Strong Pseudomonads and Premonoidal Bicategories

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Philip Saville (Oxford)

Recap of strong monads

A monad is a functor

$$T : \mathbb{C} \longrightarrow \mathbb{C}$$

equipped with

$$\mu_A : T^2 A \longrightarrow TA$$

$$\eta_A : A \longrightarrow TA \, .$$

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$$A \otimes TB \xrightarrow{t_{A,B}} T(A \otimes B)$$

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$$\begin{array}{ccc} I\,T_A & \xrightarrow{t} & T_{IA} \\ & \lambda \searrow \quad \swarrow T_\lambda & \\ & T_A & \end{array}$$

$$\begin{array}{ccccc} (AB)T_C & & \xrightarrow{t} & & T_{(AB)C} \\ \alpha \downarrow & & & & \searrow T\alpha \\ A(BT_C) & \xrightarrow{At} & AT_{BC} & \xrightarrow{t} & T_{A(BC)} \end{array}$$

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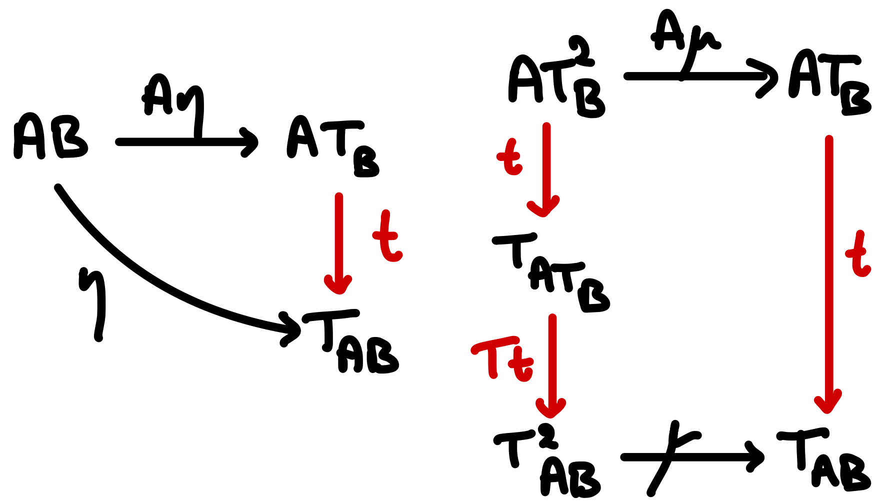

Strong monads for computation

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$$\begin{array}{c} \Gamma \vdash N : B \\ x : B \vdash M : C \end{array} \qquad \Gamma \xrightarrow{N} TB \xrightarrow{TM} T^2C \xrightarrow{\mu} TC$$

- Need a strength in general :

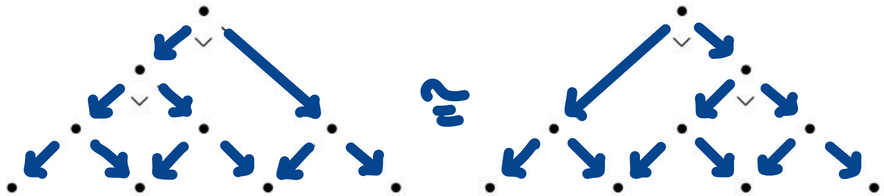
$$\begin{array}{c} \Gamma \vdash N : B \\ y : A,\ x : B \vdash M : C \end{array} \qquad \begin{array}{c} \Gamma \longrightarrow TB \\ A \otimes B \longrightarrow TC \end{array}$$

$$A \otimes \Gamma \xrightarrow{A \otimes N} A \otimes TB \xrightarrow{t} T(A \otimes B) \xrightarrow{TM} T^2C \xrightarrow{\mu} TC$$

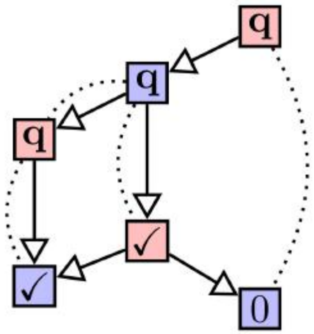
Bicategories

Recent semantic models form bicategories, not categories:

- spans or profunctors
- games and strategies
- PARA construction



$(P,\ A \otimes P \longrightarrow B)$



(composition as universal construction, morphisms are themselves structured objects, ...)

Problem: so far no explicit notion of strong monad for bicategories.

This talk

→ 1. A definition of strong pseudomonads

2. Strengths correspond to actions

3. Premonoidal bicategories

Bicategory theory

Instead of commutative diagrams we have coherent invertible 2-cells.

For example: a pseudo-natural transformation $\theta : F \to G$ consists of

$$FA \xrightarrow{\theta_A} GA \qquad \text{(for every } A\text{)}$$

$$\begin{array}{ccc} FA & \xrightarrow{\theta_A} & GA \\ {\scriptstyle Ff}\downarrow & \cong \theta_f & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\theta_B]{} & GB \end{array} \qquad \text{(for every } f\text{)}$$

$$\begin{array}{ccc} FA & \xrightarrow{\theta_A} & GA \\ {\scriptstyle Ff}\downarrow & \theta_f & \downarrow{\scriptstyle Gf} \\ FB & \longrightarrow & GB \\ {\scriptstyle Fg}\downarrow & \theta_g & \downarrow{\scriptstyle Gg} \\ FC & \xrightarrow[\theta_C]{} & GC \end{array}$$

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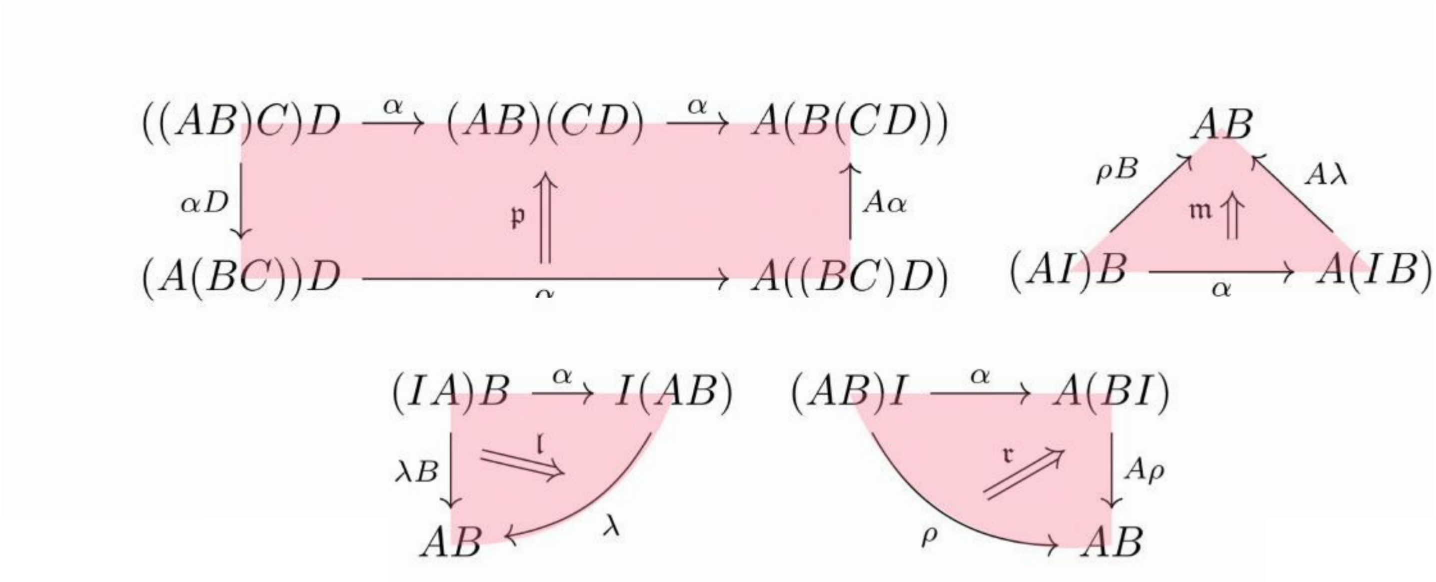
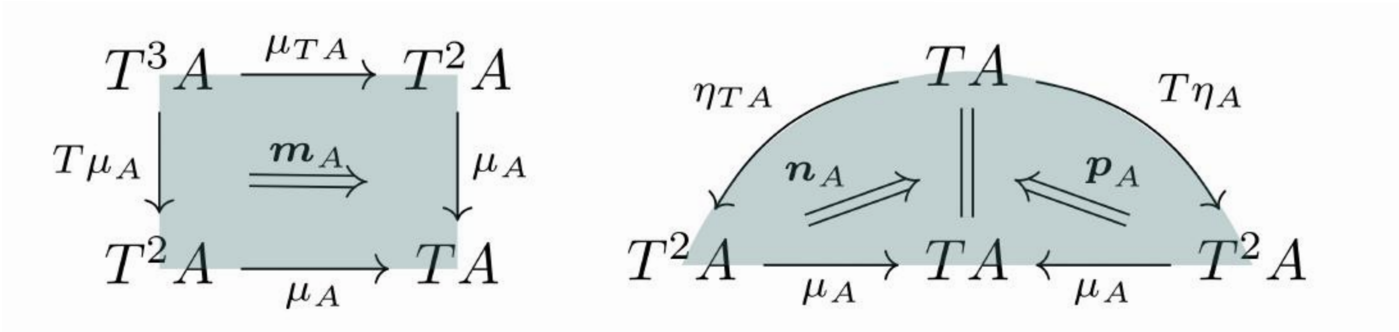
Other notions we use:

- monoidal bicategories:



- pseudomonads:



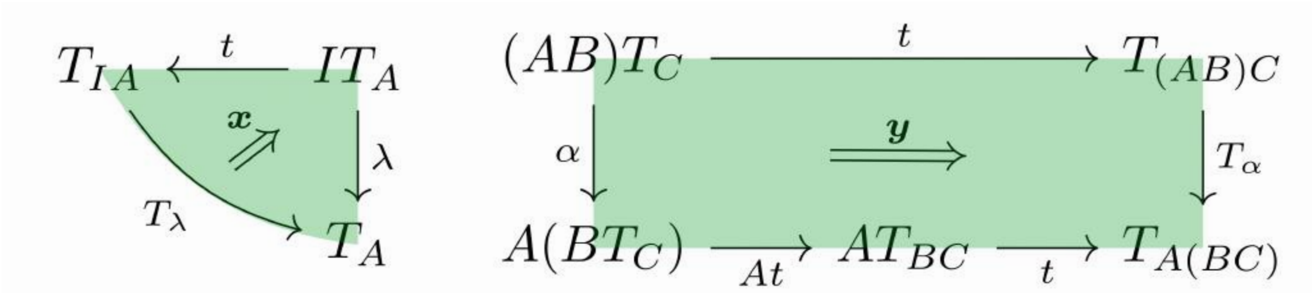
Strong Pseudomonads

A left-strength for T on $(\mathcal{B}, \otimes, I)$ is a pseudo-nat. transformation

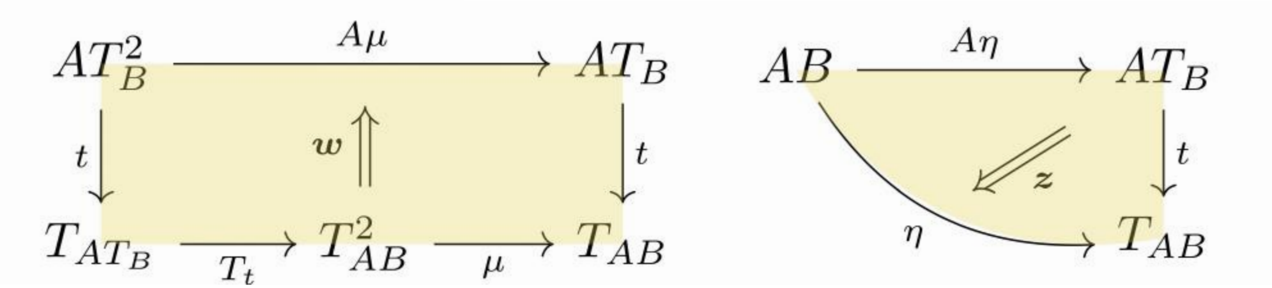
$$A \otimes TB \xrightarrow{t_{AB}} T(A \otimes B)$$

compatible with:

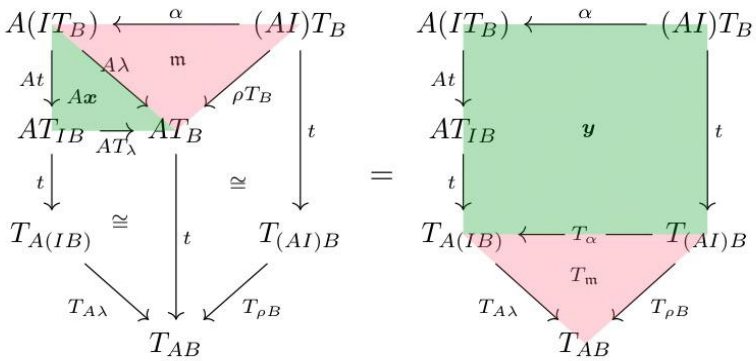
1. the monoidal structure



2. the monad structure



Satisfying :


A(IT_B) ← α — (AI)T_B
At, Aλ, 𝔪, ρT_B, Ax
AT_IB → AT_B (AT_λ)
t, ≅, t, ≅, t
T_A(IB), T_(AI)B
T_Aλ, T_ρB
T_AB
=
A(IT_B) ← α — (AI)T_B
At, y, t
AT_IB
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T_𝔪
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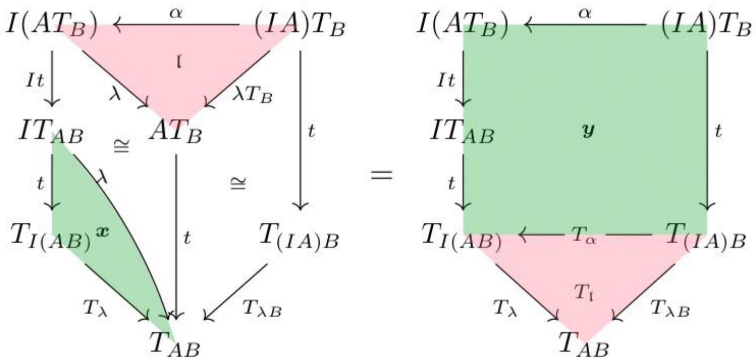
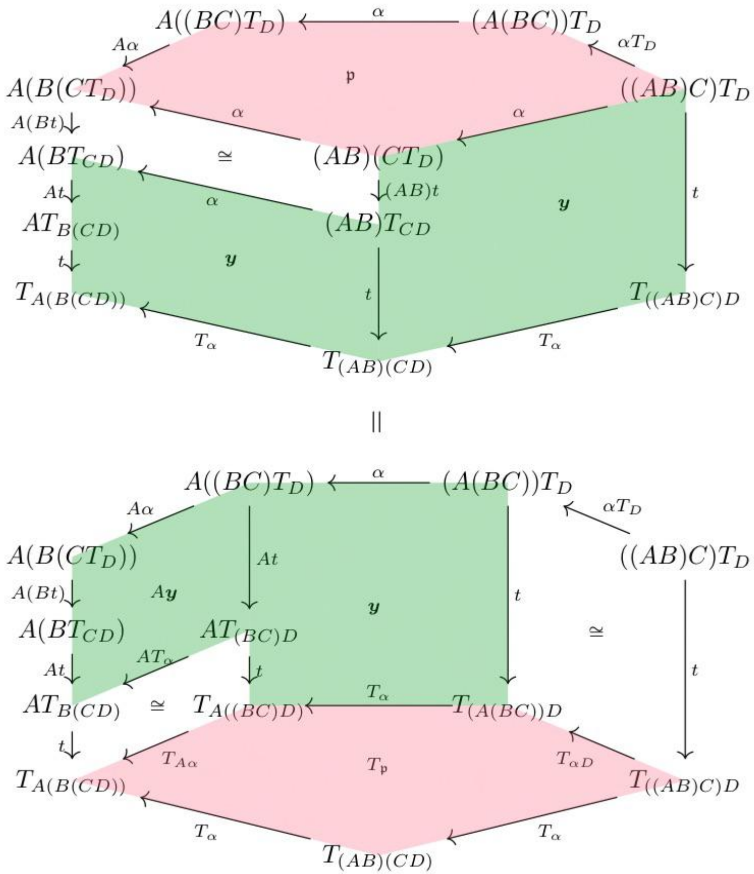
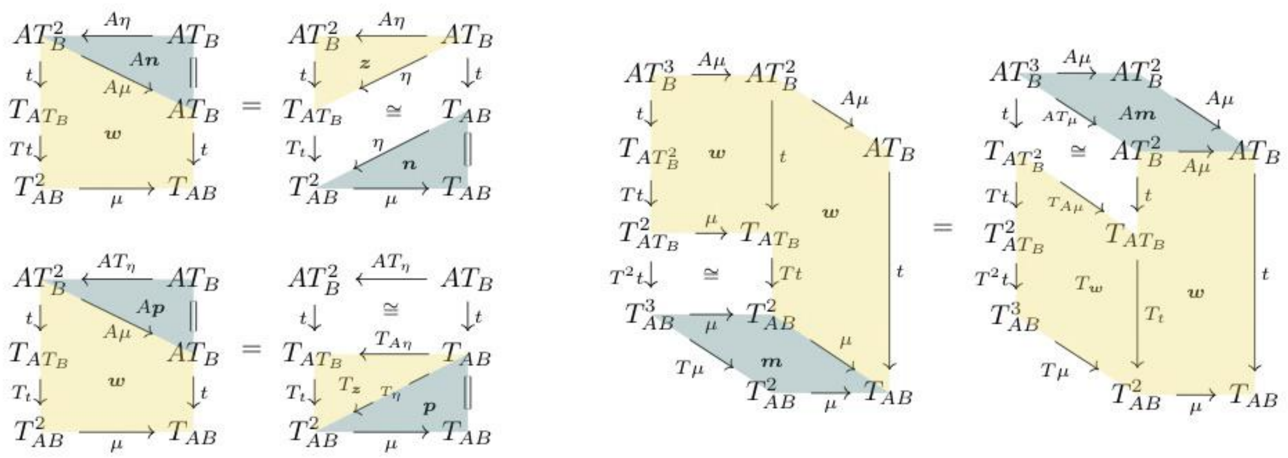
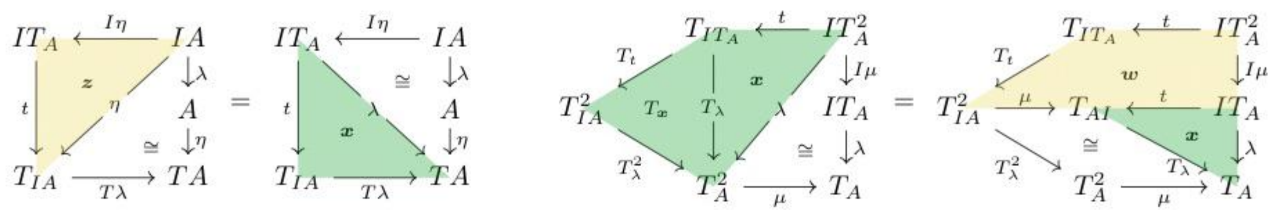

I(AT_B) ← α — (IA)T_B
It, λ, 𝔩, λT_B
IT_AB, ≅, AT_B, ≅
t, λ, x, t, t
T_I(AB), T_(IA)B
T_λ, T_λB
T_AB
=
I(AT_B) ← α — (IA)T_B
It, y, t
IT_AB
T_I(AB) ← T_α — T_(IA)B
T_𝔩
T_λ, T_λB
T_AB

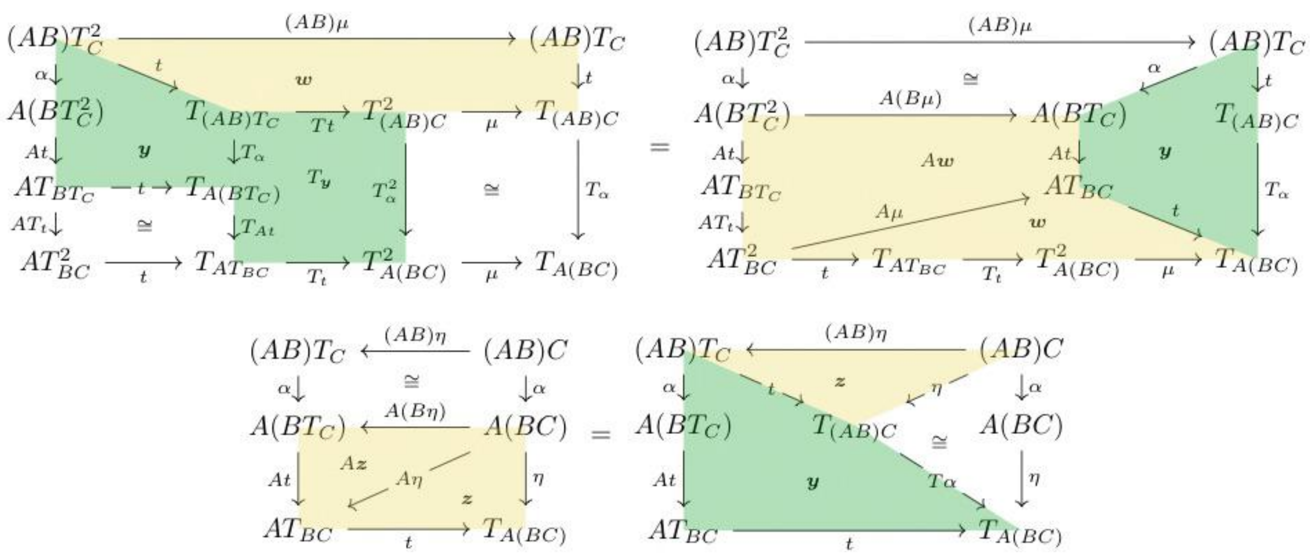

A((BC)T_D) ← α — (A(BC))T_D
Aα, 𝔭, αT_D
A(B(CT_D)), ((AB)C)T_D
A(Bt), α, α
A(BT_CD), ≅, (AB)(CT_D)
At, (AB)t, α, y, t
AT_B(CD), (AB)T_CD
t, y, t
T_A(B(CD)), T_((AB)C)D
T_α, T_α
T_(AB)(CD)
||
A((BC)T_D) ← α — (A(BC))T_D
Aα, αT_D
A(B(CT_D)), At, ((AB)C)T_D
A(Bt), Ay, y, t, ≅
A(BT_CD), AT_(BC)D
At, AT_α, t, t
AT_B(CD), ≅, T_A((BC)D) ← T_α — T_(A(BC))D
t, T_Aα, T_𝔭, T_αD
T_A(B(CD)), T_((AB)C)D
T_α, T_α
T_(AB)(CD)



AT_B^2 $A\eta$ AT_B An $A\mu$ T_{AT_B} w Tt T_{AB}^2 μ T_{AB} t
=
AT_B^2 $A\eta$ AT_B z η T_{AT_B} $\cong$ T_{AB} Tt η n T_{AB}^2 μ T_{AB}
AT_B^2 AT_η AT_B Ap $A\mu$ T_{AT_B} w T_t T_{AB}^2 μ T_{AB}
=
AT_B^2 AT_η AT_B $\cong$ T_{AT_B} $T_{A\eta}$ T_{AB} T_t T_z T_η p T_{AB}^2 μ T_{AB}
AT_B^3 $A\mu$ AT_B^2 $A\mu$ AT_B $T_{AT_B^2}$ w t Tt $T_{AT_B}^2$ μ T_{AT_B} w T^2t $\cong$ Tt T_{AB}^3 μ T_{AB}^2 μ $T\mu$ m T_{AB}^2 μ T_{AB}
=
AT_B^3 $A\mu$ AT_B^2 $A\mu$ AT_μ Am $T_{AT_B^2}$ $\cong$ AT_B^2 $A\mu$ AT_B Tt $T_{A\mu}$ t $T_{AT_B}^2$ T_{AT_B} T^2t T_w w t T_{AB}^3 T_t $T\mu$ T_{AB}^2 μ T_{AB}



IT_A $I\eta$ IA λ z t η A $\cong$ η T_{IA} $T\lambda$ TA
=
IT_A $I\eta$ IA $\cong$ λ t λ A x η T_{IA} $T\lambda$ TA
T_{IT_A} t IT_A^2 T_t $I\mu$ x T_{IA}^2 T_x T_λ λ IT_A $\cong$ λ T_λ^2 T_A^2 μ T_A
=
T_{IT_A} t IT_A^2 T_t $I\mu$ w T_{IA}^2 μ T_{AI} t IT_A $\cong$ x λ T_λ^2 T_λ T_A^2 μ T_A



$(AB)T_C^2$ $(AB)\mu$ $(AB)T_C$ α t w t $A(BT_C^2)$ $T_{(AB)T_C}$ Tt $T_{(AB)C}^2$ μ $T_{(AB)C}$ At y T_α T_y T_α^2 $\cong$ T_α AT_{BT_C} t $T_{A(BT_C)}$ AT_t $\cong$ T_{At} AT_{BC}^2 t $T_{AT_{BC}}$ T_t $T_{A(BC)}^2$ μ $T_{A(BC)}$
=
$(AB)T_C^2$ $(AB)\mu$ $(AB)T_C$ α $\cong$ α t $A(BT_C^2)$ $A(B\mu)$ $A(BT_C)$ $T_{(AB)C}$ At Aw At y T_α AT_{BT_C} AT_{BC} AT_t $A\mu$ w t AT_{BC}^2 t $T_{AT_{BC}}$ T_t $T_{A(BC)}^2$ μ $T_{A(BC)}$
$(AB)T_C$ $(AB)\eta$ $(AB)C$ α $\cong$ α $A(BT_C)$ $A(B\eta)$ $A(BC)$ Az At $A\eta$ η z AT_{BC} t $T_{A(BC)}$
=
$(AB)T_C$ $(AB)\eta$ $(AB)C$ α t z η α $A(BT_C)$ $T_{(AB)C}$ $\cong$ $A(BC)$ At y $T\alpha$ η AT_{BC} t $T_{A(BC)}$

This talk

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→ 2. Strengths correspond to actions

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Another view: strengths as actions of $(\mathcal{B}, \otimes)$ on $\mathcal{B}_T$

$$\mathcal{B} \times \mathcal{B}_T \longrightarrow \mathcal{B}_T$$

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$$\left(\begin{array}{c} A \\ \downarrow f \\ A' \end{array} , \begin{array}{c} B \\ \downarrow g \\ TB' \end{array} \right) \longmapsto \begin{array}{c} A \otimes B \\ \downarrow f \otimes g \\ A' \otimes TB' \\ \downarrow t \\ T(A' \otimes B') \end{array}$$

Another view: strengths as actions of $(\mathcal{B},\otimes)$ on $\mathcal{B}_T$

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In the other direction, if $\triangleright : \mathcal{B} \times \mathcal{B}_T \longrightarrow \mathcal{B}_T$ is an action such that $A \triangleright B = A \otimes B$, we take

$$\left(\begin{array}{c} A \otimes TB \\ \downarrow t_{A,B} \\ T(A \otimes B) \end{array} \right) := \left(\begin{array}{c} A \\ \downarrow 1_A \\ A \end{array} \right) \triangleright \left(\begin{array}{c} TB \\ \downarrow 1_{TB} \\ TB \end{array} \right)$$

The monoidal structure $\otimes : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ is a canonical action.

Def: An extension of this action to $\mathcal{B}_T$ consists of

- an action $\mathcal{B} \times \mathcal{B}_T \xrightarrow{\triangleright} \mathcal{B}_T$ (+ all data)
- an icon (= identity-on-objects pseudonatural transformation)

$$\begin{array}{ccc}
\mathcal{B} \times \mathcal{B}_T & \xrightarrow{\ \triangleright\ } & \mathcal{B}_T \\
{\scriptstyle 1_{\mathcal{B}} \times J} \uparrow & \cong_{\theta} & \uparrow {\scriptstyle J} \\
\mathcal{B} \times \mathcal{B} & \xrightarrow[\ \otimes\ ]{} & \mathcal{B}
\end{array}$$

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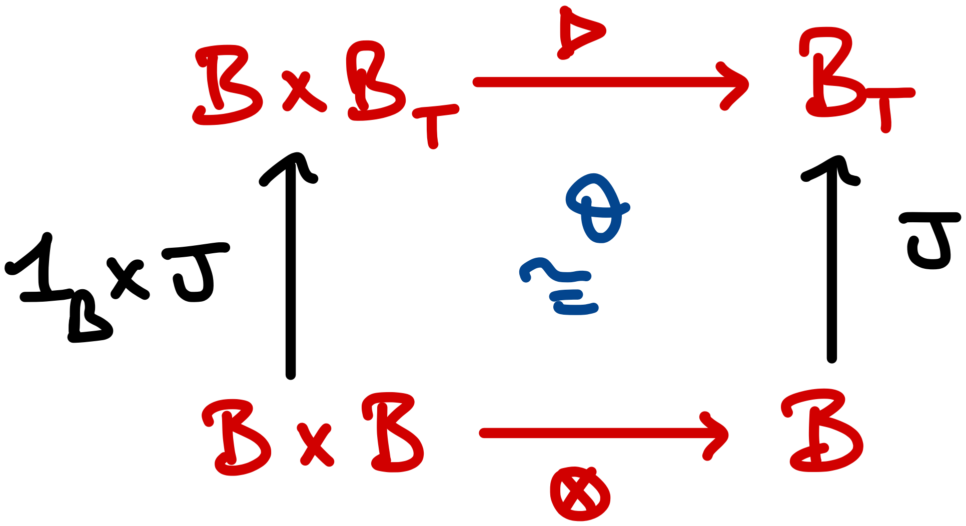
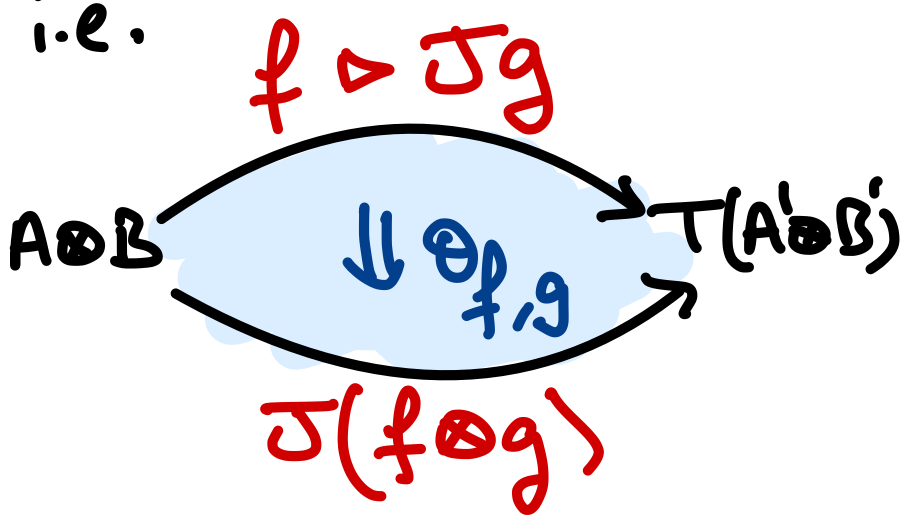


i.e.



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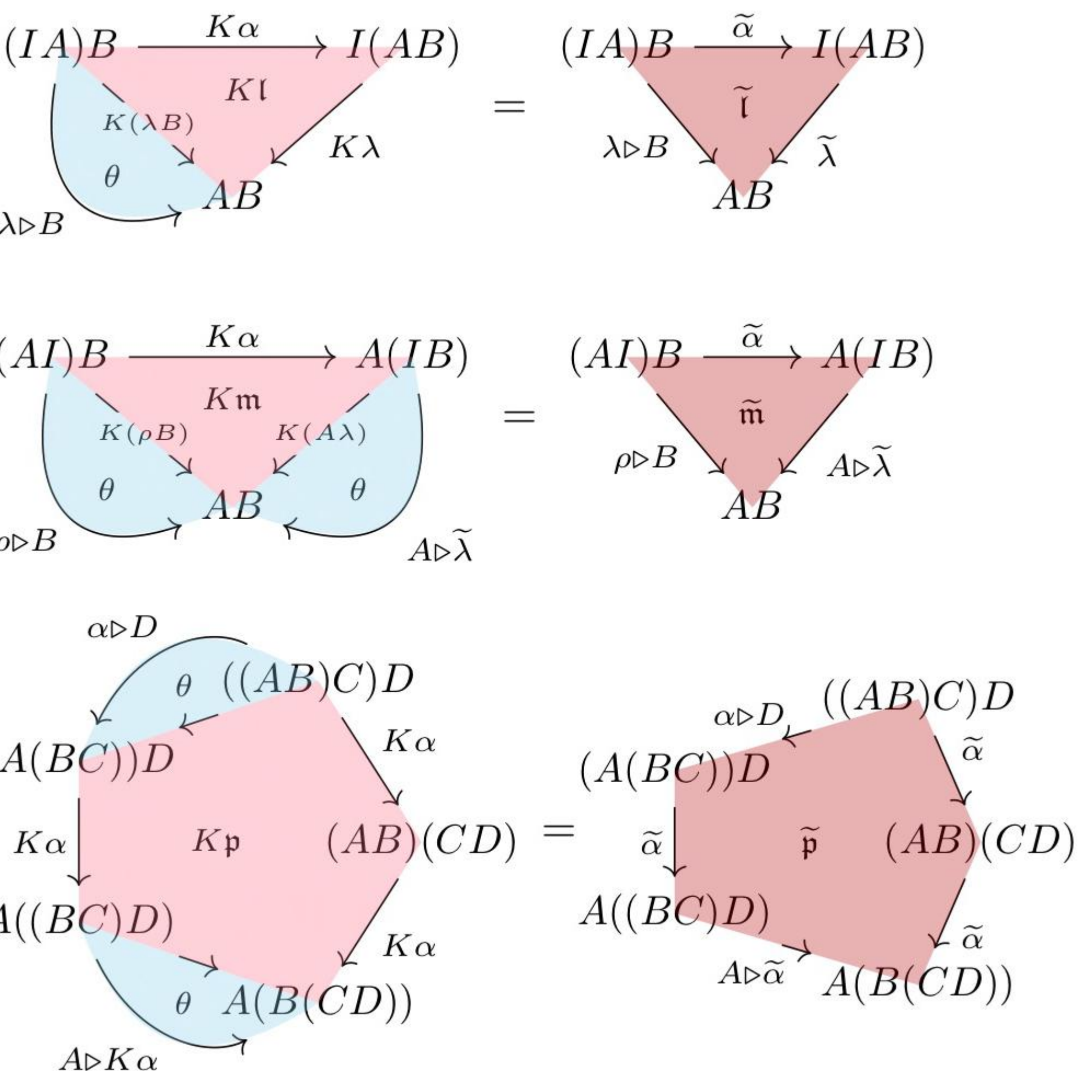

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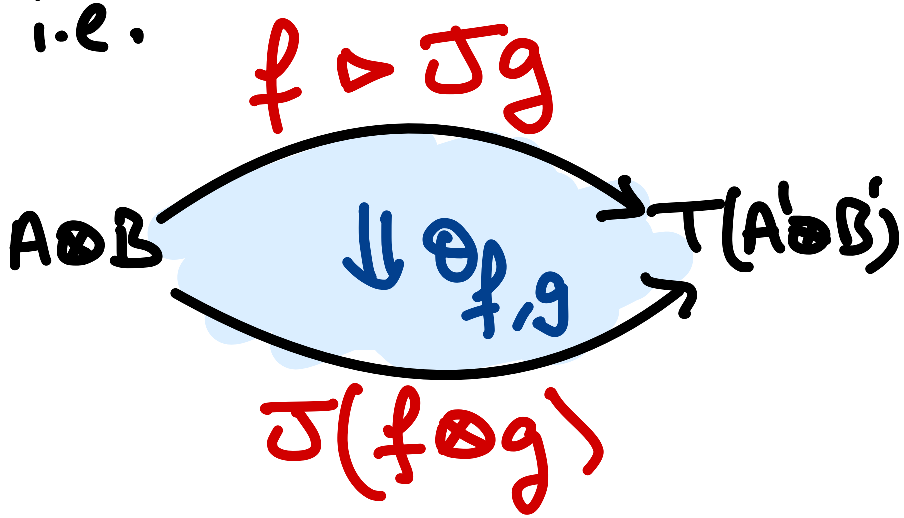

Correspondence theorem: Fix T and $(\mathcal{B}, \otimes, I)$.

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(Corresponding result for right actions / strengths)

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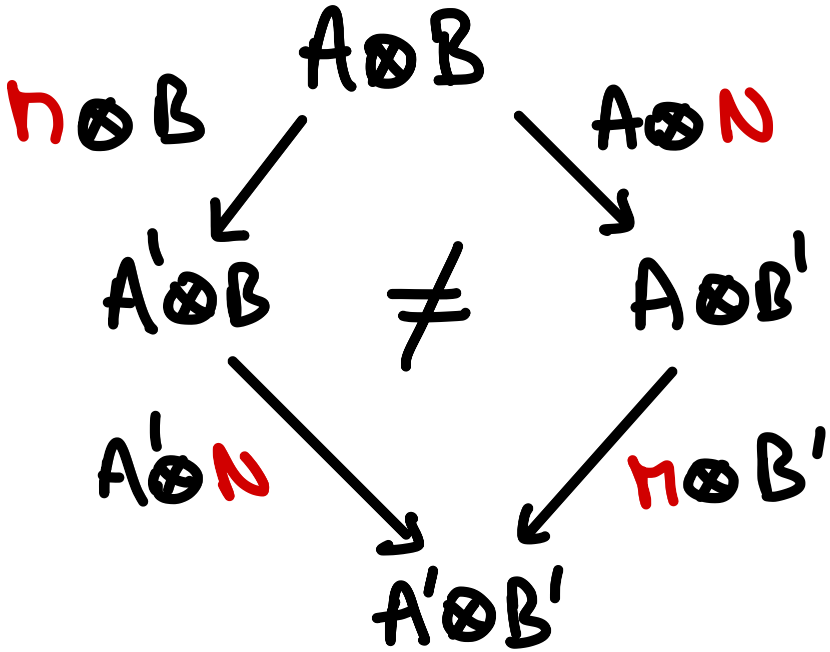
→ 3. Premonoidal bicategories

Premonoidal bicategories: another model for effectful computation

premonoidal cats.
[Power & Robinson]

- idea: directly axiomatize the structure of $\mathcal{B}_T$
- have a tensor product but no interchange:



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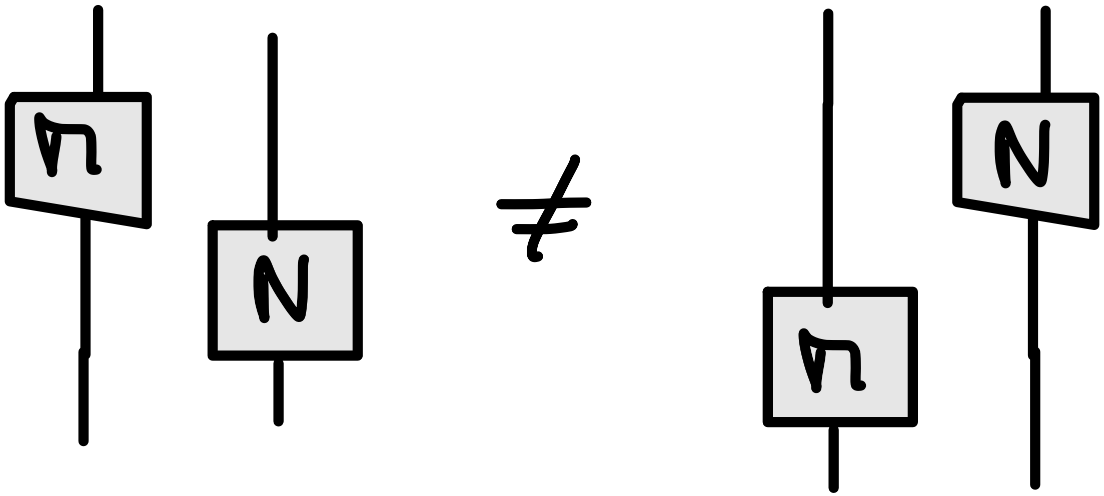

Premonoidal bicategories:

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another model for effectful computation

- idea: directly axiomatize the structure of $\mathcal{B}_T$
- have a tensor product but no interchange
- 2 functors $A \otimes -$ and $- \otimes A$ for every A
- the structural morphisms should still satisfy interchange.

Central morphisms [Power & Robinson]

In a category with $A \otimes - : \mathbb{C} \to \mathbb{C}$ (for every A)
$- \otimes A : \mathbb{C} \to \mathbb{C}$

$f : A \to A'$ is central if

$$\begin{array}{ccc} A \otimes B & \xrightarrow{f \otimes B} & A' \otimes B \\ {\scriptstyle A \otimes g}\downarrow & & \downarrow{\scriptstyle A' \otimes g} \\ A \otimes B' & \xrightarrow[f \otimes B']{} & A' \otimes B' \end{array} \qquad \begin{array}{ccc} B \otimes A & \xrightarrow{B \otimes f} & B \otimes A' \\ {\scriptstyle g \otimes A}\downarrow & & \downarrow{\scriptstyle A' \otimes g} \\ B' \otimes A & \xrightarrow[B' \otimes f]{} & B' \otimes A' \end{array}$$

for all $g : B \to B'$.

Central morphisms

In a bicategory with $A \otimes - : \mathcal{B} \longrightarrow \mathcal{B}$

$- \otimes A : \mathcal{B} \longrightarrow \mathcal{B}$ (for every A)

$f : A \to A'$ is central when equipped with

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{f \otimes B} & A' \otimes B \\
{\scriptstyle A \otimes g}\downarrow & \cong\ l^f_g & \downarrow{\scriptstyle A' \otimes g} \\
A \otimes B' & \xrightarrow[f \otimes B']{} & A' \otimes B'
\end{array}
\qquad
\begin{array}{ccc}
B \otimes A & \xrightarrow{B \otimes f} & B \otimes A' \\
{\scriptstyle g \otimes A}\downarrow & \cong\ r^f_g & \downarrow{\scriptstyle A' \otimes g} \\
B' \otimes A & \xrightarrow[B' \otimes f]{} & B' \otimes A'
\end{array}
$$

pseudonatural in $g : B \to B'$.

Def: A premonoidal bicategory $\mathcal{B}$ has

- $A \otimes -$, $- \otimes A$
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- same data and axioms as a monoidal bicategory.

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$\Rightarrow \mathcal{B}_T$ canonically premonoidal.

Conclusion

- new structures for effectful programming.
- Subtle point: the centre $Z(\mathcal{B})$ is not monoidal!
  $\rightsquigarrow$ need Freyd bicategories.