A Categorical Approach to Descriptive Complexity Theory

Damiano Mazza
CNRS, LIPN, Université Paris 13

SYCO 11
École Polytechnique, 20 April 2023

Descriptive Complexity

- Decision problem $:=$ subset of $\{0,1\}^*$.
- Complexity theory: classify decision problems according to how hard it is to decide them, in terms of resources needed.
- Theory $\mathbb{S}\mathrm{tr} :=$ binary $\leq$, axiomatized as total order, unary isOne. $\mathrm{FinMod}(\mathbb{S}\mathrm{tr}) = \{0,1\}^*$ (finite models of $\mathbb{S}\mathrm{tr}$ modulo iso).
- Descriptive complexity: classify problems according to how hard it is to describe them, in terms of logical language needed.
- Example: the following formula describes palindromes:

$$\forall x.\forall y.\forall m.\mathrm{Max}(m) \wedge \mathrm{Plus}(x,y,m) \Rightarrow (\mathrm{isOne}(x) \Leftrightarrow \mathrm{isOne}(y))$$

Boolean Categories

Boolean toposes	higher-order theories
Boolean pretoposes	first-order theories
Boolean lextensive categories	"quantifier-free" theories

- Boolean (lextensive) category [Carboni, Lack, Walters 1993]:
  - finite products (the structure below in fact implies all finite limits);
  - finite coproducts;
  - $1+1$ is disjoint, pullback-stable and the subobject classifier.

- Logical functor: functor preserving fin. prods. and fin. coprods.

- Examples (small):
  - $\mathcal{F}$: (skeleton of) finite sets and functions;
  - $\mathcal{F}_\omega$: (skeleton of) countable sets and functions;
  - syntactic categories of Boolean theories (next slide).

Boolean Categories of Finite Presentation

- Finite Boolean theory $\mathbb{T} := (\mathrm{Sort}, \mathrm{Rel}, \mathrm{Ax})$:
  - Sort finite set of *sorts;*
  - Rel finite set of *relation symbols*, $R \rightarrowtail A_1, \ldots, A_k$, with A_i sorts;
  - Ax finite set of *axioms*, quantifier-free (except for provably unique $\exists$).

  (So a Boolean theory is a multisorted, relational FO theory with equality, with closed axioms of the form $\forall \vec{x}.\varphi$ with φ quantifier-free except for provably unique $\exists$).

- $\mathcal{F}[\mathbb{T}]$: the cat of definable sets and functions in $\mathbb{T}$. It is Boolean.
  - $\mathcal{F} = \mathcal{F}[\mathbb{E}]$ where $\mathbb{E}$ is the empty theory.
  - The obj. of $\mathcal{F}[N; E \rightarrowtail N^2]$ are "polynomials" on N, E and $\overline{E}$.
  - $\mathcal{F}_\omega$ is not of finite presentation.

- $\mathcal{B}ool\mathcal{C}at_{\mathrm{fp}} :=$ fin. pres. Bool cats and logical functors modulo iso.

- $\mathcal{F}$ is the initial object of $\mathcal{B}ool\mathcal{C}at_{\mathrm{fp}}$.

Data Specifications and Complexity

- $\mathcal{D}ata := \mathcal{B}ool\mathcal{C}at_{\text{fp}}^{\text{op}}$ is lextensive. Write $\operatorname{Spec}\mathcal{B}$ for $\mathcal{B}$ as obj. of $\mathcal{D}ata$.
- Global section functor $\Gamma := \mathcal{D}ata(\operatorname{Spec}\mathcal{F}, -) : \mathcal{D}ata \to \mathbf{Set}$. $\Gamma(\operatorname{Spec}\mathcal{F}[\mathbb{T}]) = \operatorname{FinMod}(\mathbb{T})$.
- If $f : \mathcal{F}[\mathbb{T}] \to \mathcal{B}$ with $\mathcal{B}$ fin. pres., then $\mathcal{B} \cong \mathcal{F}[\mathbb{T}_f]$ with $\mathbb{T}_f$ extending $\mathbb{T}$ and $f = \mathrm{iso} \circ \mathrm{inclusion}$. We say that $f : X \to \operatorname{Spec}\mathcal{F}[\mathbb{T}]$ in $\mathcal{D}ata$ is
  - propositional if $\operatorname{Sort}(\mathbb{T}_f) = \operatorname{Sort}(\mathbb{T})$;
  - Horn if propositional + constraints on $\operatorname{Ax}(\mathbb{T}_f) \setminus \operatorname{Ax}(\mathbb{T})$;
  - Krom-Horn if Horn + other constraints on $\operatorname{Ax}(\mathbb{T}_f) \setminus \operatorname{Ax}(\mathbb{T})$.

Theorem. *$A \subseteq \{0,1\}^*$ is r.e. iff $\exists f : X \to \operatorname{Spec}\mathcal{F}[\mathbb{S}\mathrm{tr}]$ s.t. $A = \operatorname{im}\Gamma(f)$. Moreover:*

- $A \in$ NP iff f is propositional;
- $A \in$ P iff f is Horn;
- $A \in$ coNL iff f is Krom-Horn.

The Category of Reductions

- In $\mathcal{D}ata/S$, we define $f \approx f'$ iff $\operatorname{im}\Gamma(f) = \operatorname{im}\Gamma(f')$.

- Decision problem on $S :=$ $\approx$-class of morphisms of $\mathcal{D}ata/S$.

- A (quantifier-free) reduction $[g]/T \to [f]/S$ is defined by:
  - an arithmetical morphism $a : T_{+\times} \to T$;
  - a morphism $r : T_{+\times} \to S$ s.t. $a \circ r^* f \approx g$.

- Descriptive complexity allows to speak of complexity over arbitrary ordered structures. We see this as change of base.

- Usual completeness results (e.g. Cook-Levin theorem) may be reproved in this setting (as corollaries of the above Theorem).

Universal Problems via Yoneda

- Define $R, R_\bullet : \mathcal{D}ata^{\mathrm{op}} \to \mathbf{Set}$ by (on arrows, act by pullback):

$$R(\mathcal{B}) := \{\text{propositional morphisms over } \operatorname{Spec}\mathcal{B}\}$$

$$R_\bullet(\mathcal{B}) := \{(f, s) \mid f \text{ prop. morphism over } \operatorname{Spec}\mathcal{B},\ f \circ s = \mathrm{id}\}$$

- Proj. $u : R_\bullet \to R$ = "universal" NP problem:

$$\begin{array}{ccc} X & \longrightarrow & R_\bullet \\ {\scriptstyle \forall \text{ prop. morph. } f}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ S & \xrightarrow{\exists !} & R \end{array}$$

$\Gamma(R) = \{\text{prop. formulas } \varphi\}/\text{Morita equiv.},$
$\Gamma(R_\bullet) = \{(\varphi, \sigma) \mid \sigma \models \varphi\}/\text{Morita equiv.},$
$\operatorname{im}\Gamma(u) = \text{semantic version of } \textsc{Sat}.$

- Can do the same with Horn and Krom-Horn morphisms.

Perspectives

- More complexity classes?
  - L and CSPs are immediate. Uniform $AC^0 = LH = FO$ seems easy.
  - Don't know about PH or PSPACE.
  - In any case, is the "universal problem" of these classes meaningful?

- Tools from finite model theory? Structural complexity?
  A "fibrational" view of (search) problems?

- Colimits of presheaves are bad. We need sheaves.

- Algebraic geometry with Boolean cats instead of comm. rings?
  - Bool cats are intriguingly similar to algebras on a non-alg. closed field.
  - Zarisky topology? Data schemes = locally representable sheaves?
    (*Categories of spaces built from local models* [Zhen Lin Low 2016]).
  - A unifying theory? (Work in progress with Morgan Rogers).