Convex categories and contextuality

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Real Convex sets

A *real convex set* consists of a set X together with a ternary operation $< -, -, - >: [0, 1] \times X \times X \rightarrow X$ satisfying some axioms resembling the behavior of convex linear combinations in Euclidean space(like $< \alpha, x, x >= x$).

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Alternative way to define a real convex set as an algebra over the distribution monad.

The distribution monad

For a set X, we define $D_R(X)$ the set of distributions on X to be

$$D_R(X) := \{p : X \to R : \ |supp(p)| < \infty \text{ and } \sum_{x \in X} p(x) = 1\}$$

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This give us a monad D_R : **Set** $\to$ **Set** with the following structure maps:

- $\delta_X : X \to D_R(X)$ sends $x \in X$ to the distribution δ^x.
- $\mu_X : D_R^2(X) \to D_R(X)$ sends a distribution Q to the distribution

$$D_R(Q)(x) = \sum_{p \in D_R(X)} Q(p)p(x).$$

R-convex set

Given a monad T on a category **C**. A *T-algebra* consists of an object X of **C** together with a morphism $\pi : T(X) \to X$ of **C** such that the following diagrams commute

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}_X} & X \\ {\scriptstyle \delta_X}\downarrow & \nearrow{\scriptstyle \pi} & \\ T(X) & & \end{array} \qquad \begin{array}{ccc} T^2(X) & \xrightarrow{T(\pi)} & T(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ T(X) & \xrightarrow{\pi} & X \end{array}$$

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A *morphism of T-algebras* is a morphism $f : X \to Y$ of $\mathbf{C}$ such that $\pi^Y \circ T(f) = f \circ \pi^X$.

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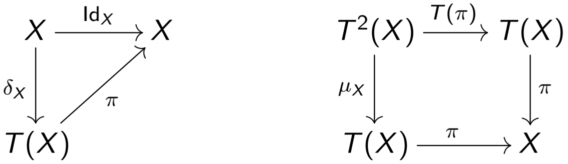

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Proposition: The category of real convex sets **Conv** is isomorphic to the category of $D_{\mathbb{R}_{\geq 0}}$-algebras.

Definition: [[Duality for convexity, Bart Jacobs]] A D_R-algebra in the category of sets is called an *R-convex set*. We will denote the category of R-convex sets by $\mathbf{Conv}_R$.

Convex categories

Let $\mathbf{C}$ be a (locally small) category. We define the category $D_R(\mathbf{C})$ to be

- $\text{Obj}(D_R(\mathbf{C})) = \text{Obj}(\mathbf{C})$
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For objects $p \in D_R(\mathbf{C}(X, Y))$ and $q \in D_R(\mathbf{C}(Y, Z))$, we define the composition $q * p \in D_R(\mathbf{C}(X, Z))$ as the following:

$$(q * p)(f) = \sum_{g_2 \circ g_1 = f} q(g_2)p(g_1)$$

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This defined a monad $D_R : \mathbf{Cat} \to \mathbf{Cat}$ with structure functors:

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Definition: A D_R-algebra in **Cat** will be called an *R-convex category*. We denote the category of R-convex categories by $\mathbf{ConvCat}_R$.

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Fact: A category that enriched by R-convex sets is an R-convex category.

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Example: The kleisli category $\mathbf{Set}_{D_R}$ is an R-convex category, but not enriched by R-convex sets.

Quantum contextuality

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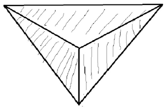
But not always the tables are coming from a global probability table:

abc	(0,0,0)	(0,0,1)	...	...	...	...	...	(1,1,1)
	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8

The tables of probabilities above can be expressed as a simplicial map $X \to D_R(N\mathbb{Z}_2)$, where X is the following space:



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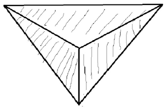

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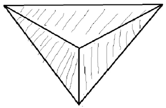

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In addition, we have the following map

$$\Theta_{X,N\mathbb{Z}_2} : D_R(s\mathbf{Set}(X, N\mathbb{Z}_2)) \to s\mathbf{Set}(X, D_R(N\mathbb{Z}_2))$$

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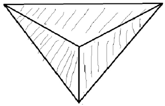

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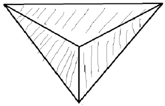

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Definition: $p \in s\mathbf{Set}(X, D_R(Y))$ is *noncontextual/classic* if it lies in the image of $\Theta_{X,Y}$.

Convex categories and contextuality

Fact: Let $\mathbf{C}^T$ be the category of T-algebras. There is an adjunction $T : \mathbf{C} \dashv \mathbf{C}^T : U$.

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Proposition

The transpose of the functor $F_{D_R} : s\mathbf{Set} \to s\mathbf{Set}_{D_R}$ with respect to the adjunction $D_R : \mathbf{Cat} \dashv \mathbf{ConvCat}_R : U$ is the functor $\Theta : D_R(s\mathbf{Set}) \to s\mathbf{Set}_{D_R}$ which is defined as identity on the objects and as the map $\Theta_{X,Y}$ on morphisms.

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Example: If Y is a simplicial group, then $s\mathbf{Set}(X, D_R(Y))$ is an R-convex monoid.

Theorem

Let R be a zero-sum-free, integral semiring R. Given a simplicial set X and a simplicial group Y, a distribution $p \in s\mathbf{Set}(X, D_R(Y))$ is non-contextual if and only if p is weakly invertible.

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Thank you for listening