Enriched ~~Co~~ RETRO functors and Lenses

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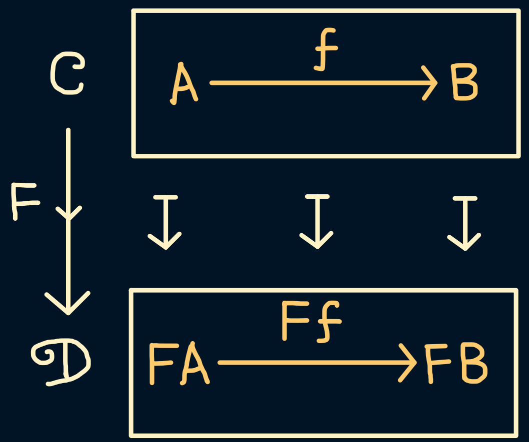
Joint work with Bryce Clarke

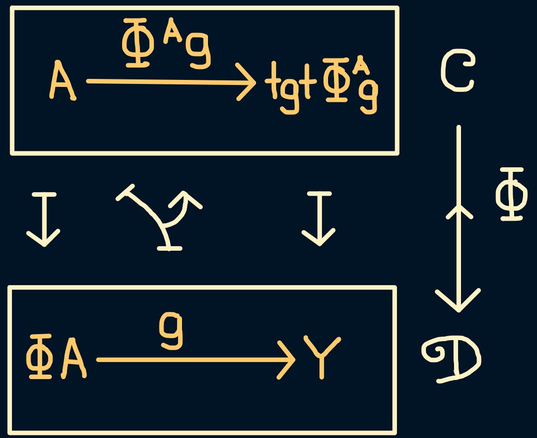
Symposium on Compositional Structures

21 April 2023

CATEGORIES



functor



retrofunctor

functor + retrofunctor = delta lens

WEIGHTED CATEGORIES

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] & A \arrow[r, "f"] \arrow[d, mapsto] & B \arrow[d, mapsto] \\ \mathcal{D} & FA \arrow[r, "Ff"] & FB \end{tikzcd} \qquad \begin{matrix} w(f) \\ \geq \\ w(Ff) \end{matrix}$$

weight in $[0,\infty]$

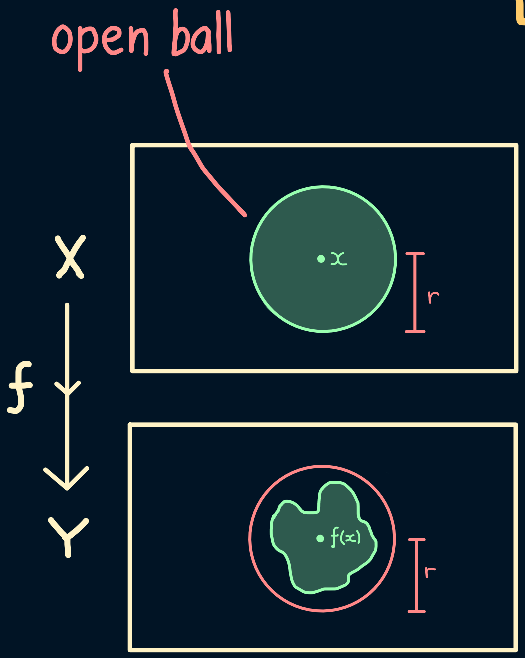
weighted functor

$$\begin{matrix} w(\Phi^A g) \\ \leq \\ w(g) \end{matrix} \qquad \begin{tikzcd} A \arrow[r, "\Phi^A g"] \arrow[d, mapsto] & \mathrm{tgt}\,\Phi^A g & \mathcal{C} \arrow[d, leftarrow, "\Phi"'] \\ \Phi A \arrow[r, "g"] & Y \arrow[u, mapsto] & \mathcal{D} \end{tikzcd}$$

weighted retrofunctor

weighted functor + weighted retrofunctor = **weighted lens**

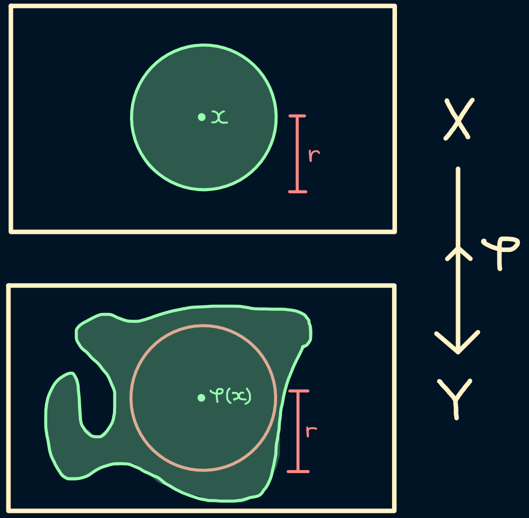
Lawvere metric spaces



$$d(x,a) \geqslant d(fx, fa)$$

non-expansive map

$$\inf_{a \in \varphi^{-1}\{y\}} d(x,a) \leqslant d(\varphi x, y)$$



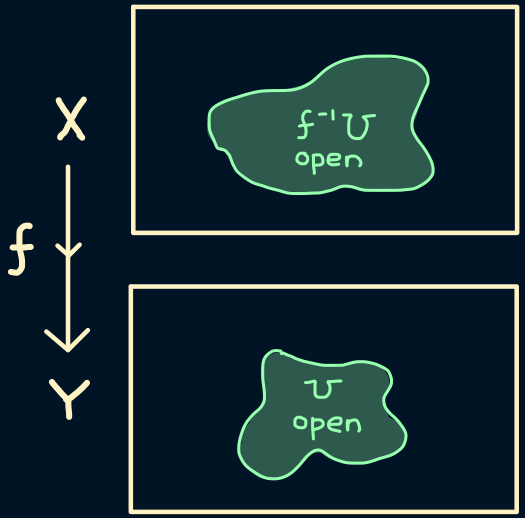
retro-non-expansive map

non-expansive map + retro-non-expansive map = weak submetry

ALEXANDROV-DISCRETE SPACES (PREORDERED SETS)

intersection of opens is open

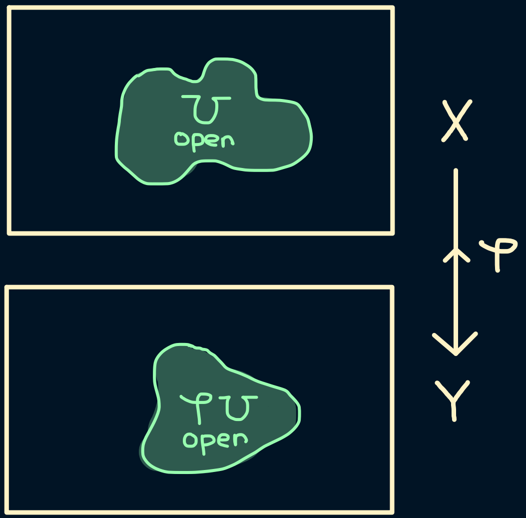
specialisation preorder



$$x \preccurlyeq x' \;\Downarrow\; fx \preccurlyeq fx'$$

continuous map

$$\exists x' : \begin{matrix} \varphi x' = y' \\ x \preccurlyeq x' \end{matrix} \;\Uparrow\; \varphi x \preccurlyeq y'$$



open map

(Generalised) Labelled Transition Systems

$$\underbrace{\mathcal{A} = \{\text{new}, \text{push}, \text{pop}\}}_{\text{actions}} \qquad \alpha \in \underbrace{\mathcal{A}^* = \{\overset{\text{silent transition}}{\tau}, \text{new}, \text{new;push}, \text{push;pop;push}, \dots\}}_{\text{free monoid on } \mathcal{A}}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{array}{c} x \xrightarrow{\alpha} x' \\ \Downarrow \\ fx \xrightarrow{\alpha} fx' \end{array}$$

functional (weak) simulation

$$\exists x'. \quad \begin{array}{c} \varphi x' = y' \\ x \xrightarrow{\alpha} x' \\ \Uparrow \\ \varphi x \xrightarrow{\alpha} y' \end{array} \qquad \begin{tikzcd} X \arrow[d, "\varphi"] \\ Y \end{tikzcd}$$

functional retrosimulation

(or zig-zag morphism)

functional simulation + functional retrosimulation = functional bisimulation

What do these examples have in common?

Enriched Categories

$\mathcal{V}$ = distributive monoidal category

$\mathcal{V}$-functor

$F : \mathcal{C} \nrightarrow \mathcal{D}$

$$\mathcal{C}(a,b) \xrightarrow{F_{a,b}} \mathcal{D}(Fa, y)$$

$\mathcal{V}$-retrofunctor

$\Phi : \mathcal{C} \nleftrightarrow \mathcal{D}$

$$\mathcal{D}(\Phi a, y) \xrightarrow{\Phi_{a,y}} \sum_{b \in \Phi^{-1}\{y\}} \mathcal{C}(a,b)$$

+ identity and composition preservation axioms

$\mathcal{V}$-lens = $\mathcal{V}$-functor + $\mathcal{V}$-retrofunctor + compatibility axioms

ENRICHED CATEGORIES

$\mathcal{V}$ = distributive monoidal category

$\mathcal{V}$-functor

$F : \mathcal{C} \longrightarrow \mathcal{D}$

$$\sum_{b \in F^{-1}\{y\}} \mathcal{C}(a,b) \xrightarrow{[F_{a,b}]_{b \in F^{-1}\{y\}}} \mathcal{D}(Fa,y)$$

$\mathcal{V}$-retrofunctor

$\Phi : \mathcal{C} \longleftrightarrow \mathcal{D}$

$$\mathcal{D}(\Phi a,y) \xrightarrow{\Phi_{a,y}} \sum_{b \in \Phi^{-1}\{y\}} \mathcal{C}(a,b)$$

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Example summary

$\mathcal{V}$	$\mathcal{V}$-category	$\mathcal{V}$-functor	$\mathcal{V}$-retrofunctor	$\mathcal{V}$-lens
Set	category	functor	retrofunctor	delta lens
wSet	weighted category	weighted functor	weighted retrofunctor	weighted lens
$([0,\infty],\geqslant)$	metric space	non-expanding map	retro non-expanding map	weak submetry
$(\{\top,\bot\},\Rightarrow)$	Alexandrov space	continuous map	open map	open continuous map
$(P(A^*),\subseteq)$	generalised labelled transition system	functional simulation	functional retrosimulation	functional bisimulation

In the article

arXiv2209.01144

- span representation of enriched retrofunctors
- natural transformations of enriched retrofunctors and lenses
- double category of enriched functors and retrofunctors

Coming soon

- proxy pullbacks and double category of enriched lens spans and bisimulation
- enriched retrofunctors and opretrofunctors as bimodules
- object of natural transformations of enriched retrofunctors

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