Geometry of Interaction for ZX-Diagrams

Kostia Chardonnet, Benoît Valiron, Renaud Vilmart

Univ. of Bologna
Univ. Paris Saclay, LMF

SYCO 11

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B & \cdots \\ a_{10}B & \ddots & \\ \vdots & & \end{pmatrix}$

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- Operation are *linear maps*

- $H := \frac{1}{\sqrt{2}} \begin{matrix} & |0\rangle & |1\rangle \\ |0\rangle & 1 & 1 \\ |1\rangle & 1 & -1 \end{matrix} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{cases}$ is unitary

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- $\text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\neg x\rangle \end{cases}$

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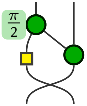

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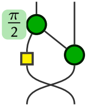
- Manipulates string diagrams e.g. 

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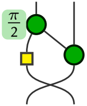
- Manipulates string diagrams e.g. 

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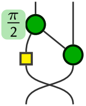
- Manipulates string diagrams e.g. 

- Relaxes unitarity
- Is Universal (can encode any linear map)
- Lack a direct operational interpretation (this work !)

Generators

(Empty) (Id) (Swap) (Cap) (Cup) (Green-Node) (Hadamard)

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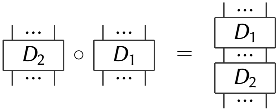

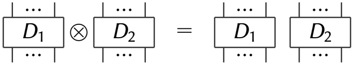

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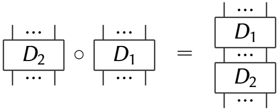
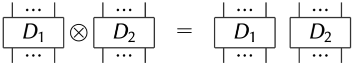
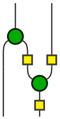




Standard Interpretation

Linear Maps : $\mathbf{ZX} \to \mathcal{M}(\mathbb{C})$



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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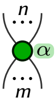
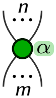
- A spider: 

$$:: \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0\dots0\rangle \mapsto |0\dots0\rangle \\ |1\dots1\rangle \mapsto e^{i\alpha}|1\dots1\rangle \\ |0\dots1\rangle \mapsto \overrightarrow{0} \end{cases}$$

- A spider: 

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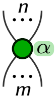
- A spider: 

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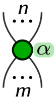
- A spider:  :: $\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0\dots0\rangle \mapsto |0\dots0\rangle \\ |1\dots1\rangle \mapsto e^{i\alpha}|1\dots1\rangle \\ |0\dots1\rangle \mapsto \vec{0} \end{cases}$

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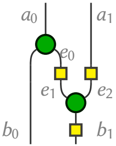
-  :: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha \mapsto \alpha|00\rangle + \alpha|11\rangle : \mathbb{C} \to \mathbb{C}^2$

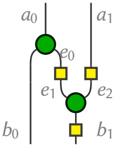
-  :: $\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = |xy\rangle \mapsto \delta_{x=y} : \mathbb{C}^2 \to \mathbb{C}$

$D =$ 

Token

3-tuple $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$ where:

- e is an edge of the ZX-Diagram D.
- d is a direction.
- b is the state of the token.

$D =$ 

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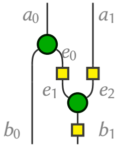
A *token state* is a **sum** of **products** of tokens with complex coefficients.

Token & Token State

$D =$ 

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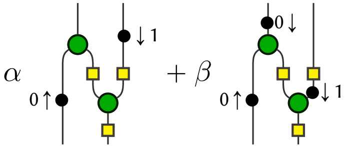
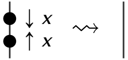
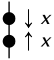
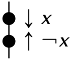


$$\langle t \,|\, t' \rangle = \begin{cases} 1 & \text{if } t = t' \\ 0 & \text{if } t \neq t' \end{cases}$$

- Collisions : 

- Collisions : 

$\rightsquigarrow$ 

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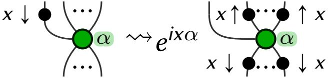
- Diffusions : 

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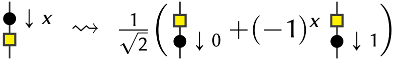

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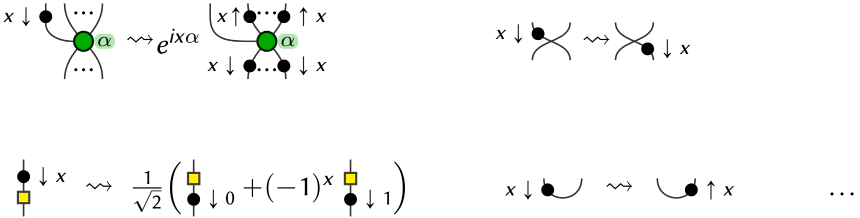
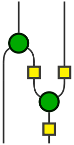


$\frac{1}{\sqrt{2}}$ CNOT =



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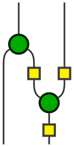
$\frac{1}{\sqrt{2}}$ CNOT =  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$

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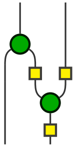

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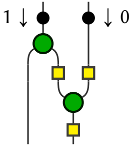
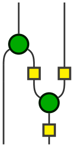
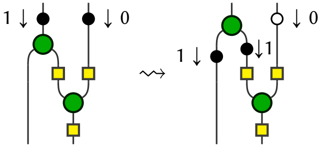


$\frac{1}{\sqrt{2}}$ CNOT $=$  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



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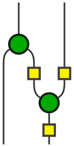

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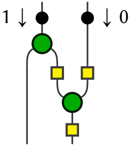

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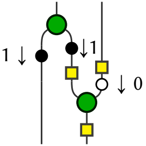

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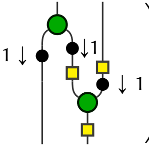

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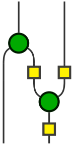
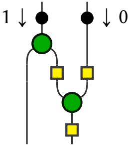
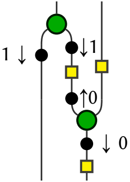
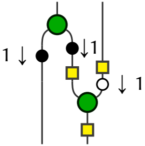
$\frac{1}{\sqrt{2}}$ CNOT =  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Longrightarrow |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



$\rightsquigarrow^* \frac{1}{\sqrt{2}} \Bigg($ 

$+$ 

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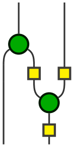
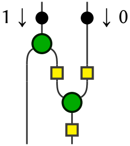
$\frac{1}{\sqrt{2}}$ CNOT $=$  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



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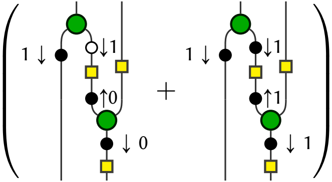
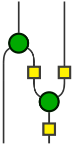
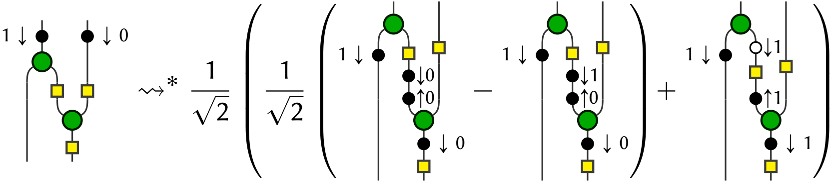
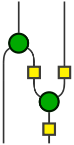
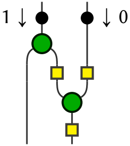
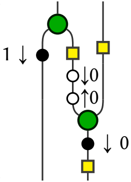
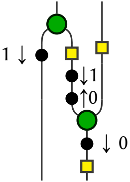
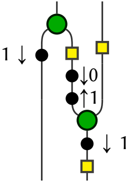
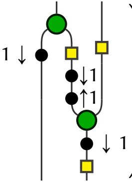


$\frac{1}{\sqrt{2}}$ CNOT =  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



$\frac{1}{\sqrt{2}}$ CNOT $=$  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$

 $\rightsquigarrow^* \ \frac{1}{2} \Bigg($  $-$  $+$  $-$  $\Bigg)$

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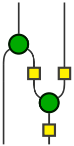

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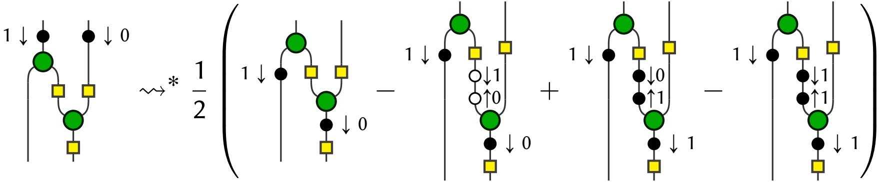

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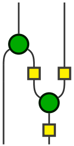

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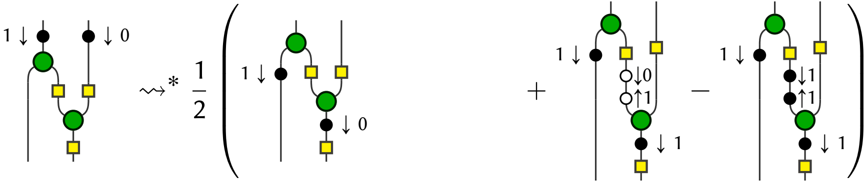

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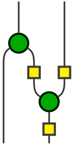

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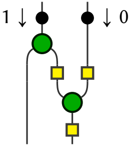

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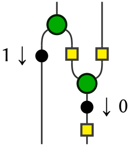

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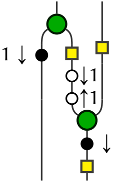

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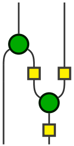
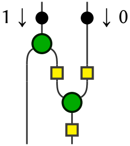
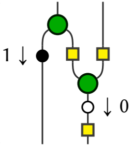
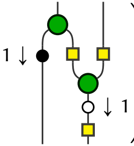
$\frac{1}{\sqrt{2}}$ CNOT =  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



$\rightsquigarrow^* \frac{1}{2} \Bigg($ 

$-$ 

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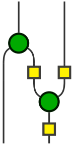
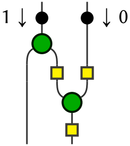
$\frac{1}{\sqrt{2}}$ CNOT $=$  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Longrightarrow |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



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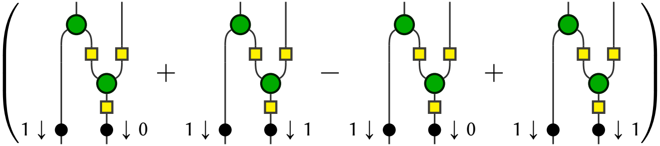
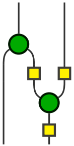


$$\frac{1}{\sqrt{2}}\,\mathrm{CNOT} =$$



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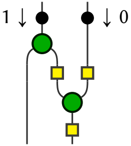

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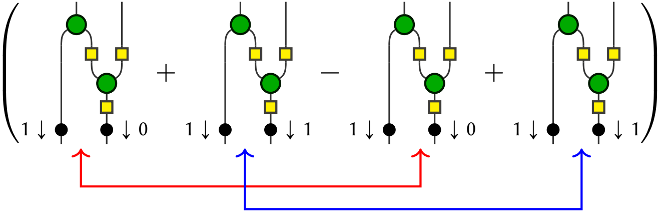
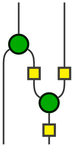
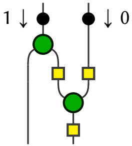


$\frac{1}{\sqrt{2}}$ CNOT $=$  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



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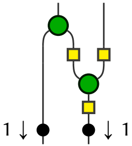
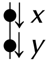


Rewriting System

We define $\rightsquigarrow$ as *exactly* one **diffusion rule** followed by all possible **collision** rules until none applies.

Want to avoid:

- Having multiple tokens on the same edge that don't collide: 

- Non-termination.

Avoiding Errors

Rewriting System

We define $\rightsquigarrow$ as *exactly* one **diffusion rule** followed by all possible **collision** rules until none applies.

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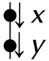
- Having multiple tokens on the same edge that don't collide: 

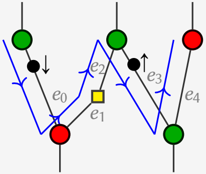
- Non-termination.

Two invariants:

- **Well-Formedness**: Avoid two tokens going in the same direction on a path.
- **Cycle-Balancedness**: Avoid tokens alone in cycles.

Polarity in a Path

$p = (e_0, e_1, e_2, e_3, e_4)$ is an oriented path.



- If a token follows the path $+1$
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity
  $P(p, (e_0 \downarrow x)(e_3 \uparrow y)) = P(p, (e_0 \downarrow x)) + P(p, (e_3 \uparrow y)) = 0$

Well-Formed Token State

Given a ZX-Diagram and a Token State, it is **Well-Formed** if for every path p its Polarity $\in \{-1, 0, 1\}$.

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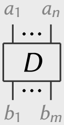
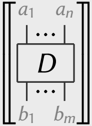
Cycle-Balanced Token State

Given a ZX-Diagram and a Token State, it is **Cycle-Balanced** if for every cycle c its Polarity $= 0$.

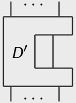
- Termination of well-formed, cycle-balanced token state.
- Local confluence of well-formed, cycle-balanced token state.

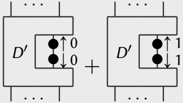
(Simulation of Standard Interpretation)

Let 

a ZX-Diagram such that 

$= \sum_{q=1}^{2^{m+n}} \lambda_q \left|y_{1,q} \dots y_{m,q}\right\rangle \left\langle x_{1,q} \dots x_{n,q}\right|$

Let $D =$ 

, consider $t =$ 

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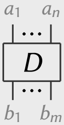
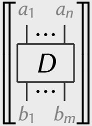
Let 

a ZX-Diagram such that 

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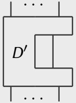
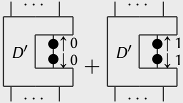
Let $D =$ 

, consider $t =$ 

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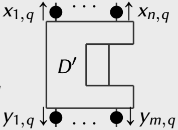
$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q$ 

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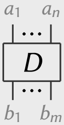
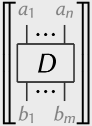
Let 

a ZX-Diagram such that 

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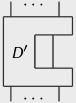
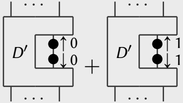
Let $D =$ 

, consider $t =$ 

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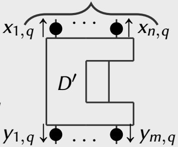
$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q$ 

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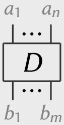
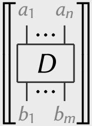
Let 

a ZX-Diagram such that 

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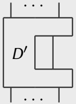
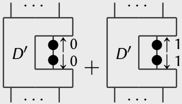
Let $D =$ 

, consider $t =$ 

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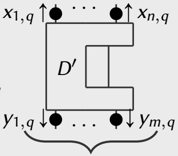
$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q$ 

Conclusion

- The Token Machine gives us a very general framework to study ZX-Calculus.
- More operational approach to ZX-Calculus.
- Can be easily adapted to extensions of ZX-Calculus (SOP, Mixed-Processes).
- General enough for any tensor networks.