Completeness for arbitrary finite dimensions of ZXW-calculus

arXiv:2302.12135

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SYCO 11

21st April 2023

QUANTINUUM

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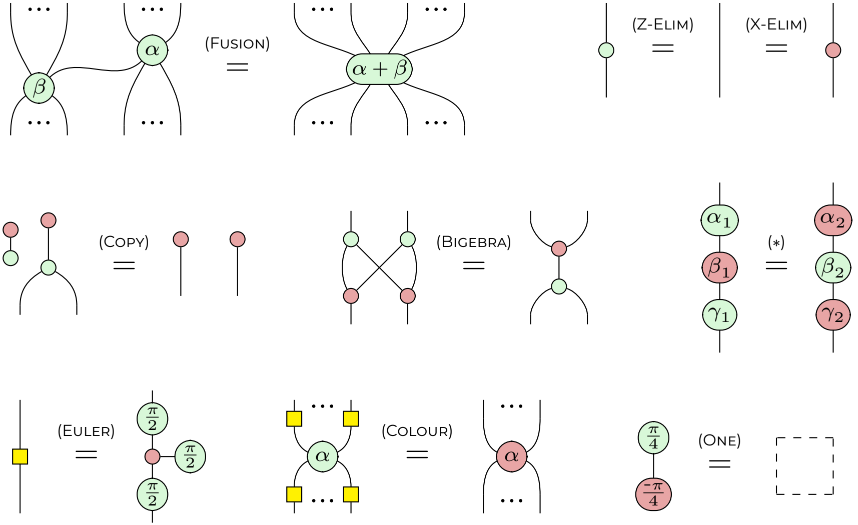
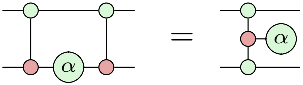
The ZX-calculus for qubits



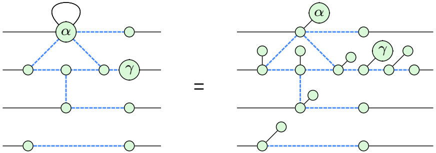
Vilmart, 2019

ZX-calculus

Quantum Circuit Optimisation



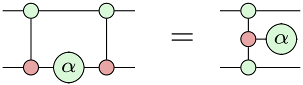
Measurement-Based Quantum Computing



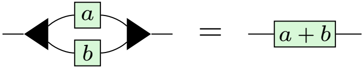
ZX-calculus

ZW-calculus

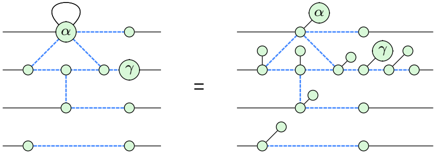
Quantum Circuit Optimisation



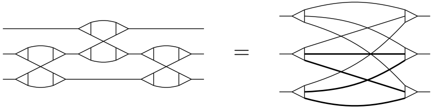
Summation



Measurement-Based Quantum Computing

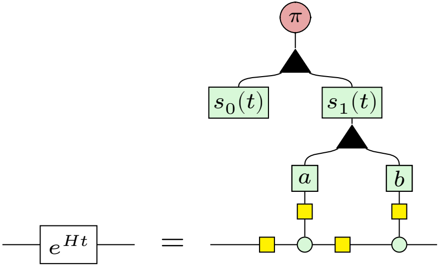


Linear Optical Quantum Computing

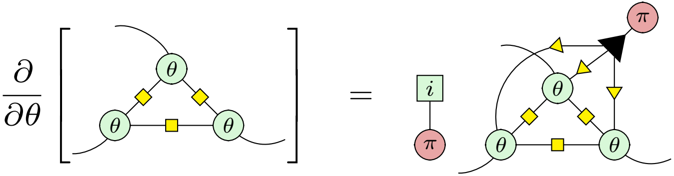


ZXW-calculus

Hamiltonian exponentiation (Shaikh et al., 2022)



Differentiation and integration (Wang et al., 2022)



What are Qudits?

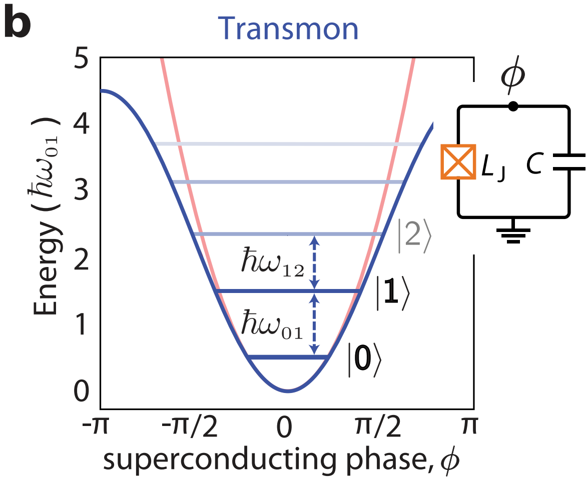
Qubits:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Qudits:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle + \cdots + a_{d-1} |d-1\rangle$$

Physical Realisation of Qudits



Kjaergaard et al., 2020

What is completeness?

A graphical calculus is complete if for any two diagrams D_1 and D_2, we can derive $D_1 = D_2$ from the rules of the calculus, given that the interpretation of D_1 and D_2 equal.

Why is completeness important?

- Everything can be shown.
- No ‘missing rules’.
- Useful structures.

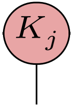
Previous completeness results

	Qubit ($d = 2$)	Qutrit ($d = 3$)	Qupit (d is prime)	Qudit ($d \in \mathbb{N}$)
Clifford	Backens, 2014	Wang, 2018	Booth and Carette, 2022	
Clifford + T	Jeandel et al., 2018			
Universal	Hadzihasanovic, 2015			This work!

The qudit ZXW-calculus

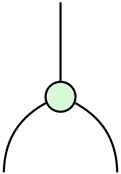
Standard basis in qudit ZXW

For $0 \leq j < d$,



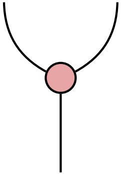
$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |d-j\rangle$$

Generator: Z spider



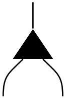
$$\xrightarrow{[\![\cdot]\!]} \quad |k\rangle \mapsto |k,\ k\rangle$$

Generator: X spider



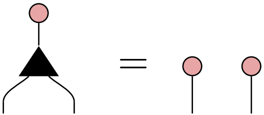
$$\xrightarrow{[\![\cdot]\!]} \quad |i,\ j\rangle \mapsto |i \oplus_d j\rangle$$

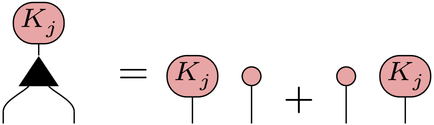
Generator: W node



$$\xrightarrow{\llbracket\cdot\rrbracket}\quad |00\rangle\langle 0| + \sum_{i=1}^{d-1}\left(|i0\rangle + |0i\rangle\right)\langle i|$$

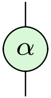
That is:





Understanding the Z box

Z spider:



$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \qquad \text{where } \alpha \in \mathbb{R}.$$

Z box:

$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \qquad \text{where } a \in \mathbb{C}.$$

Understanding the qudit Z box

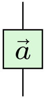
Qubit Z box: for $a \in \mathbb{C}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Qudit Z box: for $\vec{a} = (a_1, a_2, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix}$$

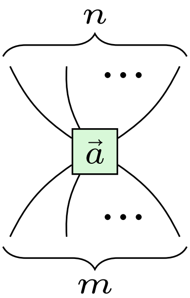
Compact description of the Z box



$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix} = \sum_{j=0}^{d-1} a_j \left|j\right\rangle \left\langle j\right| ,$$

where $\vec{a} = (a_1, a_2, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$ and $a_0 := 1$.

Generator: Z box



$$\xrightarrow{[\![\cdot]\!]} \quad \sum_{j=0}^{d-1} a_j \, |j\rangle^{\otimes m} \, \langle j|^{\otimes n} \, ,$$

where $\vec{a} = (a_1, \cdots, a_{d-1}) \in \mathbb{C}$

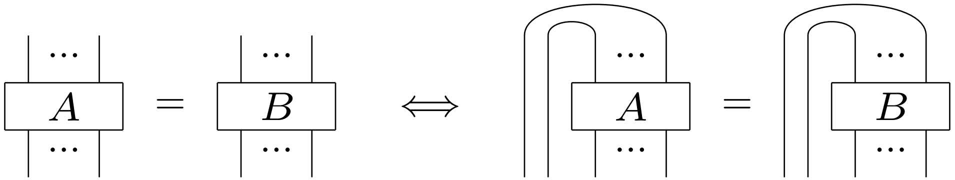
and $a_0 := 1$.

What did we prove?

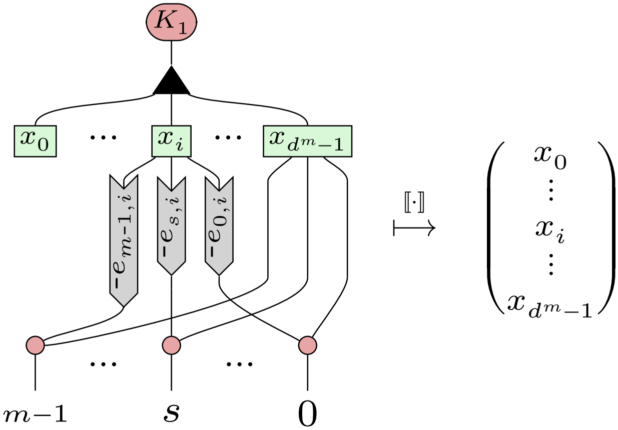
Theorem

The ZXW-calculus is universally complete for all finite dimensions.

Map-state duality

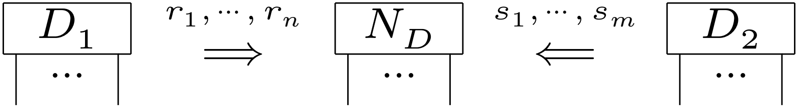


A Normal Form

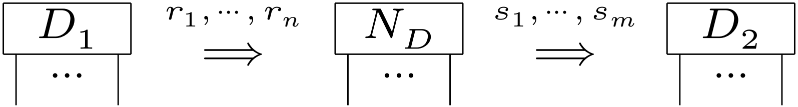


Completeness using a normal form

If D_1 and D_2 are state diagrams such that $[\![D_1]\!] = [\![D_2]\!]$, then:

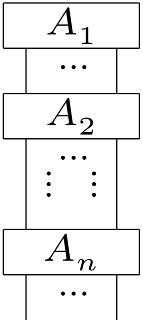


So:

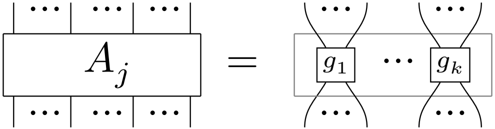


Note: Structure of states

Each state diagram has the following structure:

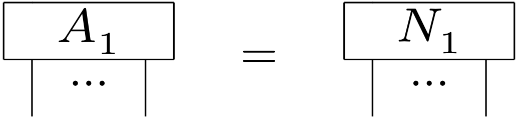


with

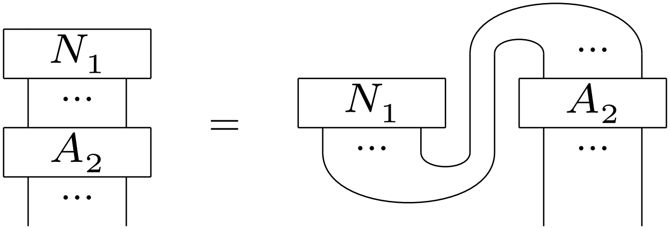


where $g_1, \cdots, g_k$ are generators.

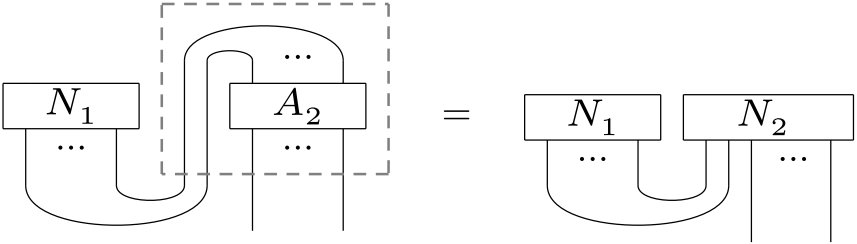
State ⟹ normal form I.



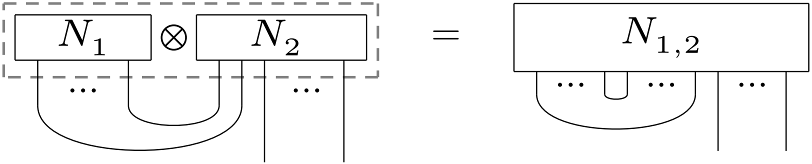
State $\Longrightarrow$ normal form II.



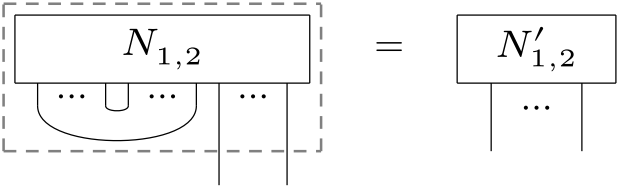
State $\Longrightarrow$ normal form III.



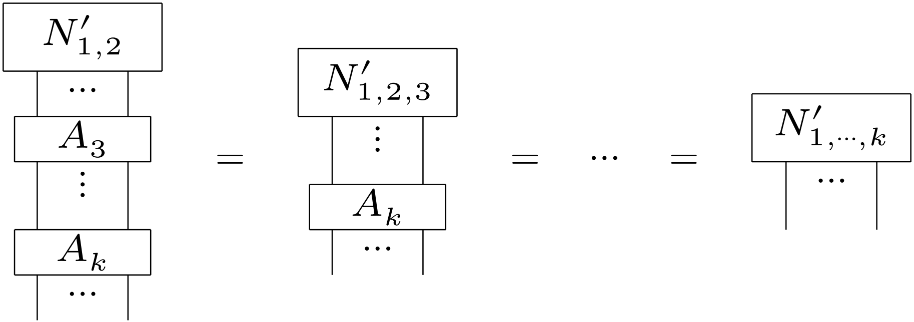
State $\Longrightarrow$ normal form IV.



State $\Longrightarrow$ normal form V.



State $\Longrightarrow$ normal form VI.



Summary: state $\Longrightarrow$ normal form

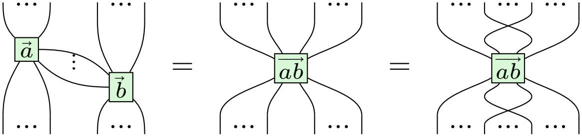
We need to rewrite the following into their normal form:

- Generators
- Tensor product of two normal forms
- Partial-traced normal form

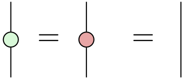
ZXW-calculus is more than just the sum of its parts

1. Rules of ZX
2. Rules of ZW
3. Rules of ZXW

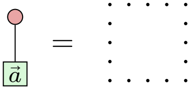
The ZX-part of the rules I



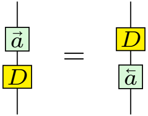
(S1)



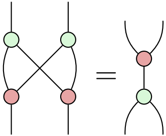
(S2)



(Ept)



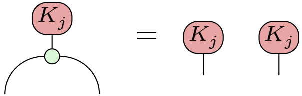
(D1)



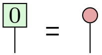
(B2)

where $\overleftarrow{a} = (a_{d-1}, \ldots, a_1), \quad \overrightarrow{ab} = (a_1 b_1, \ldots, a_{d-1} b_{d-1}).$

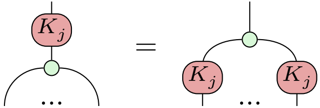
The ZX-part of the rules II



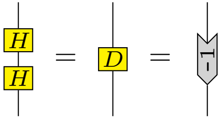
(K0)



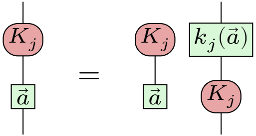
(Zer)



(K1)



(P1)

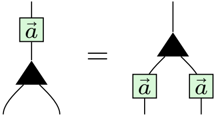


(K2)

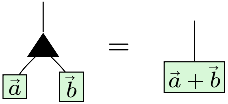
where $k_j(\vec{a}) = \left(\frac{a_{1-j}}{a_{d-j}}, \dots, \frac{a_{d-1-j}}{a_{d-j}}\right)$

Definition of multipliers

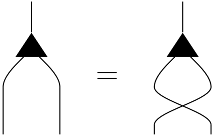
The ZW-part of the rules



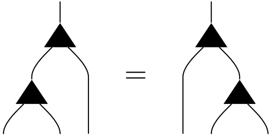
(Pcy)



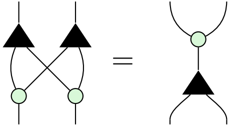
(AD)



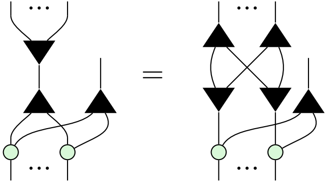
(Sym)



(Aso)

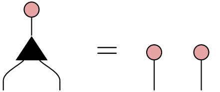


(BZW)

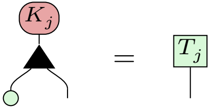


(WW)

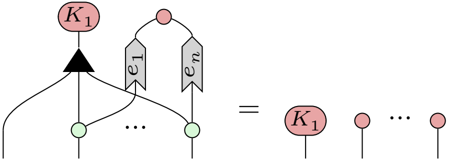
The ZXW-part of the rules I



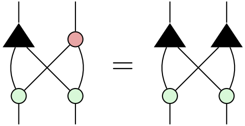
(Bs0)



(Bsj)



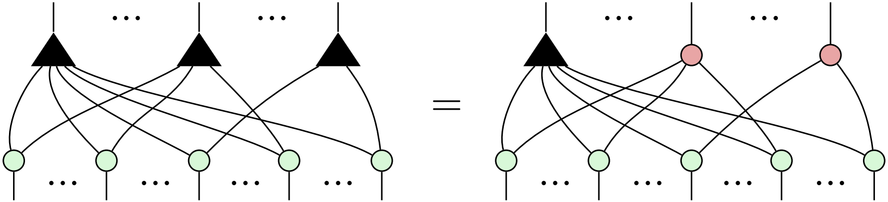
(KZ)



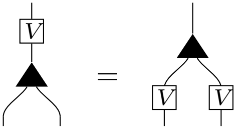
(TA)

where $T_j = (\overbrace{\underbrace{0, \dots, 1}_{d-j}, \dots, 0}^{d-1}), \quad e_1, \dots, e_n \in \{1, \dots, d-1\}.$

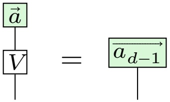
The trialgebra rule



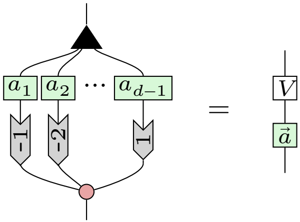
The ZXW-part of the rules II



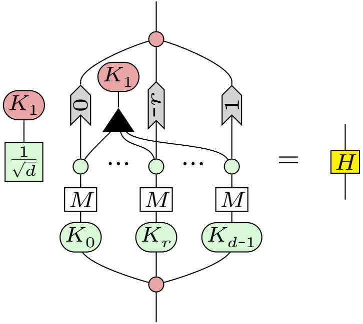
(VW)



(ZV)



(VA)



(HD)

where $\overrightarrow{a_{d-1}} = (a_{d-1}, a_{d-1}, \ldots, a_{d-1})$.

Definition of V and M

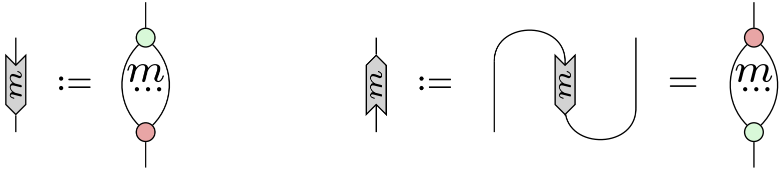
Outlook

- Speedy evaluation of ZXW-diagrams
- Prove completeness of qufinite ZXW-calculus
- More applications for photonics using ZXW
- Analyse the circuit extraction of ZXW-diagrams

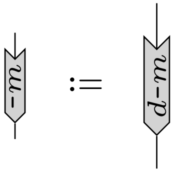
Appendix

3 Notaions

Useful notation: The multiplier

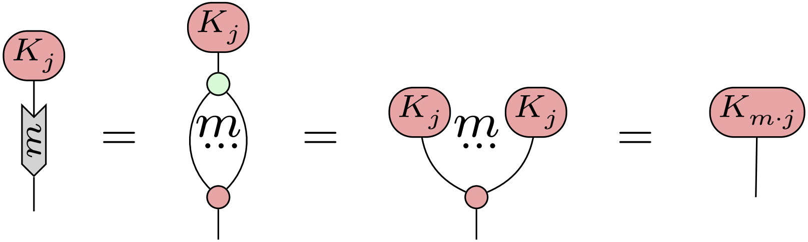


m can be labeled modulo d due to the Hopf law.



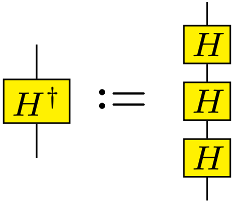
Back

Example: The multiplier



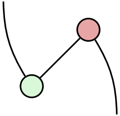
Back

Notation: The Hadamard inverse



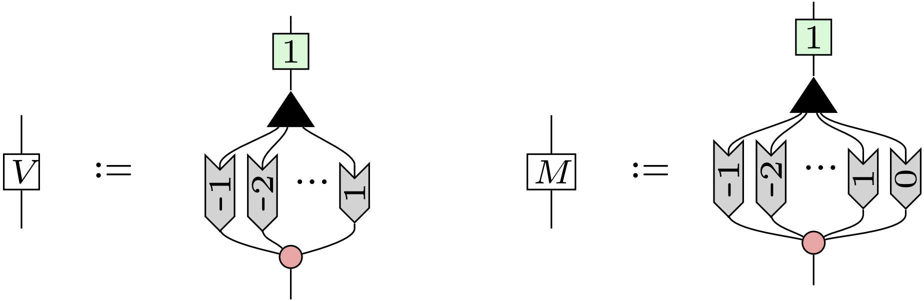
Notation: The dualiser

$:=$



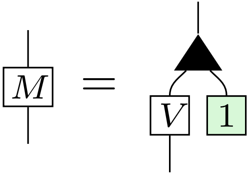
$$\xmapsto{[\![\cdot]\!]} \quad \sum_{i=0}^{d-1} |i\rangle \langle -i| \, .$$

Notation: The V and M boxes



with

$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |0\rangle\langle 0| + \sum_{i=1}^{d-1} |i\rangle\langle -1|$$



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