Compositionality of Effects in Semantics and Automata Theory

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Overview

- Motivation and context
- Monads and weak distributive laws
- Automata with effects
- The weak distributive law for combining probabilistic choice and non-determinism
- “Determinization” of automata via (weak) distributive laws
- Semialgebras and why weak laws are strong ...

Motivation and context

Composing computational effects

A computational effect is an interaction between a program and its environment.

Examples: error raising, input and output, global/local state, continuations, non-determinism and probabilistic choice.

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The approach in this talk:

1. model computational effects following the seminal work of Moggi using monads
2. consider automata with "effects"
3. consider an adapted category-theoretic tool for composing monads.

Semantic models for non-determinism and probabilistic choice

nondeterministic choice

$p \vee q$

a commutative, idempotent and associative operation

probabilistic choice

$p +_r q$

satisfying the axioms of a barycentric algebra

e.g. $p +_r q = q +_{1-r} p$.

How do we combine the two ?

Combining probabilistic and non-deterministic choice has a long history ...

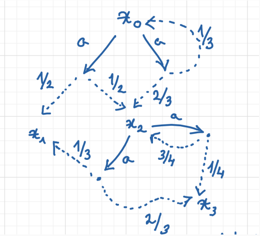
- [Jones and Plotkin]
  A probabilistic powerdomain of evaluations, LICS, 1989
- [Jung and Tix]
  The troublesome probabilistic powerdomain, ENTCS, 1998
- [Tix, Keimel, Plotkin]
  Semantic Domains for Combining Probability and Non- Determinism, ENTCS 2009
- [Mislove]
  Nondeterminism and probabilistic choice: Obeying the law, CONCUR 2000
- [Keimel, Plotkin]
  Mixed powerdomains for probability and nondeterminism, LMCS 2017
- [J. Goubault-Larrecq]
  A probabilistic and non-deterministic call-by-push-value language, LICS, 2019

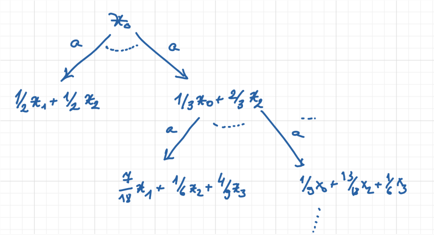
Combining probabilistic and non-deterministic choice has a long history ...

There are various approaches proposed in these studies of combinations of ordinary and probabilistic non-determinism

- power-cone models,
- prevision models
- indexed valuations
- coproducts of monads

“Determinizing” probabilistic automata yields nondeterministic automata whose states are probability distributions, i.e., belief-state transformers.





[Bonchi, Silva, Sokolova]
The Power of Convex Algebras, CONCUR 2017

[Bonchi, Sokolova, Vignudelli]
The Theory of Traces for Systems with Nondeterminism and Probability, LICS 2019

Monads

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A category-theoretic notion generalizing algebraic theories presented by operations and equations from universal algebra

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A monad (S, η^S, μ^S) consists of a functor $S : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta^S : 1_{\mathcal{C}} \Rightarrow S$, $\mu^S : S^2 \Rightarrow S$ subject to coherence axioms.

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Intuition in Set :

SX is the set of terms with variables in X for some algebraic theory

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The powerset monad $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}})$ consists of

- the powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$
  - for a set X we have $\mathcal{P}X = \{A \mid A \subseteq X\}$
  - for a function $f\colon X \to Y$, $\mathcal{P}f$ is defined as direct image
- the unit $\eta^{\mathcal{P}}_X\colon X \to \mathcal{P}X$ mapping x to the singleton $\{x\}$.

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For probabilistic choice we use the finite distribution monad, which corresponds to the algebraic theory of barycentric algebras $(X, (+_r)_{r\in[0,1]})$.

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The finite distribution monad $(\mathcal{D}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ consists of

- the finite distribution functor $\mathcal{D}\colon \mathsf{Set} \to \mathsf{Set}$ given by
  - $X \mapsto \{\varphi\colon X \to [0,1] \mid supp(\varphi) \text{ finite and } \sum_{x\in X} \varphi(x) = 1\}$
  - for a function $f\colon X \to Y$ we have $\mathcal{D}f$ is defined by $\mathcal{D}f(\varphi)(y) = \sum_{y\in f^{-1}(x)} \varphi(x)$
- the unit $\eta^{\mathcal{D}}_X\colon X \to \mathcal{D}X$ mapping x to the Dirac distribution δ_x.

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satisfying the usual axioms.

Automata with effects

Word automata in Kleisli categories

deterministic automata $\quad 1 \longrightarrow Q \overset{a}{\circlearrowleft} \longrightarrow 2 \quad$ in Set

Word automata in Kleisli categories

$$\text{deterministic automata} \qquad \begin{tikzcd} 1 \arrow[r] & Q \arrow[r] \arrow[loop above, "a"] & 2 \end{tikzcd} \qquad \text{in Set}$$

$$\text{non-deterministic automata} \qquad \begin{tikzcd} 1 \arrow[r] & Q \arrow[r] \arrow[loop above, "a"] & 1 \end{tikzcd} \qquad \text{in Rel}$$

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Subseq. transducers $$\begin{tikzcd} 1 \arrow[r] & Q \arrow[r] \arrow[loop above, "a"] & 1 \end{tikzcd}$$ in $\mathrm{Kl}(\mathcal{T})$

Determinization as a right adjoint

We use the lifting of the Kleisli adjunction for the monad $\mathcal{T}$.

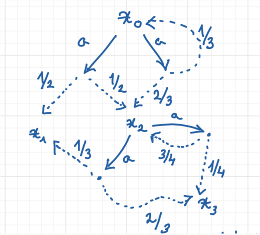
$$
\begin{tikzcd}
\mathrm{Auto}(\mathcal{L}_{\mathcal{C}}) \arrow[rr, bend left] \arrow[d, "\mathrm{States}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Auto}(\mathcal{L}_{\mathrm{Kl}(\mathcal{T})}) \arrow[ll, bend left] \arrow[d, "\mathrm{States}"] \\
\mathcal{C} \arrow[rr, bend left, "F_{\mathcal{T}}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Kl}(\mathcal{T}) \arrow[ll, bend left, "U_{\mathcal{T}}"]
\end{tikzcd}
$$

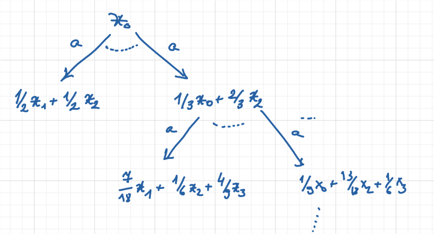
Determinization as a right adjoint

When $\mathcal{T}$ is the powerset monad, the lifting of $U_{\mathcal{T}}$ is the determinization of a non-deterministic automaton.

$$\begin{tikzcd}
\mathrm{Auto}(\mathcal{L}_{\mathrm{Set}}) \arrow[rr, bend left] \arrow[d, "\mathrm{States}" description] \arrow[rr, phantom, "\perp"] & & \mathrm{Auto}(\mathcal{L}_{\mathrm{Rel}}) \arrow[ll, bend left] \arrow[d, "\mathrm{States}" description] \\
\mathrm{Set} \arrow[rr, bend left, "F_{\mathcal{T}}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Rel} \arrow[ll, bend left, "U_{\mathcal{T}}"]
\end{tikzcd}$$

Do probabilistic automata fit in this framework?





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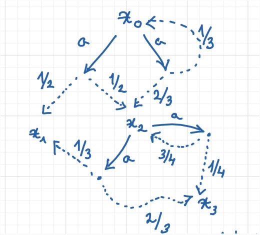

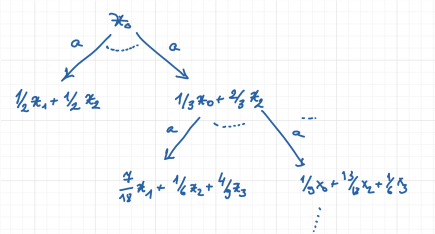

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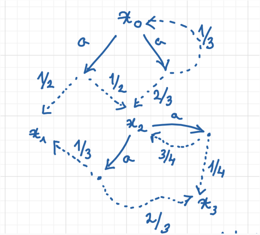

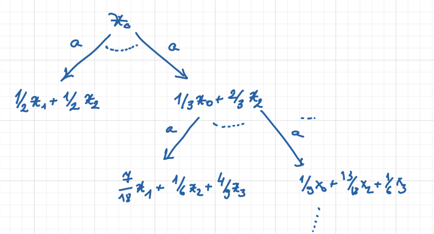

We would like to say something like

probabilistic automata are functorial automata in $Kl(\mathcal{PD})$
"determinization" is the lifting of a functor from $Kl(\mathcal{PD})$ to $Kl(P)$
but it doesn't work that nicely...

Composing monads

How can we combine monads ?

Suppose we have two monads (T, η^T, μ^T) and (S, η^S, μ^S).

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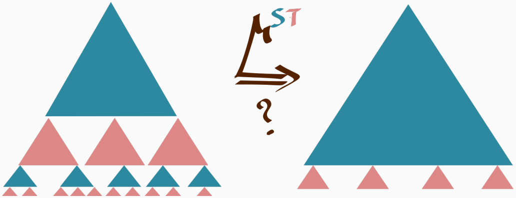

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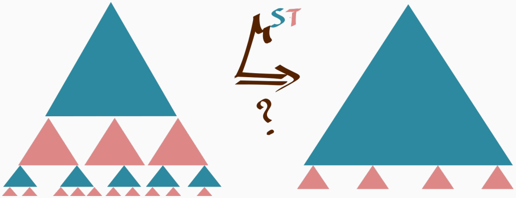

It would be nice to have a way of swapping S and T

We need a natural transformation $\gamma\colon TS \Rightarrow ST$ subject to 4 coherence conditions: compatibility with the units and the multiplications of the monads

The rich theory of distributive laws

$$
\begin{tikzcd}
 & \begin{array}{c}\text{distributive laws}\\ \gamma\colon TS \Rightarrow ST\end{array} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] \arrow[dd, Rightarrow] & \\
\begin{array}{c}\text{extensions } \hat{T}\\ \text{of } T \text{ to } Kl(S)\end{array} & & \begin{array}{c}\text{liftings } \hat{S}\\ \text{of S to } EM(T)\end{array} \\
 & \text{monad structures on } ST &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & EM(\hat{S}) \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
EM(ST) & & \gamma\text{-algebras}
\end{tikzcd}
$$

The rich theory of distributive laws

$$\begin{tikzcd} Kl(S) \arrow[r, "\hat{T}"] & Kl(S) \\ C \arrow[u, "F"] \arrow[r, "T"'] & C \arrow[u, "F"'] \end{tikzcd}$$

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$$\begin{tikzcd} EM(T) \arrow[r, "\hat{S}"] \arrow[d, "U"'] & EM(T) \arrow[d, "U"] \\ C \arrow[r, "S"'] & C \end{tikzcd}$$

distributive laws

$\gamma \colon TS \Rightarrow ST$

$\Leftrightarrow$ extensions $\hat{T}$ of T to $Kl(S)$

$\Leftrightarrow$ liftings $\hat{S}$ of S to $EM(T)$

$\Downarrow$

monad structures on ST

$$EM(ST) \Leftrightarrow EM(\hat{S}) \Leftrightarrow \gamma\text{-algebras}$$

The rich theory of distributive laws

$$\begin{tikzcd} Kl(S) \arrow[r, "\hat{T}"] & Kl(S) \\ C \arrow[u, "F"] \arrow[r, "T"'] & C \arrow[u, "F"'] \end{tikzcd}$$

extensions $\hat{T}$
of T to $Kl(S)$

$\Longleftrightarrow$

distributive laws
$\gamma \colon TS \Rightarrow ST$

$\Longleftrightarrow$

$$\begin{tikzcd} EM(T) \arrow[r, "\hat{S}"] \arrow[d, "U"'] & EM(T) \arrow[d, "U"] \\ C \arrow[r, "S"'] & C \end{tikzcd}$$

liftings $\hat{S}$
of S to $EM(T)$

$\Downarrow$

monad structures on ST

$EM(ST) \Longleftrightarrow EM(\hat{S}) \Longleftrightarrow$ γ-algebras

$$\begin{tikzcd} TSX \arrow[r, "\gamma"] \arrow[d, "Tb"'] & STX \arrow[d, "Sa"] \\ TX \arrow[dr, "a"'] & SX \arrow[d, "b"] \\ & X \end{tikzcd}$$

Furthermore, given a distributive law $TS \Rightarrow ST$, we have that $Kl(ST)$ is isomorphic to $Kl(\hat{T})$, hence we obtain adjunctions

$$\begin{tikzcd}
\mathrm{Auto}(\mathcal{L}_{Kl(S)}) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] \arrow[d, "\mathrm{States}"] & & \mathrm{Auto}(\mathcal{L}_{\mathrm{Kl}(ST)}) \arrow[ll, bend left] \arrow[d, "\mathrm{States}"] \\
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The rich theory of distributive laws

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Kl(S) \arrow[rr, bend left, "F_{\hat{T}}"] & \perp & \mathrm{Kl}(ST) \arrow[ll, bend left, "U_{\hat{T}}"]
\end{tikzcd}
$$

So, for $S = \mathcal{P}$ and $T = \mathcal{D}$ ...

There is no distributive law of $\mathcal{P}$ over $\mathcal{D}$

Plotkin's counterexample from Daniele Varacca's PhD thesis:

thus obtaining another distributive law. However, it turns out that there is no distributive law at all between the two monads. If (P, η^P, μ^P) is the finite nonempty powerset monad, and (V, η^V, μ^V) is the finite valuation monad in the category **SET**, we have

Proposition 3.1.2. *There is no distributive law of V over P.*

Proof: The idea for this proof is due to Gordon Plotkin. Assume that $d : VP \stackrel{\cdot}{\longrightarrow} PV$ is a distributive law. Consider the set $X := \{a, b, c, d\}$. Take $\Xi := \frac{1}{2}\eta_{\{a,b\}} + \frac{1}{2}\eta_{\{c,d\}} \in VP(X)$. We try to find out what $R := d_X(\Xi)$ is.

Let $Y := \{a, b\}$. Consider:

$$f : X \to Y \quad f : \begin{cases} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & a \\ d & \mapsto & b \end{cases}$$

$$f' : X \to Y \quad f' : \begin{cases} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & b \\ d & \mapsto & a\,. \end{cases}$$

Missing category theoretic understanding

In Part II we study the notion of indexed valuation, as a denotational model for probabilistic computation. This model arises from the need of combining probabilities and nondeterminism. The probabilistic powerdomain and the non-deterministic powerdomain do not combine nicely. In technical terms, there is no distributive law between the two monads. We face this mathematical problem discovering where the core of the problem lies and we propose our solution which amounts to a modification of the probabilistic powerdomain. First, we

Missing category theoretic understanding

on the underlying spaces. When specialised to domains and subprobability valuations, his results correspond to our Corollaries 4.4, 4.7, and 4.10. He worked directly with the valuation spaces rather than, as we do, making use of abstract structures such as cones and barycentric algebras. In [2, 3] Beaulieu worked algebraically; his results include free constructions of algebras satisfying the above laws over sets and partial orders, but not domains.

There has been some discussion of other ways to combine nondeterminism with probability. Categorical distributive laws provide a standard means of showing the composition of two monads form a third (see [37]). However there is no such law enabling one to compose the monad of ordinary nondeterminism with that of probabilistic nondeterminism — see the Appendix of [63]. For this reason, Varacca and Winskel [61, 62, 63] reject, or weaken, one of the axioms of (extended) probabilistic nondeterminism, viz. the distributivity law $(r + s)x = rx + sx$, and consider certain 'indexed valuations' in place of the more usual ones. As shown by Varacca in [62, Chapter 4] this approach applies to domains, where indexed valuations come in three flavours: Hoare, Smyth, and …

In Part II we s… , as a denotational model for probabilistic co… …ses from the need of combining probabilities and no… probabilistic powerdomain and the non-deterministic powerd… not combine nicely. In technical terms, there is no distributive law be…een the two monads. We face this mathematical problem discovering where the core of the problem lies and we propose our solution which amounts to a modification of the probabilistic powerdomain. First, we

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There has been so... of other ways to combine nondeterminism with probability. Categorical distribut... means of showing the composition of two monads form a third (see ... law enabling one to compose the monad of ordinary nondeter...minism — see the Appendix of [63]. For this reason, V... of the axioms of (extended) probabilistic no... and consider certain 'indexed valuat... approach app...

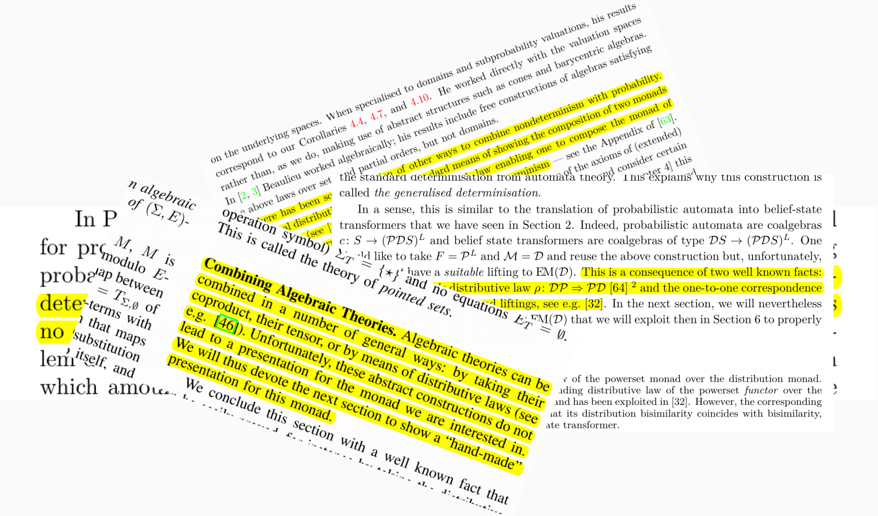
In Part II we s... for probabilistic co... probabilities and no... deterministic powerd... no distributive law be...een t... lem discovering where the co... which amounts to a modifica...

the standard determinisation from automata theory. This explains why this construction is called *the generalised determinisation*.

In a sense, this is similar to the translation of probabilistic automata into belief-state transformers that we have seen in Section 2. Indeed, probabilistic automata are coalgebras $c\colon S \to (\mathcal{P}\mathcal{D}S)^L$ and belief state transformers are coalgebras of type $\mathcal{D}S \to (\mathcal{P}\mathcal{D}S)^L$. One would like to take $F = \mathcal{P}^L$ and $\mathcal{M} = \mathcal{D}$ and reuse the above construction but, unfortunately, $\mathcal{P}^L$ does *not* have a *suitable* lifting to $\mathrm{EM}(\mathcal{D})$. This is a consequence of two well known facts: the lack of a *suitable* distributive law $\rho\colon \mathcal{D}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}$ [64] [2] and the one-to-one correspondence between distributive laws and liftings, see e.g. [32]. In the next section, we will nevertheless provide a "powerset-like" functor on $\mathrm{EM}(\mathcal{D})$ that we will exploit then in Section 6 to properly model PA as belief-state transformers.

[2] As shown in [64], there is no distributive law of the powerset monad over the distribution monad. Note that a "trivial" lifting and a corresponding distributive law of the powerset *functor* over the distribution monad exists, it is based on [11] and has been exploited in [32]. However, the corresponding "determinisation" is trivial, in the sense that its distribution bisimilarity coincides with bisimilarity, and it does not correspond to the belief-state transformer.

Missing category theoretic understanding



More negative results

[Klin, Salamanca]
Iterated Covariant Powerset is not a Monad

- there is no distributive law of the monad $\mathcal{P}$ over itself
- there is no monad structure on $\mathcal{PP}$
- there is no distributive law $T\mathcal{P} \Rightarrow \mathcal{P}T$, when T satisfies some further conditions.

[Zwart,Marsden]
Don't try this at home: No-Go Theorems for Distributive Laws, LICS 2019

- generalized Plotkin's theorem
- a fine analysis of non-existence of distributive laws

Workarounds

- use instead of $\mathcal{D}$ the monad of indexed valuations (Varacca's solution)
- define by hand a monad $\mathcal{P}_c$ on the category of Eilenberg-Moore algebras for $\mathcal{D}$ – going back to Tix et al, more recently exploited in

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The Power of Convex Algebras, CONCUR 2017

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  The Power of Convex Algebras, CONCUR 2017
- But these constructions remain a bit mysterious from a category-theoretic perspective. Are they canonical in some sense?

How far away are we from a distributive law?

- Can we at least obtain a natural transformation $\mathcal{DP} \Rightarrow \mathcal{PD}$?
- If yes, which axioms does it satisfy ?
- It turns out that we do have such a natural transformation, satisfying all but the axiom involving the unit of $\mathcal{D}$.
- This is a so called weak distributive law in the sense of [Garner, 2019].
- Garner exhibited a weak distributive law between $\mathcal{P}$ and the ultrafilter monad β and showed how the Vietoris monad on compact Hausdorff spaces can be seen as a weak lifting of $\mathcal{P}$.

The rich theory of weak distributive laws (Garner, 2019)

weak distributive laws
$\gamma\colon TS \Rightarrow ST$

weak extensions $\hat{T}$ of T to $Kl(S)$ $\Leftrightarrow$ weak distributive laws $\Leftrightarrow$ weak liftings $\hat{S}$ of S to $EM(T)$

weak distributive laws $\Downarrow$ monad structures $\widehat{ST}$

$EM(\widehat{ST}) \Leftrightarrow EM(\hat{S}) \Leftrightarrow$ γ-algebras

The rich theory of weak distributive laws (Garner, 2019)

$$\begin{tikzcd} Kl(S) \arrow[r, "\hat{T}"] & Kl(S) \\ C \arrow[u, "F"] \arrow[r, "T"'] & C \arrow[u, "F"'] \end{tikzcd}$$

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ι π

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$$\begin{tikzcd} TSX \arrow[r, "\gamma"] \arrow[d, "Tb"'] & STX \arrow[d, "Sa"] \\ TX \arrow[dr, "a"'] & SX \arrow[d, "b"] \\ & X \end{tikzcd}$$

γ-algebras

The complicated theory of weak distributive laws

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There are semi-functors between $Kl(\hat{T})$ and $Kl(\widehat{ST})$, but they do not give a semi-adjunction.

There are “obvious” functors between $Kl(\widehat{ST})$ and $Kl(S)$, but they do not give an adjunction.

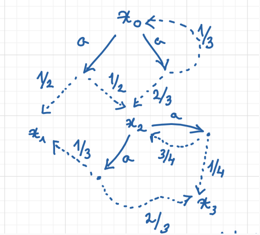
Combining nondeterminism and probabilistic choice via weak laws

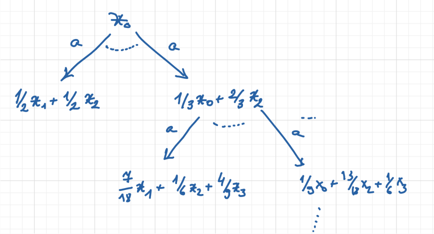
Theorem (Goy, P., LICS 2020)
There exists a weak distributive law of the powerset monad over the finite distribution monad. The corresponding weak lifting of the powerset monad to the category of convex algebras is the convex powerset monad.

- we rely on results of Barr for relational extensions of functors and natural transformations
- Rel is the Kleisli category of $\mathcal{P}$
- the functor $\mathcal{D}$ preserves weak pullbacks, hence it can be extened to Rel
- the unit of $\mathcal{D}$ is not weakly cartesian
- but the multiplication of $\mathcal{D}$ is weakly cartesian

Applications

Can we determize PAs into belief-state transformers using weak distrbutive law ?





Lemma

Consider a weak distributive law $\gamma\colon TS \Rightarrow ST$ of S over T and let $\hat{S}$ be the corresponding weak lifting *of S to $EM(T)$. Then, we have the following liftings*

$$
\begin{tikzcd}
\mathrm{Coalg}(ST) \arrow[r, "\widehat{F^T}"] \arrow[d] & \mathrm{Coalg}(\hat{S}) \arrow[d] \\
\mathcal{C} \arrow[r, "F^T"] & EM(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Coalg}(\hat{S}) \arrow[r, "\widehat{U^T}"] \arrow[d] & \mathrm{Coalg}(S) \arrow[d] \\
EM(T) \arrow[r, "U^T"] & \mathcal{C}
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This instantiates to transforming a $\mathcal{P}\mathcal{D}$-coalgebra on X into a $\mathcal{P}_c$-coalgebra on $\mathcal{D}X$, that is, to the transformation of a PA into a belief-state transformer, obtained in

[Bonchi, Silva, Sokolova]
The Power of Convex Algebras, CONCUR 2017

Combining algebraic theories of nondeterminism and probability

We obtain an immediate concrete presentation for the $\mathcal{P}_c\mathcal{D}$-algebras, i.e., convex semilattices, see

[Bonchi, Sokolova, Vignudelli]
The Theory of Traces for Systems with Nondeterminism and Probability, LICS 2019

$$\begin{tikzcd}
\mathcal{D}\mathcal{P}X \arrow[rr, "\gamma_X"] \arrow[d, "\mathcal{D}\bigvee"'] & & \mathcal{P}\mathcal{D}X \arrow[d, "\mathcal{P}a"] \\
\mathcal{D}X \arrow[dr, "a"'] & & \mathcal{P}X \arrow[dl, "\bigvee"] \\
& X &
\end{tikzcd}$$

$(X, \bigvee, (+_r)_{r \in [0,1]})$ so that

- $(X, \bigvee)$ is a complete sup-semilattice,
- $(X, (+_r)_{r \in [0,1]})$ is a convex algebra
- the distributivity axiom holds $(\bigvee x_i) +_r y = \bigvee (x_i +_r y)$.

Weak laws in a continuous setting

Theorem (Goy, Aiguier and P., ICALP 2021)
There exists a weak distributive law $\mathcal{V}\mathcal{V} \Rightarrow \mathcal{V}\mathcal{V}$ of the Vietoris monad on compact Hausdorff spaces over itself.

- we rely on KHaus being a regular category
- we use the [Carboni, Kelly and Wood, 1991] results for extending functors to relations on regular categories
- the Kleisli category $\mathrm{Kl}(\mathcal{V})$ can be seen as a category of relations satisfying additional continuity constraints
- the Vietoris functor nearly preserves pullbacks, so it can be extended to $\mathrm{Rel}(\mathrm{KHaus})$. The extension restricts to $Kl(\mathcal{V})$.
- the multiplication of $\mathcal{V}$ is nearly cartesian but the unit is not.

Semialgebras and why weak laws are strong...

Semialgebras for a monad

Given a monad T, a semialgebra for T is a morphism $a\colon TX \to X$ such that only the associativity axiom holds:

$$\begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "Ta"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$

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Semialgebras are morally algebras ....

Example: Semialgebras for the Maybe monad

Consider the Maybe monad $- + 1 \colon \mathbf{Set} \to \mathbf{Set}$.

Algebras for this monad are pointed sets, so they are presented by a constant operation $\bullet \colon 0$ and no equations.

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Seminalgebras for the maybe monad are presented by the following signature:

$$\Sigma = \{\mathrm{a} \colon 1, \bullet \colon 0\}$$

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It turns out that adding an idempotent to a given presentation of algebras for a Set-monad T, and suitably transforming the equations leads to a presentations of semialgebras.

Weak laws are strong...

Theorem (P., Sarkis, MFPS 2021)
Given a monad T on a category with coproducts, there is a monad structure on $\mathrm{id}_C + T$, called the semifree monad T^s on T, so that there is an isomorphism between Eilenberg-Moore algebras for T^s and semialgebras for T.

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Weak distributive laws $TS \Rightarrow ST$ are in one-to-one correspondence with distributive laws $T^sS \Rightarrow ST^s$ subject to an additional axiom.

[Rosset, Hansen, Endrullis, 2022] further proved an open problem we left open for the concrete algebraic presentation of the semifree monad T^s.

References and perspectives

The result presented in this talk appear in

[Goy and P.]
Combining probabilistic and non-deterministic choice via weak distributive laws, LICS 2020

[Goy, Aiguier and P.]
Powerset-Like Monads Weakly Distribute over Themselves in Toposes and Compact Hausdorff Spaces, ICALP 2021

[P. and Sarkis]
Semialgebras and Weak Distributive Laws, MFPS 2021