Endomorphisms of models: yet another categorification of model theory

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Outline

Theorem 4.1.4. *Let Ω be a set; let G be a subgroup of $\mathrm{Sym}(\Omega)$ and H a subgroup of G. Then the following are equivalent.*

(a) *H is closed in G.*

(b) *There is a structure A with $\mathrm{dom}(A) = \Omega$ such that $H = G \cap \mathrm{Aut}(A)$.*

In particular a subgroup H of $\mathrm{Sym}(\Omega)$ is of form $\mathrm{Aut}(B)$ for some structure B with domain Ω if and only if H is closed

Figure: A result from Hodges' *Model Theory*.

Main theorem preview

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ, M a **Set**-model of $\mathbb{T}$ and L (the opposite of) the endomorphism monoid of M. There is a Galois connection between:

$\{\text{Submonoids } L' \subseteq L\}$ and $\{\text{Families of (finitary) relations on } M\}$.

From left to right, a submonoid is sent to the collection of relations on M which are invariant under endomorphisms in L'.

From right to left, a family of relations is sent to the endomorphisms which preserve them.

Summary

No deep theory is needed to state this Galois connection. However, to explain where the topology comes from, we need to take a deep dive into topos theory.

I was supposed to present the *automorphism* version of this talk, but there are some subtleties I haven't yet ironed out with that version.

i.e. Beware: WIP!

Section 1

Thinking about toposes

Four perspectives

For the main results of this talk, it will be enough to think of (Grothendieck) toposes as **very structured categories**.

However, for the technical aspects we will variously be viewing them:

- **From their presentations:** toposes are generated from *sites*;
- **As categories of actions:** objects of toposes are *dynamical data*;
- **Via geometric theories:** toposes *classify* geometric theories, and
- **From the outside:** toposes are the *objects of a bicategory*.

Grothendieck sites

Definition

A **site** consists of a small category $\mathcal{C}$ and a *Grothendieck coverage* J on $\mathcal{C}$ assigning to each object $C \in \mathrm{ob}(\mathcal{C})$ a collection $J(C)$ of *covering sieves*: sets of arrows into C closed under precomposition.

We omit the requirements on the coverage J.

Example: a site from a space

Given a topological space X, let $\mathcal{C} = \mathcal{O}(X)$ be the category (poset) of open subsets of X, and $J(U)$ the set of open covers of U.

Toposes of sheaves

From a site $(\mathcal{C}, J)$, we obtain the **topos of sheaves**, denoted $\mathrm{Sh}(\mathcal{C}, J)$. It is a (reflective) subcategory of the category $\mathrm{PSh}(\mathcal{C})$ of **presheaves**:

Objects: Functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$

Morphisms: Natural transformations.

To be a sheaf, a presheaf must satisfy the *sheaf condition(s)* determined by J. We omit this too!

Examples

The category $\mathrm{Sh}(\mathcal{O}(X), J)$ is the category of (**Set**-valued) sheaves on the space X, often denoted $\mathrm{Sh}(X)$.

The category of sheaves over the one-point space is (equivalent to) **Set**.

Monoid actions

As a very special case, consider a **monoid** L as a one-object category.

Definition

A right **action** of L on a set X is a function $\alpha : X \times L \to L$ satisfying

$$\alpha(x, 1) = x \text{ and } \alpha(\alpha(x, m), n) = \alpha(x, mn)$$

for all $x \in X$ and $m, n \in L$.

Exercise!

The category of **right actions** is equivalent to $\mathrm{PSh}(L)$.

Topological monoids

Suppose we put a **topology** ρ on L. Viewing a set X as a discrete space, we can examine **continuous actions**.

Proposition

The full (coreflective) subcategory of $\mathrm{PSh}(L)$ on the *continuous right actions* is a Grothendieck topos.

Beware that continuity is a distinct condition from the sheaf condition we saw earlier!

Signatures

Logics are defined over a *signature* Σ, which defines the basic components of the syntax:

sorts, $A, B, \ldots,$

relation symbols (including equality), $R, S, \ldots,$ and

function symbols, $f, g, \ldots.$

Each of the relation and function symbols have a finite (possibly empty) sequence of sorts associated to them, denoted as:

$$R \rightarrowtail A_1, \ldots, A_n$$

$$f : A_1, \ldots, A_n \to B$$

Variables and terms

Given a signature Σ, we assume that we have a (countably infinite) supply of **variables**, $x, y, \dots,$ each having an associated sort, its **type**. We write $x : A$ or x^A to indicate that the variable x has type A.

From variables we construct **terms**, which also have types, inductively:

> The basic terms are individual variables with their associated types.
>
> If $f : A_1, \dots, A_n \to B$ is a function symbols and $t_1, \dots, t_n$ are terms such that t_i has type A_i, then $f(t_1, \dots, t_n)$ is a term of type B.

Atomic formulas

Formulas are constructed from terms using relations. If $R \rightarrowtail A_1, \ldots, A_n$ is a relation symbol in the signature and $t_1, \ldots, t_n$ are terms of the respective types, then $R(t_1, \ldots, t_n)$ is an **atomic formula**.

For example, if Σ contains a binary function symbol $\mu : A, B \to C$ (and the equality relation), then for variables $x : A, y : B, z : C$, $\mu(x, y) = z$ is an atomic formula.

Fragments of logic

General formulas are constructed from atomic formulas using **logical connectives**. The connectives permitted, and restrictions on when we may employ them, are determined by the **fragment of logic** one is working in.

Logical connective	$\top$	$\wedge$	$\exists!$	$\exists$	$\bot$	$\vee$	$\forall$	$\bigvee$	$\rightarrow$
Horn logic	✓	✓							
Cartesian logic	✓	✓	✓						
Regular logic	✓	✓	(✓)	✓					
Coherent logic	✓	✓	(✓)	✓	✓	✓			
Geometric logic	✓	✓	(✓)	✓	✓	✓		✓	
First order logic	✓	✓	(✓)	✓	✓	✓	✓		✓
Infinitary FOL	✓	✓	(✓)	✓	✓	✓	✓	✓	✓

Theories

A **geometric theory** $\mathbb{T}$ over a signature Σ consists of a collection of **axioms** in the form of sequents,

$$\phi(\vec{x}) \vdash_{\vec{x}} \psi(\vec{x}),$$

where ϕ, ψ are **geometric formulas** and $\vec{x}$ is a sequence of variables.

The intended interpretation of such an axiom is,

$$(\forall \vec{x})\,(\phi(\vec{x}) \implies \psi(\vec{x}))\,.$$

The theory of fields

Example: Fields

The (coherent) theory of fields is defined over the signature containing a single sort, two constants (0,1), two binary functions $(+, \times)$ and the equality relation. It has axioms for commutativity, units and associativity, such as:

$$\top \vdash_{x,y} x + y = y + x,$$

(this produces a Horn theory of rings), plus the axiom:

$$\top \vdash_x x = 0 \vee (\exists y)(x \times y = 1)$$

Σ-structure

Definition: Σ-structure

Given a signature Σ and a category $\mathcal{C}$, a **Σ-structure** M **in** $\mathcal{C}$ consists of an interpretation of the components of the signature in $\mathcal{C}$.

Explicitly, a Σ-structure consists of:

- An object $[\![A]\!]_M$ of $\mathcal{C}$ for each sort A,
- A relation $[\![R]\!]_M \hookrightarrow [\![A_1]\!]_M \times \cdots \times [\![A_n]\!]_M$ in $\mathcal{C}$ for each relation symbol $R \rightarrowtail A_1, \ldots, A_n$, and
- A morphism $[\![f]\!]_M : [\![A_1]\!]_M \times \cdots \times [\![A_n]\!]_M \to [\![B]\!]_M$ for each function symbol $f : A_1, \ldots, A_n \to B$.

$\mathbb{T}$-models

As long as $\mathcal{C}$ has sufficient structure, we can recursively extend the data of a Σ-structure to interpretations $[\![\vec{x}.\phi]\!]$ for each formula ϕ over Σ and context $\vec{x}$ (sequence of variables containing all free variables of ϕ). These become subobjects of products of interpretations of sorts.

Definition: $\mathbb{T}$-model

A Σ-structure M in $\mathcal{C}$ is a **model** of a theory $\mathbb{T}$ (in a given fragment of logic) if for each axiom $\phi \vdash_{\vec{x}} \psi$ of $\mathbb{T}$ we have

$$[\![\phi]\!]_M \subseteq [\![\psi]\!]_M$$

as subobjects of the product of interpretations of types of the variables $\vec{x}$.

$\mathbb{T}$-models

For our puroses, it suffices to know that a topos $\mathcal{F}$ has all of the structure required to interpret **geometric logic**[1]. So for each theory $\mathbb{T}$ we have a category $\mathbb{T}$-mod$(\mathcal{F})$:

Objects: $\mathbb{T}$-models in $\mathcal{F}$.

Morphisms: Σ-structure homomorphisms.

[1] In fact, much more can be interpreted in toposes!

Syntactic categories

From a geometric theory $\mathbb{T}$ we can construct a **syntactic category**, $\mathcal{C}_{\mathbb{T}}$:

Objects: formulae in context over the signature, $\{\vec{x}.\phi\}$.

Morphisms: '$\mathbb{T}$-provably functional' formulas.

We can identify $\mathbb{T}$-models in $\mathcal{F}$ with **geometric functors** $\mathcal{C}_{\mathbb{T}} \to \mathcal{F}$, and Σ-structure homomorphisms are natural transformations between such functors.

Syntactic sites

We obtain a **syntactic site** from a syntactic category by defining a Grothendieck coverage $J_{\mathbb{T}}$ whose covering sieves correspond to sequents of the form:

$$\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y}) \psi_i,$$

with the right-hand side constrained according to the fragment of logic under consideration.

We will shortly see how the sheaf topos $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, which we denote $\mathbf{Set}[\mathbb{T}]$, 'classifies' the theory $\mathbb{T}$.

Geometric morphisms

Definition: geometric morphisms

Given toposes $\mathcal{F}$ and $\mathcal{E}$, a **geometric morphism** $f : \mathcal{F} \to \mathcal{E}$ consists of an adjoint pair of functors,

$$\mathcal{F} \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}$$

such that f^* preserves finite limits.

Why this definition?

Example: geometric morphisms from maps

For sober spaces X, Y, there is a correspondence between continuous maps $X \to Y$ and geometric morphisms $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Definition: geometric transformations

Given toposes $\mathcal{F}$ and $\mathcal{E}$, and geometric morphisms $f, g : \mathcal{F} \rightrightarrows \mathcal{E}$, a **geometric transformation** from f to g is simply a natural transformation $\beta : f^* \Rightarrow g^*$ between their *inverse image functors*.

With this, we can form a bicategory $\mathfrak{Top}$:

Objects: Grothendieck toposes.

Morphisms: Geometric morphisms.

2-morphisms: Geometric transformations.

Representable (pseudo)functors

For toposes $\mathcal{F}$, $\mathcal{E}$ the collection of geometric morphisms $\mathrm{Geom}(\mathcal{F}, \mathcal{E})$ is a category. Allowing $\mathcal{F}$ to vary, we obtain a **representable pseudofunctor**

$$\mathrm{Geom}(-, \mathcal{E}) : \mathfrak{Top}^{\mathrm{op}} \to \mathbf{CAT}.$$

Meanwhile, since geometric logic is the fragment which is preserved by *inverse image functors* f^*. That is, for each geometric theory $\mathbb{T}$, we have a pseudofunctor:

$$\mathbb{T}\text{-mod}(-) : \mathfrak{Top}^{\mathrm{op}} \to \mathbf{CAT}.$$

Fundamental Theorem of Classifying Toposes

The pseudofunctor $\mathbb{T}$-mod$(-)$ is represented by the topos $\mathbf{Set}[\mathbb{T}]$.

Section 2

A model as a point

Points as models

We can identify a point x of a space X with a map from the one-point space:

$$\{*\} \xrightarrow{x} X$$

As such, we define a **point** of a topos $\mathcal{E}$ to be a geometric morphism,

$$\mathbf{Set} \simeq \mathrm{Sh}(\{*\}) \longrightarrow \mathcal{E}.$$

Example

By the theorem just given, a **point** of $\mathbf{Set}[\mathbb{T}]$ corresponds to a **model** of $\mathbb{T}$ in $\mathbf{Set}$; these are the focus of model theory!

Recognizing continuous actions

Definition

A geometric morphism h is **hyperconnected** if h^* is full and faithful and its image is closed under subobjects and quotients.

The coreflection $\mathrm{PSh}(L) \to \mathrm{Cont}(L, \rho)$ encountered earlier is an example of a hyperconnected morphism.

Proposition

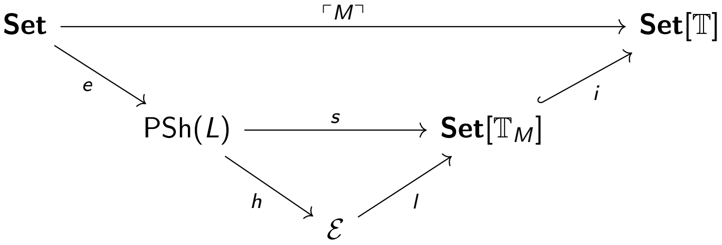
If $\mathcal{E}$ admits a hyperconnected geometric morphism $h : \mathrm{PSh}(L) \to \mathcal{E}$ then $\mathcal{E} \simeq \mathrm{Cont}(N, \tau)$, where N is (the opposite of) the endomorphism monoid of the *point* $\mathbf{Set} \to \mathrm{PSh}(L) \to \mathcal{E}$ of $\mathcal{E}$, and τ is the coarsest topology making sets coming from $\mathcal{E}$ continuous.

Factorizing the point

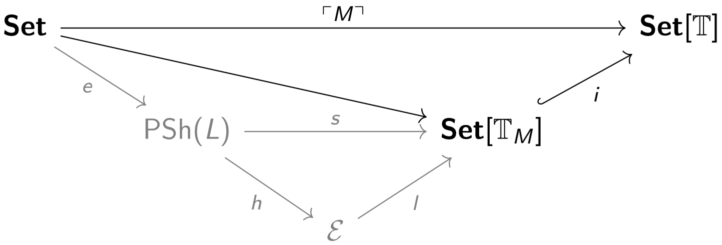
Let's fix a theory $\mathbb{T}$ over a signature Σ and a **Set**-model M of $\mathbb{T}$, and look at the corresponding geometric morphism:

$$\ulcorner M \urcorner : \mathbf{Set} \to \mathbf{Set}[\mathbb{T}]$$

There are various well-known orthogonal factorization systems for geometric morphisms, and a factorization coming from monoid actions, which we shall apply to this morphism. The final picture will look like this:

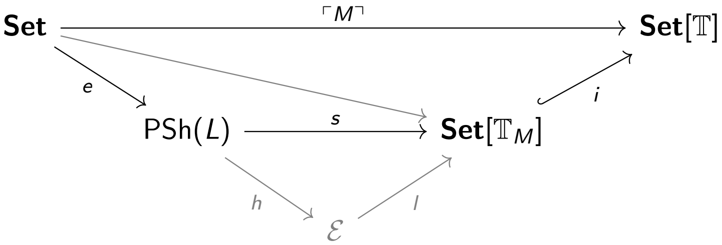


Surjection–inclusion factorization



The **surjection–inclusion** factorization constructs the largest subtopos of **Set**$[\mathbb{T}]$ through which $\ulcorner M \urcorner$ factors. This can be presented as the classifying topos of the **theory of** M, the extension of $\mathbb{T}$ obtained by adding all axioms valid in M.

Endomorphism factorization



Let L be the *opposite* of the monoid of **geometric endomorphisms** of $\ulcorner M \urcorner$, i.e. of natural transformations $\ulcorner M \urcorner^* \Rightarrow \ulcorner M \urcorner$.

We may factorize $\ulcorner M \urcorner$ through $\mathrm{PSh}(L)$. Indeed, for each object X in $\mathbf{Set}[\mathbb{T}]$, $\ulcorner M \urcorner^*(X)$ is a set, and the natural transformations' components at X assemble into a left action of the endomorphism monoid, and hence a right action of L.

Here e is an essential surjection and s is a surjection.

Extracting the monoid

Lemma

Let $(\mathcal{C}, J)$ be any site for $\mathbf{Set}[\mathbb{T}_M]$. Denote by $\int_{\mathcal{C}} s^*$ the category of elements of the restriction of s^* to the representable sheaves. The monoid L, as an object of $\mathrm{PSh}(L)$, is expressible as the limit:

$$L \cong \lim_{(X,x) \in \int_{\mathcal{C}} s^*} s^*(X).$$

We will mostly use the syntactic site, but the statement is intended to indicate that there is nothing special about the site involved.

This doesn't help us calculate, but it lets us prove abstract properties of s.

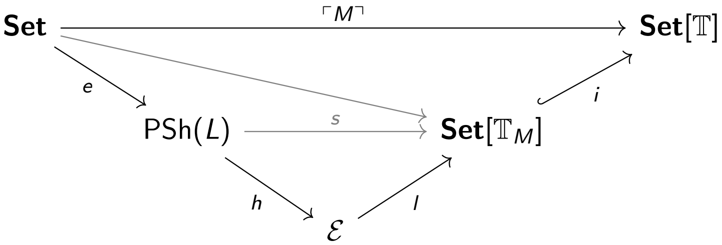
An intrinsic property of s

Lemma

The geometric morphism s is **representably full and faithful** on essential geometric morphisms: for any (Grothendieck) topos $\mathcal{F}$, the following functor is full and faithful.

$$s \circ - : \mathrm{EssGeom}(\mathcal{F}, \mathrm{PSh}(L)) \to \mathrm{Geom}(\mathcal{F}, \mathbf{Set}[\mathbb{T}_M])$$

Hyperconnected–localic factorization



Consider the **hyperconnected–localic** factorization of s. On one hand, this defines a constructs the largest subtopos of **Set**$[\mathbb{T}]$ through which $\ulcorner M \urcorner$ factors. This can be presented as the classifying topos of the **theory of** M, the extension of $\mathbb{T}$ obtained by adding all axioms valid in M.

(De)compositionality

Corollary

The geometric morphism h is **representably full and faithful** on essential geometric morphisms.

Proof: Suppose we are given essential geometric morphisms $x, y : \mathcal{F} \rightrightarrows \mathrm{PSh}(L)$. Consider the maps:

$$\mathrm{Hom}(x, y) \xrightarrow{h\circ -} \mathrm{Hom}(h \circ x, h \circ y) \xrightarrow{l\circ -} \mathrm{Hom}(s \circ x, s \circ y).$$

By the previous Lemma, the composite map (corresponding to composition with s) is a bijection, so the right-hand map must be surjective. Meanwhile, localic geometric morphisms are representably faithful, so the right-hand map is also injective. It follows that both functions are bijections.

Definitions

Let $\mathbb{T}$ be a geometric theory over a signature Σ, M a **Set**-model of $\mathbb{T}$ and L the monoid of Σ-structure endomorphisms of M.

Definition: equivariant theory

The **equivariant theory of** M is the theory obtained from $\mathbb{T}$ by adding a relation symbol $R \rightarrowtail A_1, \dots, A_n$ for each L-equivariant relation $R \hookrightarrow [\![A_1]\!]_M \times \dots [\![A_n]\!]_M$ and all axioms relating these and geometric formulas over Σ to one another which are valid in M. This is a localic extension of the theory of M, $\mathbb{T}_M$, introduced earlier. We denote it by $\mathbb{T}_{\looparrowright M}$, FWOBN.

Definitions

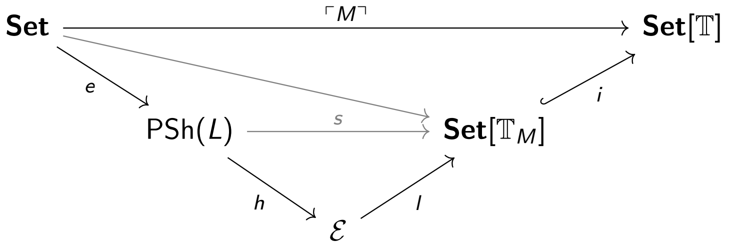
Let $\mathbb{T}$ be a geometric theory over a signature Σ, M a **Set**-model of $\mathbb{T}$ and L the monoid of Σ-structure endomorphisms of M.

Definition: pointwise convergence

The **pointwise convergence topology** on L induced by M has as basis of neighbourhoods of an element $m \in L$ the sets

$$U_{x_1,\dots,x_k}(m) = \{m' \in L \mid m'(x_1) = m(x_1), \dots, m'(x_k) = m(x_k)\},$$

where k varies over the natural numbers and $x_i \in [\![A_i]\!]_M$ for sorts $A_1, \dots, A_k$ in Σ. We denote this topology by ρ.



Theorem

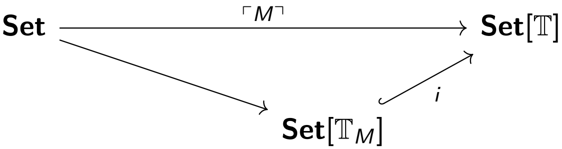
$$\mathrm{Cont}(L, \rho) \simeq \mathcal{E} \simeq \mathbf{Set}[\mathbb{T}_{\hookrightarrow M}].$$

This result employs the Corollary to conclude that the endomorphism monoid of $h \circ e$ is isomorphic to L.

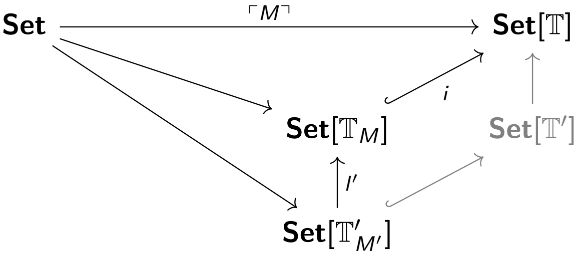
Section 3

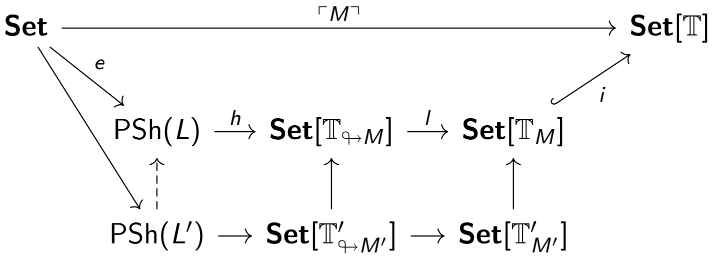
Relational extensions

Adding relations...



Consider the above fragment of the diagram. If we add some relations on M without adding any sorts and add to $\mathbb{T}_M$ all of the axioms valid for the resulting structure M', we get a localic surjection,

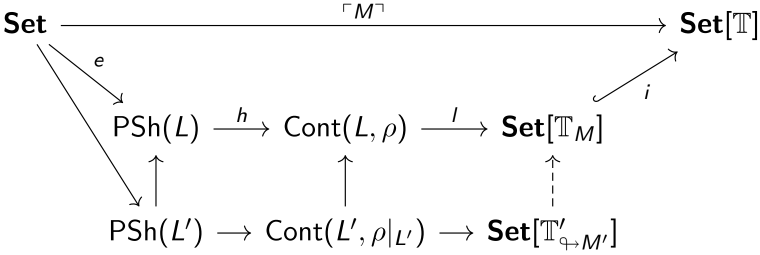




After applying the factorization to the extended model, we easily see that this defines a submonoid L' of L. In fact, this is a *closed* submonoid of (L, ρ), *precisely because the middle morphism is localic*[2].

[2] The relationship is not contained in these slides, ask for more detail if interested.

Choosing a submonoid produces relations



Conversely, given a submonoid L' of L, we can expand $\mathbb{T}_M$ with the class of relations on M which are L'-equivariant (and all of the valid axioms).

Main theorem part 2

Let us return to the theorem statement from the introduction.

Theorem

Let $\mathbb{T}, \Sigma, M, L$ be as previously. There is a Galois connection between:
$\{$Submonoids of $L\}$ and $\{$Families of (finitary) relations on $M\}$.

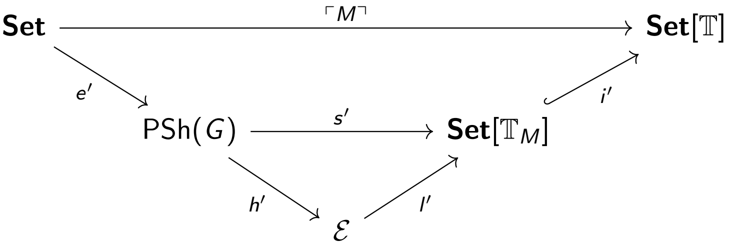
We now know the origin of the *pointwise convergence topology* for which the fixed elements on the left are precisely the **closed** submonoids. We could call the fixed families on the right **closed**, too.

Fin

Thanks for listening!

Any questions?

What happens for groups?



Let $\mathbb{T}, \Sigma, M$ be as before, but now let G be the opposite of $\mathrm{Aut}(M)$.

We expect that s' here is now **representably faithful** and **representably full on isomorphisms** for essential geometric morphisms.

We expect the relational extensions corresponding to closed subgroups to be the *decidable* ones: families of relations closed under negation/complementation.