

Geometry of Interaction for ZX-Diagrams

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SYCO 11

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B & \cdots \\ a_{10}B & \ddots & \\ \vdots & & \end{pmatrix}$

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- Operation are *linear maps*

- $H := \frac{1}{\sqrt{2}} \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \begin{array}{cc} \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{array}{c} |1\rangle \end{array} \\ \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) = \left\{ \begin{array}{l} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{array} \right. \text{ is unitary}$

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- $|+\rangle := H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
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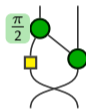
- $\text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\neg x\rangle \end{cases}$

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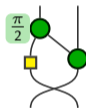
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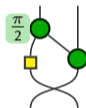
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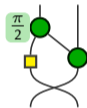
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- Is Universal (can encode any linear map)

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- Relaxes unitarity
- Is Universal (can encode any linear map)
- Lack a direct operational interpretation (this work !)

Generators



(Empty)



(Id)



(Swap)



(Cap)



(Cup)

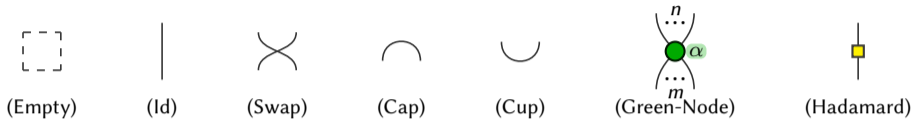


(Green-Node)

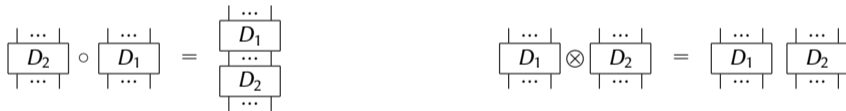


(Hadamard)

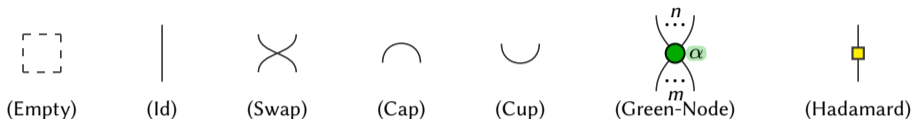
Generators



Compositions



Generators




Compositions



Standard Interpretation

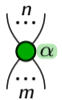
Linear Maps : $\mathbf{ZX} \rightarrow \mathcal{M}(\mathbb{C})$


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- A spider:  α $::$
$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \dots & \dots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0 \dots 0\rangle \mapsto |0 \dots 0\rangle \\ |1 \dots 1\rangle \mapsto e^{i\alpha} |1 \dots 1\rangle \\ |0 \dots 1\rangle \mapsto \vec{0} \end{cases}$$


• A spider: $\begin{array}{c} \left. \begin{array}{c} \dots \\ n \end{array} \right\} \\ \text{---} \alpha \\ \left. \begin{array}{c} \dots \\ m \end{array} \right\} \end{array} :: \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \dots & \dots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0 \dots 0\rangle \mapsto |0 \dots 0\rangle \\ |1 \dots 1\rangle \mapsto e^{i\alpha} |1 \dots 1\rangle \\ |0 \dots 1\rangle \mapsto \vec{0} \end{cases}$


• A change of basis: $\begin{array}{c} | \\ \text{---} \square \\ | \end{array} :: \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$

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- Wires: $\begin{vmatrix} | \\ | \end{vmatrix} :: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |x\rangle \mapsto |x\rangle$, $\begin{vmatrix} | & | \\ | & | \end{vmatrix} :: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = |xy\rangle \mapsto |yx\rangle$

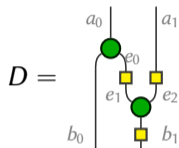
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-  :: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha \mapsto \alpha |00\rangle + \alpha |11\rangle : \mathbb{C} \rightarrow \mathbb{C}^2$

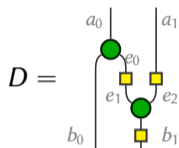
-  :: $(1 \ 0 \ 0 \ 1) = |xy\rangle \mapsto \delta_{x=y} : \mathbb{C}^2 \rightarrow \mathbb{C}$



Token

3-tuple $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$ where:

- e is an edge of the ZX-Diagram D .
- d is a direction.
- b is the state of the token.



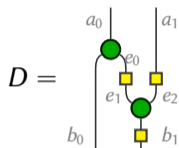
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Token State

A *token state* is a **sum** of **products** of tokens with complex coefficients.



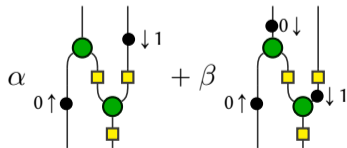
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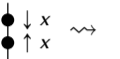
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$$\langle t | t' \rangle = \begin{cases} 1 & \text{if } t = t' \\ 0 & \text{if } t \neq t' \end{cases}$$

- Collisions : 

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• Collisions : $\begin{array}{c} \bullet \\ \downarrow x \\ \bullet \\ \uparrow x \end{array} \rightsquigarrow \left| \begin{array}{c} \bullet \\ \downarrow x \\ \bullet \\ \uparrow -x \end{array} \right. \rightsquigarrow 0$

• Diffusions : $\begin{array}{c} x \downarrow \bullet \\ \curvearrowright \\ \bullet \alpha \\ \curvearrowleft \\ \bullet \dots \\ \bullet \dots \end{array} \rightsquigarrow e^{ix\alpha} \begin{array}{c} x \uparrow \bullet \dots \bullet \uparrow x \\ \curvearrowright \\ \bullet \alpha \\ \curvearrowleft \\ \bullet \dots \bullet \downarrow x \\ \bullet \dots \bullet \downarrow x \end{array}$

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$$\begin{array}{c} \bullet \\ \downarrow x \\ \square \end{array} \rightsquigarrow \frac{1}{\sqrt{2}} \left(\begin{array}{c} \square \\ \bullet \downarrow 0 \end{array} + (-1)^x \begin{array}{c} \square \\ \bullet \downarrow 1 \end{array} \right)$$

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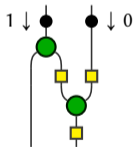
$\begin{array}{c} x \downarrow \bullet \\ \curvearrowright \end{array} \rightsquigarrow \begin{array}{c} \bullet \\ \curvearrowright \\ \bullet \\ \uparrow x \end{array} \dots$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

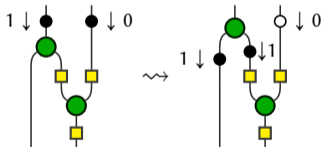
The diagram shows a quantum circuit with two vertical lines representing qubits. The top line has a green circle (CNOT control) connected to a yellow square (CNOT target) on the bottom line. The bottom line has a yellow square (CNOT target) connected to a green circle (CNOT control) on the top line. The circuit is symmetric and represents a CNOT gate with a $\frac{1}{\sqrt{2}}$ scaling factor.

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{---} \\ | \\ \bullet \\ \text{---} \\ | \\ \square \\ \text{---} \\ \bullet \\ | \\ \square \\ \text{---} \\ | \\ \text{---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

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$$\frac{1}{\sqrt{2}} \text{CNOT} = \left[\text{Quantum Circuit Diagram} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\left[\text{Quantum Circuit Diagram} \right] \rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\left[\text{Quantum Circuit Diagram 1} \right] \downarrow 0 + \left[\text{Quantum Circuit Diagram 2} \right] \downarrow 1 \right)$$

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$$\left[\text{Quantum Circuit Diagram} \right] \rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \left(\left[\text{Circuit 1} \right] - \left[\text{Circuit 2} \right] \right) + \left[\text{Circuit 3} \right] \right)$$

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$$\left[\text{Circuit Diagram} \right] \rightsquigarrow^* \frac{1}{2} \left(\left[\text{Circuit 1} \right] - \left[\text{Circuit 2} \right] + \left[\text{Circuit 3} \right] - \left[\text{Circuit 4} \right] \right)$$

The diagram shows the decomposition of a CNOT gate into a sum of four terms. Each term is a quantum circuit with two qubits. The first qubit starts in state $|1\rangle$ and the second in state $|0\rangle$. The terms are:

- Term 1:** A CNOT gate with control on the first qubit and target on the second. The second qubit ends in state $|0\rangle$.
- Term 2:** A CNOT gate with control on the second qubit and target on the first. The second qubit ends in state $|0\rangle$.
- Term 3:** A CNOT gate with control on the first qubit and target on the second. The second qubit ends in state $|1\rangle$.
- Term 4:** A CNOT gate with control on the second qubit and target on the first. The second qubit ends in state $|1\rangle$.

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$$\left[\text{Quantum Circuit Diagram} \right] \rightsquigarrow^* \frac{1}{2} \left(\left[\text{Quantum Circuit Diagram 1} \right] - \left[\text{Quantum Circuit Diagram 2} \right] \right)$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \left(\text{Quantum Circuit Diagram} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

The quantum circuit diagram shows two qubits. The top qubit has a control dot connected to a target dot on the bottom qubit. The target dot is connected to a CNOT gate (green circle). The control dot is also connected to a CNOT gate (green circle) on the bottom qubit. The target dot is connected to a CNOT gate (green circle) on the top qubit. The control dot is also connected to a CNOT gate (green circle) on the top qubit. The target dot is connected to a CNOT gate (green circle) on the bottom qubit.

$$\left(\text{Quantum Circuit Diagram} \right) \rightsquigarrow^* \frac{1}{2\sqrt{2}} \left(\begin{array}{c} \text{Circuit 1} \\ \text{Circuit 2} \\ - \text{Circuit 3} \\ + \text{Circuit 4} \end{array} \right)$$

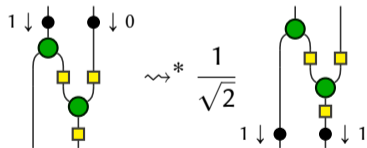
The diagram shows a quantum circuit with two qubits. The top qubit starts in state $|1\rangle$ and the bottom qubit starts in state $|0\rangle$. The circuit consists of a CNOT gate with control on the top qubit and target on the bottom qubit, followed by a CNOT gate with control on the bottom qubit and target on the top qubit. The resulting state is a superposition of four terms, each represented by a quantum circuit diagram with its own initial state:

- $|1\rangle$ (top), $|0\rangle$ (bottom)
- $|1\rangle$ (top), $|1\rangle$ (bottom)
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- $|1\rangle$ (top), $|1\rangle$ (bottom)

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} 1 \downarrow \bullet \quad \bullet \downarrow 0 \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \end{array} \rightsquigarrow^* \frac{1}{2\sqrt{2}} \left(\begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} - \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ | \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \square \text{---} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} \right)$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



Rewriting System

We define \rightsquigarrow as *exactly* one **diffusion rule** followed by all possible **collision** rules until none applies.

Want to avoid:

- Having multiple tokens on the same edge that don't collide:
- Non-termination.



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Two invariants:

- **Well-Formedness:** Avoid two tokens going in the same direction on a path.
- **Cycle-Balancedness:** Avoid tokens alone in cycles.

Polarity in a Path

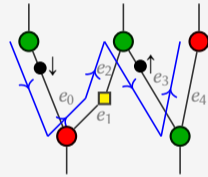
$p = (e_0, e_1, e_2, e_3, e_4)$ is an oriented path.

- If a token follows the path +1
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity

$$P(p, (e_0 \downarrow x)(e_3 \uparrow y)) = P(p, (e_0 \downarrow x)) + P(p, (e_3 \uparrow y)) = 0$$



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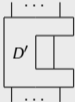
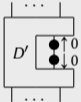
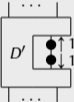
Given a ZX-Diagram and a Token State, it is **Cycle-Balanced** if for every cycle c its Polarity = 0.

- Termination of well-formed, cycle-balanced token state.
- Local confluence of well-formed, cycle-balanced token state.

(Simulation of Standard Interpretation)

Let D a ZX-Diagram such that

$$\left[\begin{array}{c} a_1 \quad a_n \\ \dots \\ \dots \\ b_1 \quad b_m \end{array} \right] D = \sum_{q=1}^{2^{m+n}} \lambda_q |y_{1,q} \dots y_{m,q}\rangle \langle x_{1,q} \dots x_{n,q}|$$

Let $D =$  , consider $t =$  $+$ 

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Let $D = \begin{array}{c} \dots \\ \text{---} \\ D' \\ \text{---} \\ \dots \end{array}$, consider $t = \begin{array}{c} \dots \\ \text{---} \\ D' \\ \bullet \uparrow \downarrow \bullet \\ \bullet \uparrow \downarrow \bullet \\ \text{---} \\ \dots \end{array}^0 + \begin{array}{c} \dots \\ \text{---} \\ D' \\ \bullet \uparrow \downarrow \bullet \\ \bullet \uparrow \downarrow \bullet \\ \text{---} \\ \dots \end{array}^1$

Then

$$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q \begin{array}{c} x_{1,q} \uparrow \quad \dots \quad \bullet \uparrow \quad x_{n,q} \\ \text{---} \\ D' \\ \text{---} \\ y_{1,q} \downarrow \quad \dots \quad \bullet \downarrow \quad y_{m,q} \end{array}$$

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(Simulation of Standard Interpretation)

Let D a ZX-Diagram such that $\left[\begin{array}{c} a_1 \ a_n \\ \dots \\ b_1 \ b_m \end{array} \right] D = \sum_{q=1}^{2^{m+n}} \lambda_q \underbrace{|y_{1,q} \dots y_{m,q}\rangle}_{\text{state}} \langle x_{1,q} \dots x_{n,q}|$

Let $D = \begin{array}{c} \dots \\ \text{---} \\ D' \\ \text{---} \\ \dots \end{array}$, consider $t = \begin{array}{c} \dots \\ \text{---} \\ D' \\ \text{---} \\ \dots \end{array} \begin{array}{c} \bullet \\ \updownarrow 0 \\ \bullet \\ \downarrow 0 \end{array} + \begin{array}{c} \dots \\ \text{---} \\ D' \\ \text{---} \\ \dots \end{array} \begin{array}{c} \bullet \\ \updownarrow 1 \\ \bullet \\ \downarrow 1 \end{array}$

Then

$$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q \begin{array}{c} x_{1,q} \uparrow \dots \uparrow x_{n,q} \\ \text{---} \\ D' \\ \text{---} \\ y_{1,q} \downarrow \dots \downarrow y_{m,q} \end{array}$$

Conclusion

- The Token Machine gives us a very general framework to study ZX-Calculus.
- More operational approach to ZX-Calculus.
- Can be easily adapted to extensions of ZX-Calculus (SOP, Mixed-Processes).
- General enough for any tensor networks.