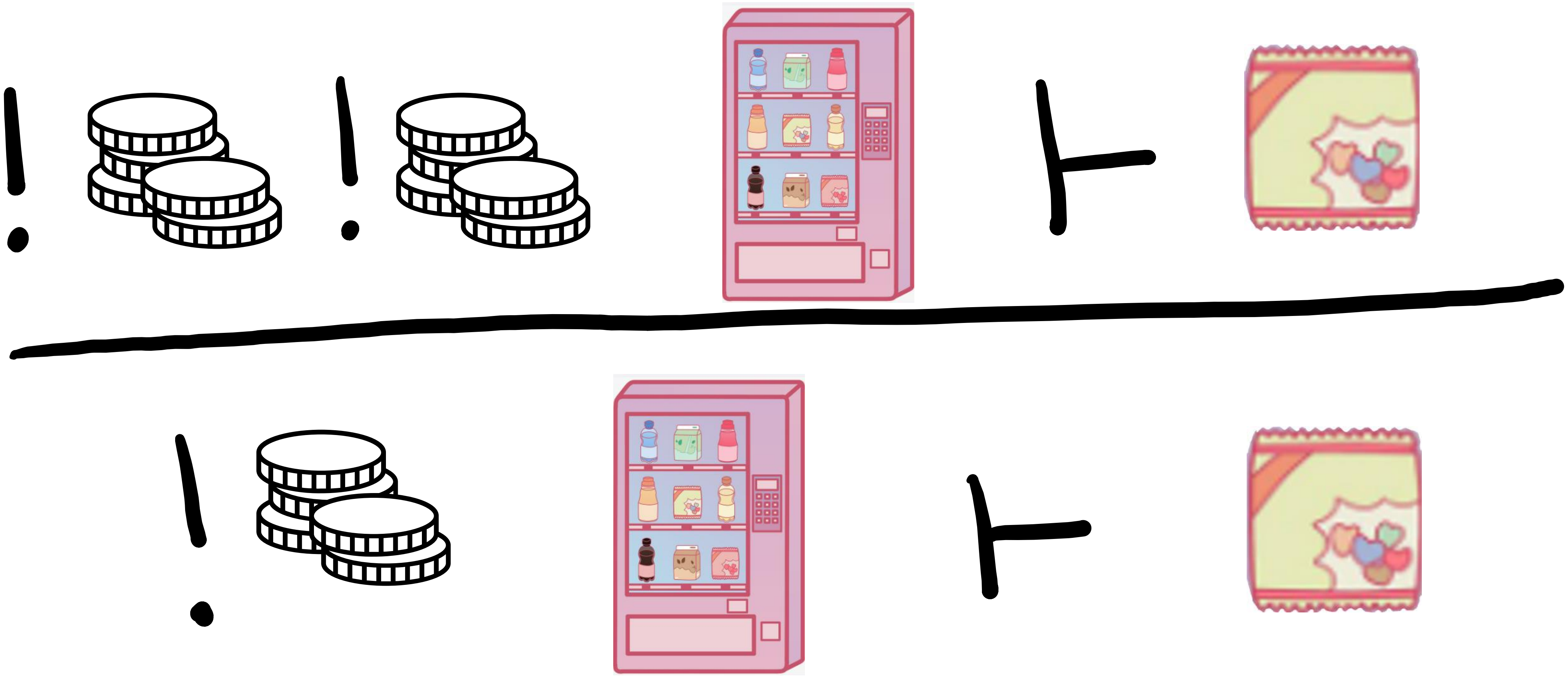


Bring structure back (with a bang!)



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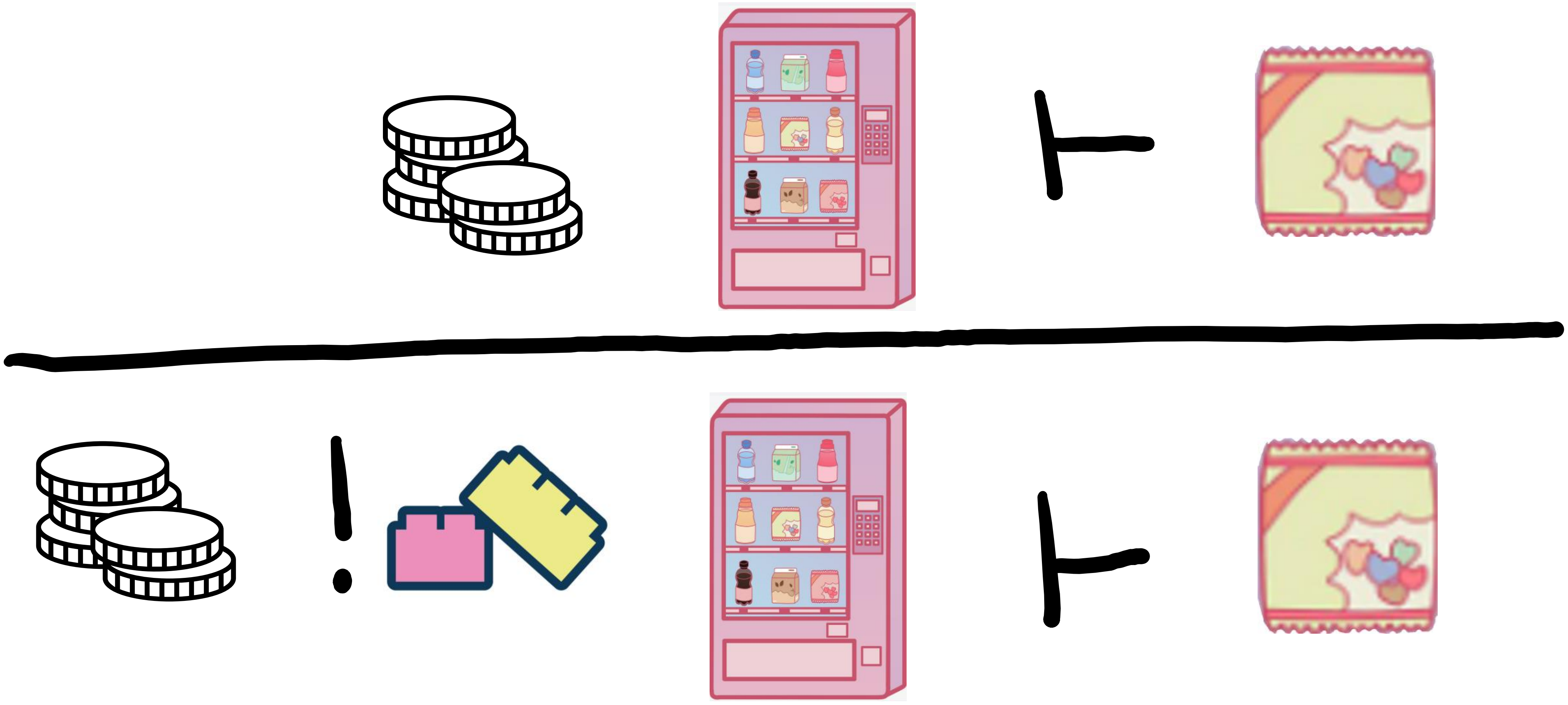
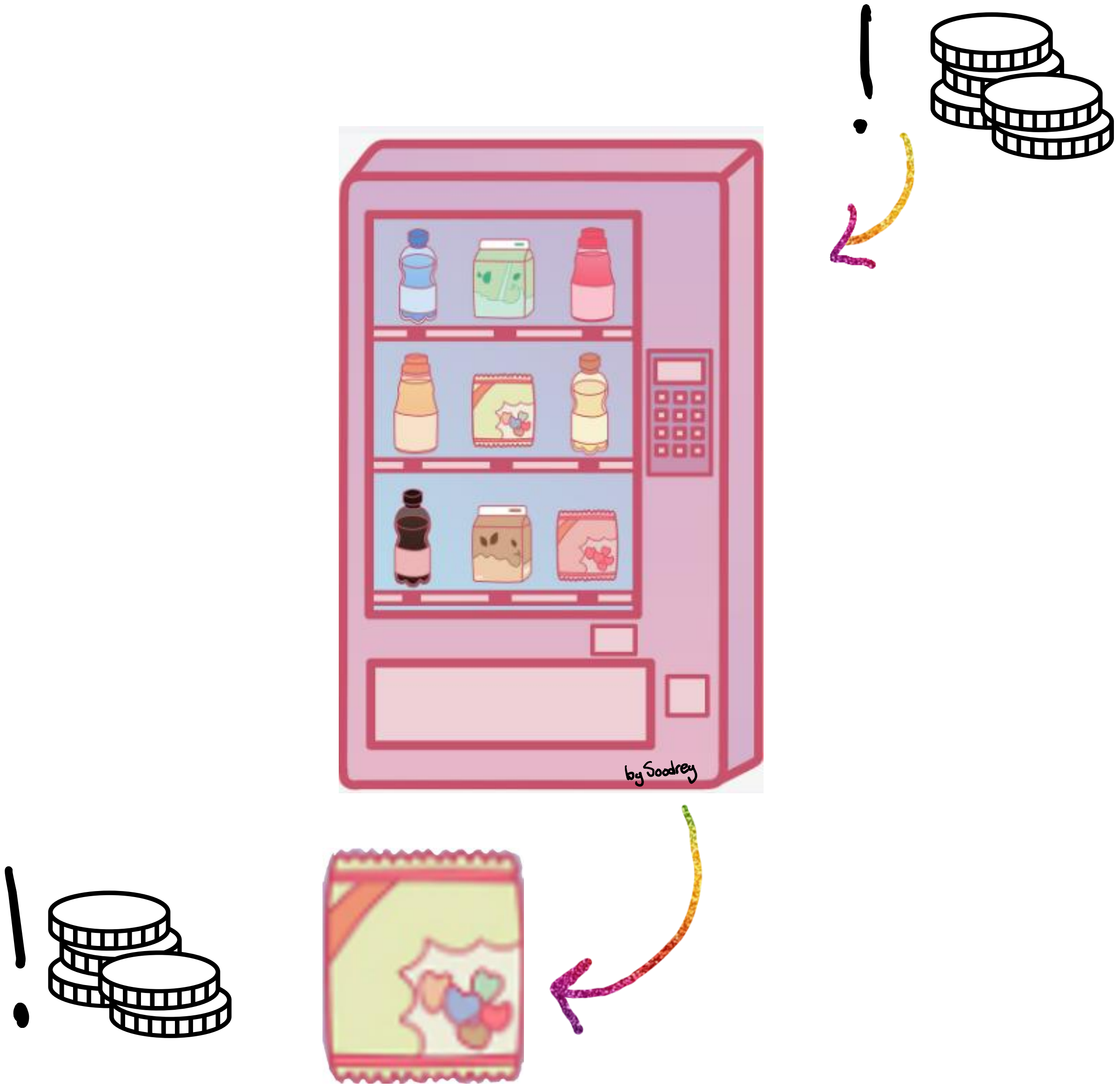
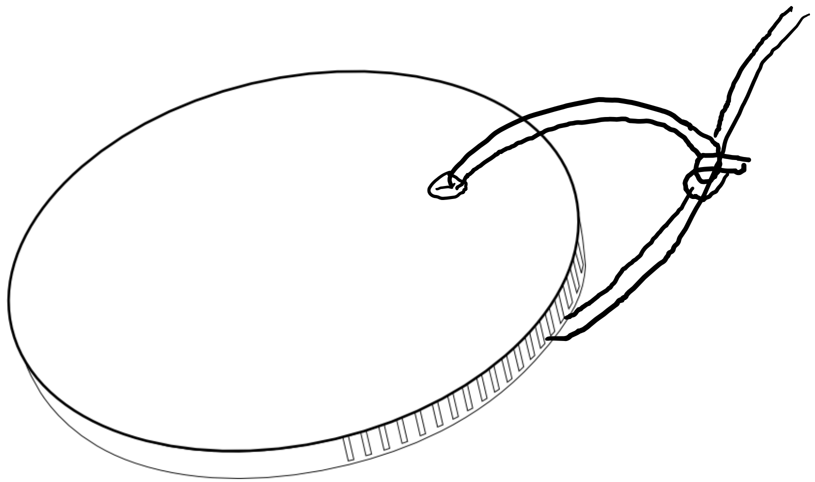


Intuitionistic Linear logic





Intuitionistic Linear logic

Symmetric Monoidal closed category

SMCC

Intuitionistic Linear logic

Symmetric Monoidal closed category

Comonad

$! \circlearrowright \mathrm{SMCC}$

Intuitionistic linear logic

Symmetric monoidal closed category

Linear exponential comonad

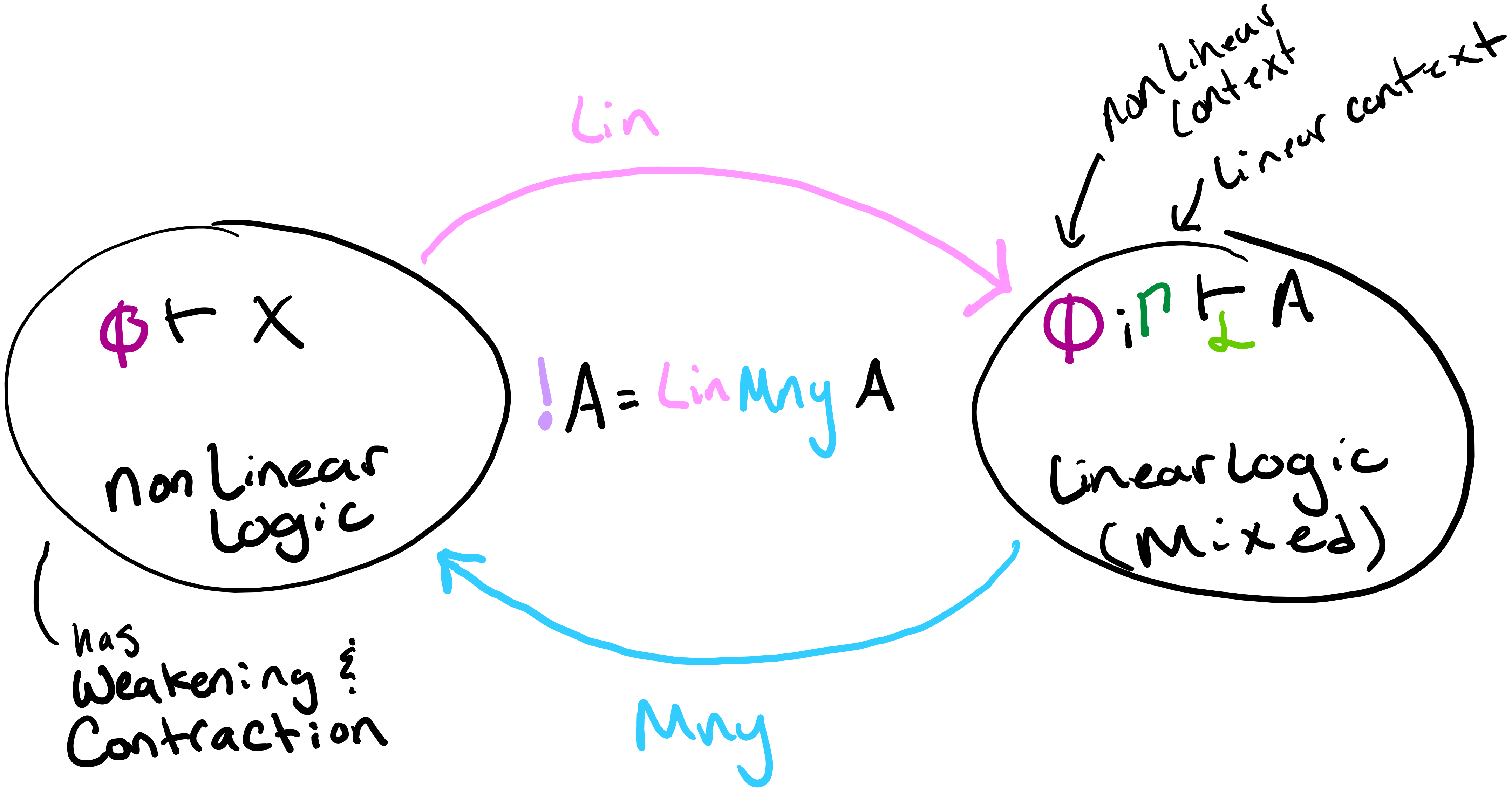
given $F \dashv G$ with counit ϵ and unit η we make the comonad:

$$(FG, F\eta_G, \epsilon)$$

Benton's LNL

$$\begin{tikzcd} \mathrm{CCC} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{SMCC} \arrow[l, bend left] \end{tikzcd}$$

Benton's LNL



Lin
$\phi \vdash X$
non Linear logic
has Weakening & Contraction
$!A = \text{Lin}\,\text{Mny}\,A$
non linear context
linear context
$\Phi ; \Gamma \vdash_L A$
Linear logic (Mixed)
Mny

Graded modal logic

traditional vs. graded

$\Box A$ $\qquad$ $\Box_r A$

$r \in (\mathbb{R}, \times, +, 1, 0, \leq)$

Graded modal logic

traditional vs. graded

$\Box A$

Modality formula

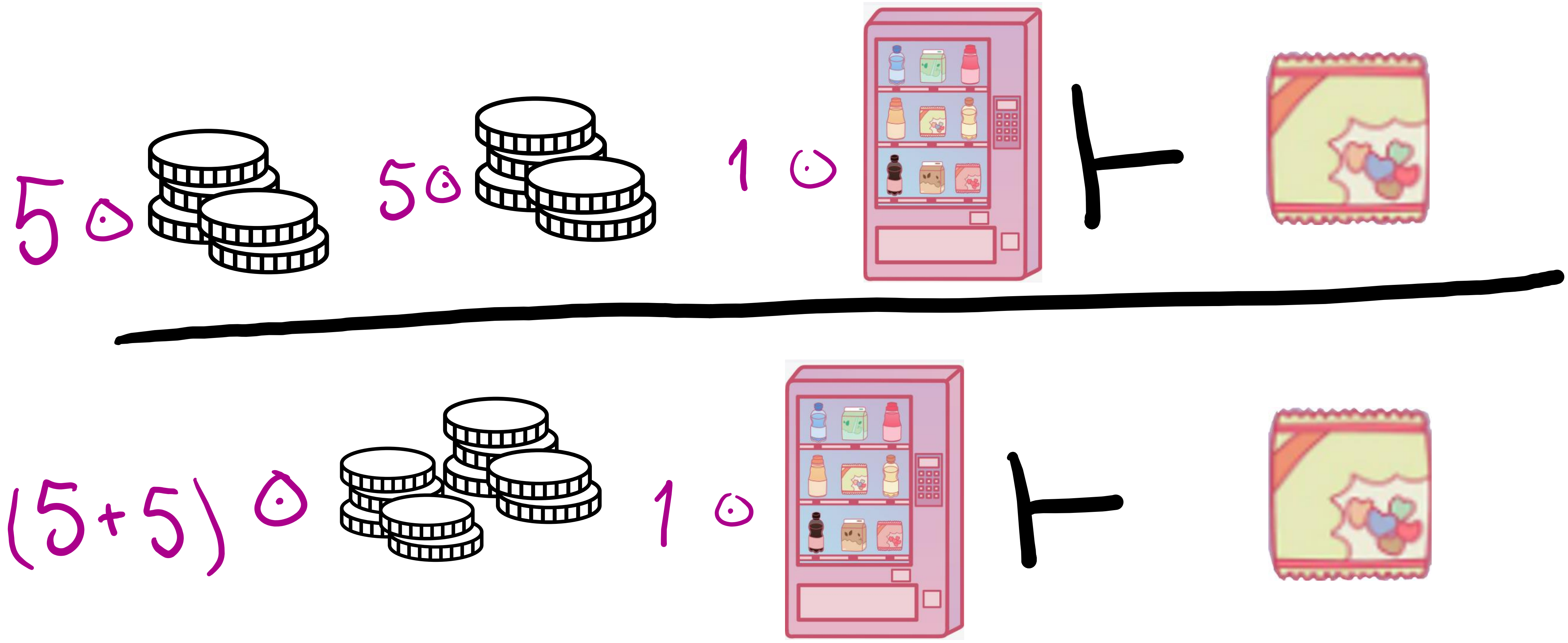
$\Box_r A$

Modality is indexed

$r \in (R, \times, +, 1, 0, \leq)$

ordered algebra

Graded modal logic



Mixed Graded Linear Logic (mGL)

graded

$X, Y ::= J \mid X \boxtimes Y$

$\Delta ::= (x : X, \Delta)$ $\quad \delta ::= (r, \delta)$ $\longleftarrow$ $r \in (R, *, 1, +, 0, \leq)$

$r \odot x : X$ and $\delta \odot A$

MGL

Linear

$A, B \ ::= I \mid A \otimes B$

MGL

Linear/Mixed

$A, B \quad ::= I \mid A \otimes B$

$\delta \odot \Delta ; \Gamma \vdash_{MS} t : A$

mGL

graded

$X, Y, Z ::= J \mid X \boxtimes Y$

$\Delta ::= (t : X, \Delta)$ $\quad \delta ::= (r, \delta)$

$r \odot x : X$ and $\delta \odot A$

$\delta \odot \Delta \vdash_{GS} t : X$

Linear/Mixed

$A, B, C ::= I \mid A \otimes B$

$r \in (R, *, 1, +, 0, \leq)$

$\delta \odot \Delta ; \Gamma \vdash_{MS} t : A$

mGL

graded

$X, Y, Z ::= J \mid X \boxtimes Y \mid \mathrm{Lin}\ A$

$\Delta ::= (t : X, \Delta) \quad \delta ::= (r, \delta)$

$r \odot x : X$ and $\delta \odot A$

$\delta \odot \Delta \vdash_{GS} t : X$

Linear/Mixed

$A, B, C ::= I \mid A \otimes B \mid \mathrm{Grd}_r\ X$

$r \in (R, *, 1, +, 0, \leq)$

$\delta \odot \Delta ; \Gamma \vdash_{MS} t : A$

MGL

Lin A

$\mathrm{Grd}_r\, X$

Some background math (to inform our later choices)

$$\square : R \to [M, M] \quad \text{where } R \text{ is } (R, 1, *, 0, +, \leq)$$

$$\delta_{r,s} : \square_{r*s} \to \square_r \square_s$$

$$\varepsilon : \square_1 \to id_M$$

Some background math (to inform our later choices)

the cofree coalgebras for $\Box_r$?

$(\Box_r A, \delta)$?

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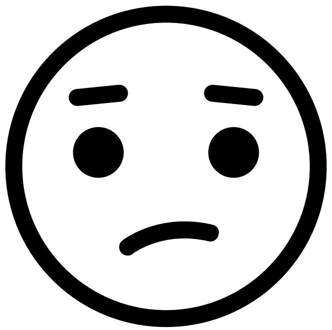
the cofree coalgebras for $\Box_r$?

$(\Box_r A, \delta)$?

We'd expect $\delta : \Box_r A \to \Box_r \Box_r A$

But $\delta : \Box_{r*s} A \to \Box_r \Box_s A$



Some background math (to inform our later choices)

$$F : [R, M] \to [R, M]$$

graded F-coalgebras (X, h)

$$X : R \to M$$

h is a family of morphisms

$$h_{r_1 r_2} : X(r_1 * r_2) \longrightarrow F_{r_1} X(r_2)$$

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Some background math (to inform our later choices)

for $(\square : R \to [M, M], \delta, \varepsilon)$

graded F-coalgebras $(\square_{\_} A, \delta_{-,-,A})$

$\square_{\_} A : R \longrightarrow M$

$\delta_{-,-,A}$ is a family of morphisms

$\delta_{r_1 r_2 A} : \square_{r_1 * r_2} A \longrightarrow \square_{r_1} \square_{r_2} A$

hey!

Fujii, Soichiro, Shin-ya Katsumata, and Paul-André Mellies. "Towards a formal theory of graded monads." *FOSSACS 2016*

Some background math (to inform our later choices)

for $(\Box : R \to [M, M], \delta, \varepsilon)$

graded F-coalgebras $(\Box\_A, \delta_{-,-,A})$

$\Box\_A : R \longrightarrow M$

$\delta_{-,-,A}$ is a family of morphisms

$\delta_{r_1 r_2 A} : \Box_{r_1 * r_2} A \longrightarrow \Box_{r_1} \Box_{r_2} A$

$\boxed{M^{\Box}}$

Can we make an adjunction now?

$$M^{\Box} : L \dashv \mathrm{Box} : M$$

forgetful

$$\mathrm{Box}(A) = \lambda r.\ \Box_r A,\ \delta$$

Goal: this way that way is $\square_r$

$$L(Box(A)) = L(\lambda r . \square_r A, \delta)$$

$$= \square_1 A$$

always 1

Goal: thisway thatway is $\square_r$

$$\odot : R \times M^{\square} \longrightarrow M^{\square}$$

$$r \odot (X, h) = (\lambda s.\ X(s * r),\ \lambda r.\ \lambda r_2.\ h_{r_1, r_2 * r})$$

Goal: thisway thatway is $\Box_r$

$$r \odot (X, h) = (\lambda s.\, X(s * r),\ \lambda r.\, \lambda r_2.\, h_{r_1, r_2 * r})$$

$$L(r \odot \mathrm{Box}(A)) = L(r \odot (\lambda r.\, \Box_r A, \delta))$$

Goal: thisway thatway is $\square_r$

$$r \odot (X, h) = (\lambda s.\ X(s * r),\ \lambda r.\ \lambda r_2.\ h_{r_1, r_2 * r})$$

$$L(r \odot \mathrm{Box}(A)) = L(r \odot (\lambda r.\ \square_r A, \delta))$$

$$= L(\lambda s.\ \square_{s*r} A, \delta)$$

Goal: this way $\xrightarrow{\text{that way}}$ is $\square_r$

$$r \odot (X, h) = (\lambda s.\ X(s * r),\ \lambda r_1.\ \lambda r_2.\ h_{r_1, r_2 * r})$$

$$L(r \odot \mathrm{Box}(A)) = L(r \odot (\lambda r.\ \square_r A, \delta))$$

$$= L(\lambda s.\ \square_{s*r} A, \delta)$$

$$= \square_{1*r} A = \square_r A \quad \checkmark$$

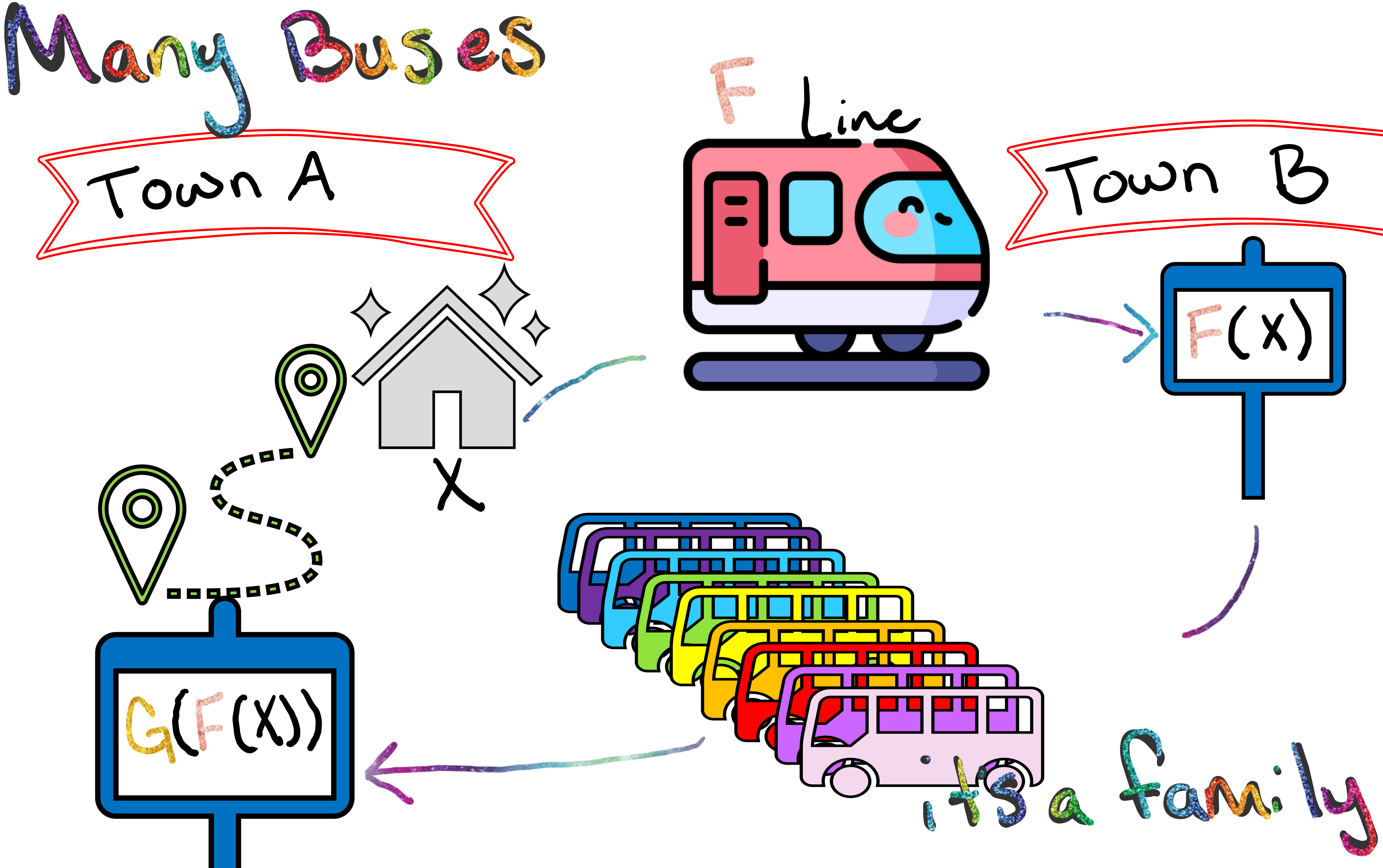
Fun Fact

symmetric monoidal adjunction

+

strict monoidal action

$\rightsquigarrow$ family of adjunctions


Many Buses
Town A
F
Line
Town B
F(X)
X
G(F(X))
it's a family

Graded multicategory

$(R, 1, *, \leq)$ $\qquad O$ objects

$Gr(R, O)$

$$\underbrace{(r_1, X_1) \ldots (r_n, X_n)}_{\Phi} \xrightarrow{f} Z$$

Graded multicategory

$(R, 1, *, \leq)$ $\qquad$ O objects

$\mathrm{Gr}(R, O)$

$$\underbrace{(r_1, X_1) \ldots (r_n,}_{\Phi} X_n) \xrightarrow{f} Z$$

:(

Structural natural transformations

$$\frac{\langle \Phi, \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, 0 * \Phi', \Psi \rangle \xrightarrow{\mathsf{weak}_{\Phi,\Phi',\Psi,Z}(f)} Z}$$

$$\frac{\langle \Phi, (r, X), (s, X), \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, (r + s, X), \Psi \rangle \xrightarrow{\mathsf{contr}_{\Phi,r,s,X,\Psi,Z}(f)} Z}$$

$$\frac{\langle \Phi, \Phi_1, \Phi_2, \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, \Phi_2, \Phi_1, \Psi \rangle \xrightarrow{\mathsf{ex}_{\Phi,\Phi_1,\Phi_2,\Psi,Z}(f)} Z}$$

Structural natural transformations

$$\frac{\langle \Phi, \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, 0 * \Phi', \Psi \rangle \xrightarrow{\mathsf{weak}_{\Phi,\Phi',\Psi,Z}(f)} Z}$$

$$\frac{\langle \Phi, (r, X), (s, X), \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, (r + s, X), \Psi \rangle \xrightarrow{\mathsf{contr}_{\Phi,r,s,X,\Psi,Z}(f)} Z}$$

$$\frac{\langle \Phi, \Phi_1, \Phi_2, \Psi \rangle \xrightarrow{f} Z}{\langle \Phi, \Phi_2, \Phi_1, \Psi \rangle \xrightarrow{\mathsf{ex}_{\Phi,\Phi_1,\Phi_2,\Psi,Z}(f)} Z}$$

MGL Models

$C : \mathrm{Mny} \dashv \mathrm{Lin} : M$

MGL Models

$$\mathcal{C} : \mathrm{Mnd} \dashv \mathrm{Lin} : \mathcal{M}$$

$$\rightsquigarrow$$

$$\mathrm{Gr}(\mathcal{R}, \mathcal{C})$$

MGL Models

$$C : \mathrm{Mny} \dashv \mathrm{Lin} : M$$

$$\rightsquigarrow$$

$$\mathrm{Gr}(R, C) : \mathrm{Grd}(r, -) \dashv \mathrm{Lin}^{\odot} : M$$

MGL Models

$$\mathrm{Gr}(R, C) : \mathrm{Grd}(r, -) \dashv \mathrm{Lin}^{\odot} : M$$

$$\mathrm{Gr}(R, c) : \mathrm{Mny}(x(r)) \dashv \lambda s. s \odot \mathrm{Lin} : M$$

This slide is
just for me
to ramble during :)